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An Algorithm for Solving the System $-e \leq Ax \leq e,\ \|x\|_1 \geq 1$

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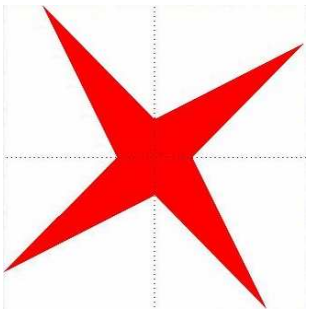
Abstract:

We describe a not-a-priori-exponential algorithm for solving the system $-e \leq Ax \leq e$, $\|x\|_1 \geq 1$. This system, despite its apparent simplicity, can be considered the basic NP-complete problem of interval computations.[2]



Keywords:
Linear inequalities, absolute value, NP-completeness, algorithm.

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[2]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction

In this paper we describe an algorithm for solving the system of inequalities

$$-e \le Ax \le e, \tag{1.1}$$

$$\|x\|_1 \ge 1, \tag{1.2}$$

where $A \in \mathbb{R}^{n\times n}$, $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ and $\|x\|_1 = \sum_i |x_i| = e^T|x|$, which can also be written in the equivalent "shorthand" form

$$|Ax| \le e, \qquad \|x\|_1 \ge 1. \tag{1.3}$$

The choice of the system may seem surprising: why just this system, and why such a specific form (using e and 1)? There are three reasons for this formulation.

First, in [2], Theorem 2.3 it was proved that *the problem of checking solvability of the system (1.1), (1.2) is NP-complete for nonnegative symmetric positive definite rational matrices*, and this result was further used there for proving NP-hardness or (co-)NP-completeness of nine other problems (see Theorems 2.12, 2.15, 2.18, 2.21, 2.30, 2.33, 2.38, 3.15 and 3.17 in [2]), thus having demonstrated that it is an ideal tool for establishing complexity results for problems with interval data. This is why this problem was called "the basic NP-complete problem of interval computations" in [4].

Second, it turns out that one of the basic NP-complete problems termed as such by Garey and Johnson [3] can be transformed to our problem.

And third – and this the topic of the present paper – there exists a not-a-priori-exponential algorithm for solving (1.1), (1.2) which, in turn, yields a not-a-priori-exponential algorithm for solving one of the basic NP-hard problems mentioned in the previous paragraph; formulation of the latter algorithm will possibly appear elsewhere.

The algorithm for solving (1.1), (1.2), which in a finite number of steps either finds a solution to (1.1), (1.2) or proves its nonexistence, is listed in the form of several interconnected MATLAB-like functions in the last Section 4. The preceding two sections bring the theoretical background and some examples.

2 Description

In order that the algorithm, whose description stretches over several pages, could be presented as a whole and not intertwined with the text, it is given in the last Section 4.

Theorem 1. *For each square matrix A the algorithm* **basintnpprob** *(Fig. 4.1) in a finite, but not-a-priori-exponential number of steps either finds a solution x of the system (1.1), (1.2) (the case of $x \ne [\,]$), or proves that no such a solution exists (the case of $x = [\,]$).*

Proof. As proved in [5], the algorithm **singreg** (Fig. 4.2), when applied to the interval matrix $[A - ee^T, A + ee^T]$ (Fig. 4.1, lines (06)-(07)) in a finite, but not-a-priori-exponential number of steps either yields a singular matrix S satisfying $|A-S| \le ee^T$ (the case of $S \ne [\,]$), or states that such a singular matrix S does not exist (the case of $S = [\,]$).

In the first case, taking an arbitrary $x \neq 0$ satisfying $Sx = 0$ (which exists because S is singular), we have

$$|Ax| = |(A - S)x| \leq |A - S||x| \leq ee^T|x| = \|x\|_1 e$$

so that for $x' = x/\|x\|_1$ we have $|Ax'| \leq e$ and $\|x'\|_1 = 1$, which means that x' solves (1.3) (Fig. 4.1, lines (09)-(10)).

In the second case there does not exist a singular matrix S satisfying $|A - S| \leq ee^T$. We shall prove that in this case the system (1.3) has no solution. Suppose to the contrary that (1.3) has a solution x. Then

$$|Ax| \leq e \leq e\|x\|_1 = ee^T|x|,$$

so that the interval matrix $[A - ee^T, A + ee^T]$ is singular, i.e., there exists a singular matrix S satisfying $|A - S| \leq ee^T$, a contradiction (Fig. 4.1, line (08)). □

3 Examples

In this section we give two examples with 100×100 matrices. In the first one a solution is found, whereas the second one has no solution.

```
>> tic, rand('state',1); n=100; A=rand(n,n); x=basintnpprob(A);
>> x', min(ones(n,1)-abs(A*x)), norm(x,1), toc
ans =
  Columns 1 through 10
 -0.0293 -0.0154 -0.0091  0.0138  0.0099 -0.0185  0.0186 -0.0082 -0.0076  0.0213
  Columns 11 through 20
  0.0074 -0.0204 -0.0157  0.0054  0.0303  0.0005  0.0155 -0.0003  0.0026 -0.0037
  Columns 21 through 30
 -0.0023  0.0111  0.0045 -0.0043  0.0043 -0.0027  0.0032 -0.0157  0.0070 -0.0069
  Columns 31 through 40
 -0.0067  0.0135  0.0097  0.0004 -0.0200  0.0013  0.0137 -0.0030 -0.0003  0.0033
  Columns 41 through 50
  0.0009 -0.0148 -0.0051  0.0008  0.0059 -0.0047  0.0054  0.0229 -0.0133  0.0294
  Columns 51 through 60
  0.0103  0.0101  0.0036  0.0028  0.0146  0.0215 -0.0288 -0.0113  0.0229 -0.0021
  Columns 61 through 70
 -0.0035 -0.0065  0.0161  0.0094  0.0051 -0.0048  0.0053 -0.0094 -0.0082 -0.0002
  Columns 71 through 80
  0.0046 -0.0094 -0.0128  0.0062 -0.0271 -0.0053  0.0013 -0.0169  0.0014 -0.0203
  Columns 81 through 90
  0.0225 -0.0145 -0.0092 -0.0110 -0.0008  0.0045 -0.0143  0.0081  0.0115  0.0201
  Columns 91 through 100
 -0.0168  0.0108  0.0026  0.0143  0.0050  0.0055 -0.0094 -0.0123 -0.0028 -0.0111
ans =
  0.9981
```

```
ans =
  1.0000
Elapsed time is 0.804711 seconds.

>> tic, rand('state',1); n=100; A=10000*rand(n,n); x=basintnpprob(A);
>> x', min(ones(n,1)-abs(A*x)), norm(x,1), toc
x =
     []
ans =
     []
ans =
     []
Elapsed time is 0.407147 seconds.
```

4 The algorithm

(01) **function** $x =$ **basintnpprob** (A)
(02) % BASic INTerval NP PROBlem.
(03) % $x \neq [\,]$: x solves $-e \leq Ax \leq e$, $\|x\|_1 \geq 1$.
(04) % $x = [\,]$: $-e \leq Ax \leq e$, $\|x\|_1 \geq 1$ has no solution.
(05) $n = \text{size}(A, 1)$; $e = \text{ones}(n, 1)$;
(06) $\mathbf{A} = [A - ee^T, A + ee^T]$;
(07) $S =$ **singreg** $(\mathbf{A})$;
(08) **if** $S = [\,]$, $x = [\,]$; **return**, **end**
(09) find an $x \neq 0$ satisfying $Sx = 0$;
(10) $x = x/\|x\|_1$;

Figure 4.1: An algorithm for solving the basic interval NP-complete problem.

```
(01)  function S = singreg (A)
(02)  % S ≠ []: S is a singular matrix in A.
(03)  % S = []: no singular matrix in A exists.
(04)   S = []; n = size(A, 1); e = (1, ..., 1)^T ∈ R^n;
(05)   if A_c is singular, S = A_c; return, end
(06)   R = A_c^{-1}; D = Δ|R|;
(07)   if D_kk = max_j D_jj ≥ 1
(08)      x = R_•k;
(09)      for i = 1 : n
(10)         if (Δ|x|)_i > 0, y_i = (A_c x)_i/(Δ|x|)_i; else y_i = 1; end
(11)         if x_i ≥ 0, z_i = 1; else z_i = −1; end
(12)      end
(13)      S = A_c − T_y Δ T_z; return
(14)   end
(15)   if ϱ(D) < 1, return, end
(16)   b = e;
(17)   x = Rb; γ = min_k |x_k|;
(18)   for i = 1 : n
(19)      for j = 1 : n
(20)         x' = x − 2b_j R_•j;
(21)         if min_k |x'_k| > γ, γ = min_k |x'_k|; x = x'; b_j = −b_j; end
(22)      end
(23)   end
(24)   [x, S] = intervalhull (A, [b, b]);
```

Figure 4.2: An algorithm for finding a singular matrix in an interval matrix.

```
(01)  function [x, S] = intervalhull (A, b)
(02)  % Computes either the interval hull x
(03)  % of the solution set of Ax = b,
(04)  % or a singular matrix S ∈ A.
(05)   x = [ ]; S = [ ];
(06)   if A_c is singular, S = A_c; return, end
(07)   x_c = A_c^{-1} b_c; z = sgn(x_c); x̲ = x_c; x̄ = x_c;
(08)   Z = {z}; D = ∅;
(09)   while Z ≠ ∅
(10)      select z ∈ Z; Z = Z − {z}; D = D ∪ {z};
(11)      [Q_z, S] = qzmatrix (A, z);
(12)      if S ≠ [ ], x = [ ]; return, end
(13)      [Q_{−z}, S] = qzmatrix (A, −z);
(14)      if S ≠ [ ], x = [ ]; return, end
(15)      x̄_z = Q_z b_c + |Q_z|δ;
(16)      x̲_z = Q_{−z} b_c − |Q_{−z}|δ;
(17)      if x̲_z ≤ x̄_z
(18)         x̲ = min(x̲, x̲_z); x̄ = max(x̄, x̄_z);
(19)         for j = 1 : n
(20)            z' = z; z'_j = −z'_j;
(21)            if ((x̲_z)_j (x̄_z)_j ≤ 0 and z' ∉ Z ∪ D)
(22)               Z = Z ∪ {z'};
(23)            end
(24)         end
(25)      end
(26)   end
(27)   x = [x̲, x̄];
(01)  function [Q_z, S] = qzmatrix (A, z)
(02)  % Computes either a solution Q_z
(03)  % of the equation QA_c − |Q|ΔT_z = I,
(04)  % or a singular matrix S ∈ A.
(05)  for i = 1 : n
(06)      [x, S] = absvaleqn (A_c^T, −T_z Δ^T, e_i);
(07)      if S ≠ [ ], S = S^T; Q_z = [ ]; return
(08)      end
(09)      (Q_z)_{i•} = x^T;
(10)  end
(11)  S = [ ];
```

Figure 4.3: An algorithm for computing the interval hull.

```
(01)  function [x, S] = absvaleqn (A, B, b)
(02)  % Finds either a solution x to Ax + B|x| = b, or
(03)  % a singular matrix S satisfying |S − A| ≤ |B|.
(04)  x = [ ]; S = [ ]; i = 0; r = 0 ∈ R^n; X = 0 ∈ R^{n×n};
(05)  if A is singular, S = A; return, end
(06)  z = sgn(A^{-1}b);
(07)  if A + BT_z is singular, S = A + BT_z; return, end
(08)  x = (A + BT_z)^{-1}b;
(09)  C = −(A + BT_z)^{-1}B;
(10)  while z_j x_j < 0 for some j
(11)     i = i + 1;
(12)     k = min{j | z_j x_j < 0};
(13)     if 1 + 2z_k C_kk ≤ 0
(14)        S = A + B(T_z + (1/C_kk)e_k e_k^T);
(15)        x = [ ]; return
(16)     end
(17)     if ((k < n and r_k > max_{k<j} r_j) or (k = n and r_n > 0))
(18)        x = x − X_•k;
(19)        for j = 1 : n
(20)           if (|B||x|)_j > 0, y_j = (Ax)_j/(|B||x|)_j; else y_j = 1; end
(21)        end
(22)        z = sgn(x);
(23)        S = A − T_y|B|T_z;
(24)        x = [ ]; return
(25)     end
(26)     r_k = i;
(27)     X_•k = x;
(28)     z_k = −z_k;
(29)     α = 2z_k/(1 − 2z_k C_kk);
(30)     x = x + αx_k C_•k;
(31)     C = C + αC_•k C_k•;
(32)  end
```

Figure 4.4: An algorithm for solving an absolute value equation.

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