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2012

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Abstract:

The note presents a new definition of graded fuzzy logic, different from that of Běhounek et al. Some few properties of graded fuzzy logic (in our new sense) are proven.

Keywords:
graded theories, mathematical fuzzy logic

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December 5, 2011

This short remark wants to be a contribution to the discussion on various possibilities of defining graded fuzzy logic. The reader is assumed to know basics of the mathematical fuzzy logic as developed in [4], in particular the basic (propositional and predicate) mathematical fuzzy logic BL and $BL\forall$. Several definitions of graded notions related to (t-norm based) fuzzy logic are analyzed in papers by Běhounek et al., some of them being cited below. For example, in [1] the formula $\bigwedge_{\alpha,\beta}(T\alpha\beta \equiv T\beta\alpha)$ (with $\equiv$ being fuzzy equivalence) is shown as a definition of degree of commutativity of T. The present approach is different from Běhounek's notions; several things are proven but a serious discussion of the offered notion is not contained here. Any comments on this are very welcome.

Let $\&, \rightarrow, 0$ be interpreted in the usual way using a continuous t-norm, let c be an element of the real interval $(0, 1]$. For $a, b, c \in [0, 1]$ let
$(a \rightarrow^c b) \equiv (c \rightarrow (a \rightarrow b))$,
$(a \equiv^c b) \equiv (c \rightarrow (a \equiv b))$.

Investigate connectives $\&^c$, $\rightarrow^c$, 0^c satisfying
$(a\&^c b) \equiv c \rightarrow (a\&b)$, $(a \rightarrow^c b) \equiv c \rightarrow (a \rightarrow b)$, $z \equiv 0^c \equiv^c z \equiv 0$.

Lemma. The following formulas are valid:
(1) $((a \equiv^c b)\&(u \equiv^c v)) \rightarrow ((a\&u) \equiv^{c^2} (b\&v))$
(2) $((a \equiv^{c^1} b)\&(b \equiv c^2 d)) \rightarrow (a \equiv^{c^1 \cdot c^2} d)$
(3) $(x\&^c 0^c) \equiv^{c^3} 0^c$
(4) $(1 \rightarrow^c y) \equiv^c y$
(5) $(x\&^c(x \rightarrow^c y)) \rightarrow^{c^2} y$

Proof.
(1) $((c \rightarrow (a \equiv b))\&(c \rightarrow (u \rightarrow v))) \rightarrow (c^2 \rightarrow ((a\&u) \equiv (b\&v)))$
(2) Similar proof.
(3) $(x\&^c 0^c) \equiv^c (x\&0^c) \equiv^c (x\&0) \equiv^1 0 \equiv^c 0^c$
(4) easy;
(5) $(x\&^c(x \rightarrow^c y)) \equiv^c (x\&(x \rightarrow^c y)) \equiv^c (x\&(x \rightarrow y)) \equiv^1 y$

Now let us analyze the axioms from [4] 2.2.4.

Lemma. The following formulas are valid:
$(A1^c)$ $(x \to^c y) \to^c ((y \to^c z) \to^c (x \to^c z))$
$(A2^c)$ $(x \&^c y) \to^c x$
$(A3^c)$ $(x \&^c y) \to^c (y \&^c x)$
$(A4^c)$ $(x \&^c (x \to^c y)) \to^c (y \&^c (y \to^c x))$
$(A51^c)$ $(x \to^c (y \to^c z)) \to^c (((x \&^c y) \to^c z)$
$(A6^c)$ $((x \to^c y) \to^c z) \to^c (((y \to^c x) \to^c z) \to^c z)$
$(A7^c)$ $0^c \to^c x$

Proof.
$(A1^c)$
$(x \to y) \to ((y \to z) \to (x \to z))$
$(x \to y) \to ((c \to (y \to z) \to (c \to (x \to z))$
$(c \to (x \to y)) \to (c \to ((c \to (y \to z) \to (c \to (x \to z)))$
$(x \to^c y) \to ((y \to^c z) \to^c (x \to^c z))$
$c \to [(x \to^c y) \to (((y \to^c z) \to^c (x \to^c z)))]$
$(x \to^c y) \to^c ((y \to^c z) \to^c (x \to^c z))$

$(A2^c)$
$(x \&^c y) \equiv^c (x \& y)$, $(x \& y) \to^1 x$, thus
$c \to ((x \&^c y) \to x)$
$(x \&^c y) \to^c x$

$(A3^c)$
$(c \to (x \& y)) \to (c \to (y \& x))$
$(x \&^c y) \to (y \&^c x)$
$(x \&^c y) \to^c (y \&^c x)$

$(A4^c)$
cx is an abbreviation of $c \& x$.
Each line implies the next one.
$x \&^c (x \to^c y))$
$c \to (x \& (x \to^c y))$
$c \to (x \& (c \to (x \to y)))$
$c^2 \to (cx \& (cx \to y))$
$c^2 \to ((y \to cx) \& y)$
$c \to (c \to ((y \to cx) \& y))$
$c \to (((y \to^c cx) \& y))$
$((y \to^c cx) \&^c y)$
$((y \to^c x) \&^c y)$

$(A51^c)$
$[x \to^c (y \to^c z)] \to [c \to (x \to (y \to (c \to z))] \to$
$\to [c \to (c \to (x \& y) \to z)] \to [c \to (c \to (x \& y) \to (c \to z))] \to$
$\to [c \to ((x \&^c y \to^c z)$, thus
$(x \to^c (y \to^c z)) \to^c (((x \&^c y) \to^c z))$.

$(A6^c)$
$((x \to y) \to z) \to (((y \to x) \to z) \to z)$
$((x \to y) \to z) \to^c (((y \to x) \to z) \to^c z)$
$((x \to^c y) \to^c z) \to^c (((y \to^c x) \to^c z) \to^c z)$

$(A7^c)$
$c \to (c \to 0) \to x$, thus $0^c \to^c x$.

Lemma (modus ponens.) ϕ^c a $(\phi \to \psi)^c$ implies $c \to \psi^c$.
Proof. $(\phi \to \psi)^c$ is $\phi^c \to^c \psi^c$, thus $c \to (\phi^c \to \psi^c)$.

Corollary. If BL proves ϕ then for some n BL^c proves $c^n \to \phi^c$.

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