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Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:rohn@cs.cas.cz

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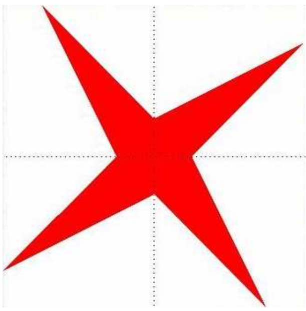
Abstract:

We describe a not-a-priori-exponential necessary and sufficient condition for regularity of interval matrices which is an easy consequence of an earlier result on interval linear equations.



Keywords:
Interval matrix, regularity, necessary and sufficient condition.[4]

[1]Department of Applied Mathematics and Computer Sciences, Hamedan Branch, Islamic Azad University, Hamedan, Iran, e-mail: vhoshiar@iauh.ac.ir.

[2]Young Researchers Club, Hamedan Branch, Islamic Azad University, Hamedan, Iran, e-mail: rfarhad@iauh.ac.ir.

[3]Institute of Computer Science, Czech Academy of Sciences, Prague, Czech Republic, e-mail: rohn@cs.cas.cz. This author's work was supported by the Institutional Research Plan AV0Z10300504.

[4]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2, 4]x_1 + [-2, 1]x_2 = [-2, 2]$, $[-1, 2]x_1 + [2, 4]x_2 = [-2, 2]$ (Barth and Nuding [1])).

Checking regularity of interval matrices is a known NP-hard problem. Forty necessary and sufficient regularity conditions are summed up in [2]; all of them are exponential because they explicitly or implicitly contain the quantifier "for each $z \in \{-1,1\}^n$". The condition given below is, to these authors' knowledge, the *first* ever published not-a-priori-exponential regularity condition because instead of $\{-1,1\}^n$ it employs only a subset Z of it. Cardinality of the set Z varies with the data, but its minimal value is 1. Notation used: e_j is the jth column of the $n \times n$ identity matrix, $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$, $\mathrm{diag}(z)$ is the $n \times n$ diagonal matrix with diagonal vector z and for an $x \in \mathbb{R}^n$, $\mathrm{sgn}(x)$ is defined by $(\mathrm{sgn}(x))_i = 1$ if $x_i \geq 0$ and $(\mathrm{sgn}(x))_i = -1$ otherwise.

Theorem 1. *An $n \times n$ interval matrix $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$ is regular if and only if A_c is nonsingular and there exists a subset Z of $\{-1,1\}^n$ having the following properties:*

(a) $\mathrm{sgn}(A_c^{-1}e) \in Z$,

(b) for each $z \in Z$ the inequalities

$$(QA_c - I)\,\mathrm{diag}(z) \geq |Q|\Delta, \qquad (0.1)$$
$$(QA_c - I)\,\mathrm{diag}(-z) \geq |Q|\Delta \qquad (0.2)$$

have matrix solutions Q_z and Q_{-z}, respectively,

(c) if $z \in Z$, $Q_{-z}e \leq Q_z e$, and $(Q_{-z}e)_j(Q_z e)_j \leq 0$ for some j, then $z - 2z_j e_j \in Z$.

Proof. "If": The assumptions (a)-(c) imply that the three assumptions of Theorem 3 in [3] are met for the system of interval linear equations $\boldsymbol{A}x = [e, e]$ whose solution set in virtue of the same theorem is bounded, hence $\boldsymbol{A}$ is regular. "Only if": If $\boldsymbol{A}$ is regular, then (a) and (c) are satisfied for $Z = \{-1,1\}^n$ and for each $z \in \{-1,1\}^n$ the equations

$$(QA_c - I)\,\mathrm{diag}(z) = |Q|\Delta,$$
$$(QA_c - I)\,\mathrm{diag}(-z) = |Q|\Delta$$

have (even unique) solutions, see [2]. ∎

Hence we can also formulate the theorem in the following way:

Theorem 2. *An $n \times n$ interval matrix $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$ is regular if and only if A_c is nonsingular and there exists a subset Z of $\{-1,1\}^n$ having the following properties:*

(a) $\mathrm{sgn}(A_c^{-1}e) \in Z$,

(b) for each $z \in Z$ the equations

$$(QA_c - I)\,\mathrm{diag}(z) = |Q|\Delta, \qquad (0.3)$$
$$(QA_c - I)\,\mathrm{diag}(-z) = |Q|\Delta \qquad (0.4)$$

have matrix solutions Q_z and Q_{-z}, respectively,

(c) if $z \in Z$, $Q_{-z}e \leq Q_z e$, and $(Q_{-z}e)_j(Q_z e)_j \leq 0$ for some j, then $z - 2z_j e_j \in Z$.

Notice that if $z \in \{-1,1\}^n$, then $z - 2z_j e_j \in \{-1,1\}^n$ (in (c)), so that $Z \subseteq \{-1,1\}^n$; thus Z is defined recursively by (a) and (c). In practical computations, equations (0.3), (0.4) are solved instead of inequalities (0.1), (0.2) as it was done in the function **qzmatrix**, using the subfunction **absvaleqn**, in [4].

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