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**Institute of Computer Science**  
**Academy of Sciences of the Czech Republic**

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and Jiri Rohn

Technical report No. V-1145

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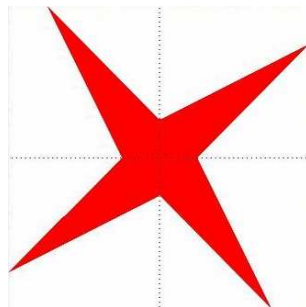
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Abstract:

We describe an iterative method for solving absolute value equations. The usual spectral condition is replaced by assumption of existence of two matrices satisfying certain matrix inequality.



Keywords:

Absolute value equation, iterative method.<sup>5</sup>

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<sup>5</sup>Above: logo of interval computations and related areas (depiction of the solution set of the system  $[2, 4]x_1 + [-2, 1]x_2 = [-2, 2]$ ,  $[-1, 2]x_1 + [2, 4]x_2 = [-2, 2]$  (Barth and Nuding [1])).

# 1 Introduction

The absolute value equation

$$Ax + B|x| = b \quad (1.1)$$

(where  $A, B \in \mathbb{R}^{n \times n}$  and  $b \in \mathbb{R}^n$ ) has been recently studied by several authors, cf. e.g. Caccetta, Qu and Zhou [2], Hu and Huang [3], Karademir and Prokopyev [4], Mangasarian [5], [6], Mangasarian and Meyer [7], Prokopyev [8], Rohn [9], [10], and Zhang and Wei [11]. Little attention was, however, dedicated so far to iterative methods for solving (1.1). In this report we formulate such a method (Theorem 1) working under assumption of existence of matrices  $M \geq 0$  and  $R$  satisfying the inequality

$$M(I - |I - RA| - |RB|) \geq I \quad (1.2)$$

which replaces the traditional spectral condition. This inequality is discussed in more detail in Sections 4 and 5, and interval iterations are proposed in Section 3.

# 2 The method

The following theorem is the basic result of this report.

**Theorem 1.** *Let  $M \geq 0$  and  $R$  satisfy (1.2). Then the sequence  $\{x^i\}_{i=0}^{\infty}$  given by  $x^0 = Rb$  and*

$$x^{i+1} = (I - RA)x^i - RB|x^i| + Rb \quad (i = 0, 1, \dots) \quad (2.1)$$

*converges to the unique solution  $x^*$  of (1.1) and for each  $i \geq 1$  there holds*

$$|x^* - x^i| \leq (M - I)|x^i - x^{i-1}|. \quad (2.2)$$

*Proof.* Let (1.2) have a solution  $M \geq 0$  and  $R$ . Denote

$$G = |I - RA| + |RB|,$$

then  $G \geq 0$  and the condition (1.2) can be written as

$$I + MG \leq M. \quad (2.3)$$

Postmultiplying this inequality by  $G$  and adding  $I$  to both sides we obtain

$$I + G + MG^2 \leq I + MG \leq M$$

and by induction

$$\sum_{j=0}^k G^j + MG^{k+1} \leq M$$

for  $k = 0, 1, 2, \dots$ . In view of nonnegativity of  $M$ , this shows that the nonnegative matrix series  $\sum_{j=0}^{\infty} G^j$  satisfies

$$\sum_{j=0}^{\infty} G^j \leq M, \quad (2.4)$$

hence it is convergent, so that  $G^j \rightarrow 0$  and consequently

$$\varrho(G) < 1. \quad (2.5)$$

Now we have

$$I - RA \leq |I - RA| \leq G,$$

hence

$$\varrho(I - RA) \leq \varrho(|I - RA|) \leq \varrho(G) < 1. \quad (2.6)$$

Since  $\varrho(I - RA) < 1$ , the matrix

$$RA = I - (I - RA) \quad (2.7)$$

is nonsingular, which gives that both  $R$  and  $A$  are nonsingular.

Let  $i \geq 1$ . Subtracting the equations

$$\begin{aligned} x^{i+1} &= (I - RA)x^i - RB|x^i| + Rb, \\ x^i &= (I - RA)x^{i-1} - RB|x^{i-1}| + Rb, \end{aligned}$$

we get

$$|x^{i+1} - x^i| \leq |I - RA||x^i - x^{i-1}| + |RB|||x^i| - |x^{i-1}|| \leq G|x^i - x^{i-1}|$$

and for each  $m \geq 1$  by induction

$$\begin{aligned} |x^{i+m} - x^i| &= \left| \sum_{j=0}^{m-1} (x^{i+j+1} - x^{i+j}) \right| \leq \sum_{j=0}^{m-1} |x^{i+j+1} - x^{i+j}| \leq \sum_{j=0}^{m-1} G^{j+1} |x^i - x^{i-1}| \\ &\leq \left( \sum_{j=0}^{\infty} G^{j+1} \right) |x^i - x^{i-1}| \leq (M - I) |x^i - x^{i-1}| \leq (M - I) G^{i-1} |x^1 - x^0|. \end{aligned}$$

From the final inequality

$$|x^{i+m} - x^i| \leq (M - I) G^{i-1} |x^1 - x^0|,$$

in view of the fact that  $G^{i-1} \rightarrow 0$  as  $i \rightarrow \infty$ , we can see that the sequence  $\{x^i\}$  is Cauchian, thus convergent,  $x^i \rightarrow x^*$ . Taking the limit in (2.1) we get

$$x^* = (I - RA)x^* - RB|x^*| + Rb$$

which gives

$$0 = R(Ax^* + B|x^*| - b),$$

and employing the above-proved nonsingularity of  $R$ , we can see that  $x^*$  is a solution of (1.1). The estimation (2.2) follows from the above-established inequality

$$|x^{i+m} - x^i| \leq (M - I) |x^i - x^{i-1}|$$

by taking  $m \rightarrow \infty$ .

To prove uniqueness, assume that (1.1) has a solution  $x'$ . From the equations

$$\begin{aligned} Ax^* + B|x^*| &= b, \\ Ax' + B|x'| &= b \end{aligned}$$

written as

$$\begin{aligned} x^* &= (I - RA)x^* - RB|x^*| + Rb, \\ x' &= (I - RA)x' - RB|x'| + Rb \end{aligned}$$

we obtain

$$|x^* - x'| \leq G|x^* - x'|,$$

hence

$$(I - G)|x^* - x'| \leq 0$$

and by (2.3),

$$|x^* - x'| \leq M(I - G)|x^* - x'| \leq 0,$$

which shows that  $x' = x^*$ . This proves uniqueness. ▀

Employing the approach used in the last part of the proof, we can derive an estimation of the distance of the solution from an arbitrary point in  $\mathbb{R}^n$ .

**Theorem 2.** *Let  $M \geq 0$  and  $R$  satisfy (1.2). Then for the unique solution  $x^*$  of (1.1) there holds*

$$|x^* - x| \leq M|R(Ax + B|x| - b)| \quad (2.8)$$

for each  $x \in \mathbb{R}^n$ .

*Proof.* Let (1.2) have a solution  $M \geq 0$  and  $R$ . Denote

$$G = |I - RA| + |RB|.$$

Then  $G \geq 0$  and the condition (1.2) can be written as

$$I \leq M(I - G). \quad (2.9)$$

Now take an arbitrary  $x \in \mathbb{R}^n$  and put

$$Ax + B|x| - b = r,$$

then premultiplying this equality by  $R$  and adding  $x$  to both sides we obtain

$$x = (I - RA)x - RB|x| + Rb + Rr.$$

Since  $x^*$  is the unique solution of (1.1) we have

$$Ax^* + B|x^*| = b,$$

which similarly implies that

$$x^* = (I - RA)x^* - RB|x^*| + Rb.$$

Hence

$$|x^* - x| \leq |I - RA||x^* - x| + |RB||x^* - x| + |Rr| = G|x^* - x| + |Rr|.$$

Using relation 2.9 we have

$$|x^* - x| \leq M(I - G)|x^* - x| \leq M|Rr|.$$

Consequently

$$|x^* - x| \leq M|R(Ax + B|x| - b)|$$

which was to be proved. ■

The result can also be formulated in an explicit interval form.

**Theorem 3.** *Under assumptions and notation of Theorem 2 we have*

$$x^* \in [x - M|R(Ax + B|x| - b)|, x + M|R(Ax + B|x| - b)|] \quad (2.10)$$

for each  $x \in \mathbb{R}^n$ .

*Proof.* Clearly, the relation (2.8) can be equivalently rewritten as

$$x - M|R(Ax + B|x| - b)| \leq x^* \leq x + M|R(Ax + B|x| - b)|,$$

which is (2.10). ■

### 3 Interval iterations

When generating the sequence (2.1) on computer, instead of the true sequence

$$x^0, x^1, \dots, x^i, \dots$$

we compute a floating-point sequence

$$\tilde{x}^0, \tilde{x}^1, \dots, \tilde{x}^i, \dots$$

for which the estimation (2.2) no longer remains in force. To overcome this difficulty, we may resort to interval iterations. For each  $i \geq 0$ , using outward rounding (e.g. in INTLAB), first compute verified enclosures

$$\tilde{x}^i - M|R(A\tilde{x}^i + B|\tilde{x}^i| - b)| \in [\underline{y}^i, \bar{y}^i],$$

$$\tilde{x}^i + M|R(A\tilde{x}^i + B|\tilde{x}^i| - b)| \in [\underline{z}^i, \bar{z}^i],$$

where  $\underline{y}^i, \bar{y}^i, \underline{z}^i, \bar{z}^i$  are floating-point vectors; then by Theorem 3 we have the verified enclosure

$$x^* \in [\underline{y}^i, \bar{z}^i]$$

for each  $i = 0, 1, 2, \dots$ . Now define

$$\mathbf{x}^0 = [\underline{y}^0, \bar{z}^0]$$

and

$$\mathbf{x}^{i+1} = [\underline{y}^{i+1}, \bar{z}^{i+1}] \cap \mathbf{x}^i \quad (i = 0, 1, 2, \dots),$$

then

$$\mathbf{x}^0 \supseteq \mathbf{x}^1 \supseteq \dots \supseteq \mathbf{x}^i \supseteq \dots \ni x^*$$

so that we get a nested sequence of floating-point interval vectors each of whom is verified to contain the solution  $x^*$ .

## 4 The inequality (1.2)

Let us write the condition (1.2), as before, in the form

$$M(I - G) \geq I, \quad (4.1)$$

$$M \geq 0, \quad (4.2)$$

where

$$G = |I - RA| + |RB|.$$

In this section we shall be interested in finding  $M$  and  $R$  satisfying (4.1), (4.2). Obviously, such  $M$  and  $R$  do not exist always because their existence implies unique solvability of (1.1) (Theorem 1). We show how  $M$  and  $R$  satisfying (4.1), (4.2) can be found.

**Theorem 4.** *If (assuming both inverses to exist)*

$$(I - |A^{-1}B|)^{-1} \geq 0,$$

then

$$R = A^{-1}, \quad (4.3)$$

$$M = (I - |A^{-1}B|)^{-1} \quad (4.4)$$

satisfy (4.1), (4.2).

*Proof.* Indeed,  $M \geq 0$  by assumption, and  $M(I - |I - RA| - |RB|) = M(I - |A^{-1}B|) = I$ .  $\blacksquare$

This gives us the clue: in practical computations choose  $R$  and  $M$  as the *computed* values of (4.3), (4.4).

## 5 Condensed form of conditions (4.1), (4.2)

The two conditions (4.1), (4.2) can be merged into a single “condensed” condition

$$M \geq I + |M|G. \quad (5.1)$$

In fact, if (5.1) holds, then  $M \geq 0$  (because  $G \geq 0$ ), hence  $|M| = M$  and (5.1) turns into (4.1). Conversely, if (4.1), (4.2) hold, then  $M = |M|$  and the condition (4.1), if written as  $M \geq I + MG$ , becomes (5.1).

This, of course, is only a technical trick, which, however, spares us of necessity of emphasizing nonnegativity of  $M$  whenever mentioning the condition (4.1).



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