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# Graded dominance and related graded properties of fuzzy connectives

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Abstract:

This text is a further elaboration of the previous research report [5]. Graded properties of binary and unary connectives valued in $\mathrm{MTL}_\triangle$-algebras are studied in the framework of Fuzzy Class Theory (or higher-order fuzzy logic) FCT, which serves as a tool for easy derivation of graded theorems. The properties studied include graded monotonicity, a generalized Lipschitz property, commutativity, associativity, unit elements, and dominance. Graded generalizations of known non-graded theorems as well as new graded theorems on the graded properties are proved in FCT. The graded notion of dominance is applied to the transmission of graded transitivity and extensionality of fuzzy relations.

Keywords:
Fuzzy relation, Fuzzy connective, Fuzzy Class Theory, Dominance

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# 1 Introduction

This report summarizes and further elaborates the results of the previous papers and reports on the topic: [5, 3, 4]. For the motivation of the investigation carried out in this report see the introductory sections of [5, 3, 4]. For the general context of Fuzzy Class Theory and graded fuzzy mathematics see esp. [6, 7].

# 2 Preliminaries

The general results of this paper are derived in the framework of higher-order fuzzy logic, also known as Fuzzy Class Theory (FCT). FCT is an axiomatic theory of Zadeh's notions of fuzzy set [18] and fuzzy relation [19] in formal fuzzy logic (in the sense of [14]). For reference, a (slightly simplified) definition of FCT is given below; for more details see the original paper [6] or the freely available primer [8].

We shall use the variant of FCT over the logic $\mathrm{MTL}_\triangle$ of all left-continuous t-norms [13], one of the weakest fuzzy logics suitable for this type of graded fuzzy mathematics. We assume the reader's familiarity with the first-order logic $\mathrm{MTL}_\triangle$; here we shall just briefly recall the standard semantics of its connectives and quantifiers over the real unit interval $[0, 1]$:

| | | |
|---|---|---|
| $\&$ | ... | any left-continuous t-norm $*$ |
| $\to$ | ... | the residuum $\Rightarrow_*$ of $*$, defined as $x \Rightarrow_* y =_{\mathrm{df}} \sup\{z \mid z * x \le y\}$ |
| $\wedge, \vee$ | ... | min, max |
| $\neg$ | ... | $\neg x =_{\mathrm{df}} x \Rightarrow_* 0$ |
| $\leftrightarrow$ | ... | bi-residuum: $(x \Rightarrow_* y) \wedge (y \Rightarrow_* x)$ |
| $\triangle$ | ... | $\triangle x = 1$ if $x = 1$; $\triangle x = 0$ otherwise |
| $\forall, \exists$ | ... | inf, sup |

By means of this 'dictionary', the formal results formulated and proved in $\mathrm{MTL}_\triangle$ can be translated into the more common semantic notions: for instance, the defining formula of Definition 3.1,

$$\mathrm{Cng}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall \alpha\beta)((\alpha \leftrightarrow \beta) \to (\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)),$$

expresses the following semantic definition of the degree of the fuzzy property Cng for a unary fuzzy connective $\mathbf{u}$:

$$\mathrm{Cng}(\mathbf{u}) = \bigwedge_{\alpha,\beta} \Big( \big((\alpha \Rightarrow_* \beta) \wedge (\beta \Rightarrow_* \alpha)\big) \Rightarrow_* \big((\mathbf{u}(\alpha) \Rightarrow_* \mathbf{u}(\beta)) \wedge (\mathbf{u}(\beta) \Rightarrow_* \mathbf{u}(\alpha))\big) \Big),$$

for any given left-continuous t-norm $*$. In this manner, all formulae encountered in this paper can be understood as denoting the corresponding semantic facts about standard fuzzy sets and fuzzy relations.

Recall further that since $x \Rightarrow_* y$ equals 1 iff $x \le y$, theorems with implication as the principal connective express the comparison of degrees. Thus, e.g., in the claim (R9) of Theorem 5.5 on page 17 below, the formula $\mathrm{Com}(\mathbf{c}) \ \Rightarrow\ \mathrm{Sym}(R \,\underline{\mathbf{c}}\, R^{-1})$ expresses the semantic fact that the degree of $\mathrm{Com}(\mathbf{c})$ is less than or equal to the degree of $\mathrm{Sym}(R \,\underline{\mathbf{c}}\, R^{-1})$. (Notice that sometimes we use the sign $\Rightarrow$ for $\to$ and $\Leftrightarrow$ for $\leftrightarrow$, by Convention 2.1 below.) Similarly theorems with $\leftrightarrow$ as the principal connective express the identity of degrees; so for example the claim (C1) of Theorem 3.2 on page 4 below, $\mathrm{MonCng}(\mathbf{u}) \ \Leftrightarrow\ \mathrm{Mon}(\mathbf{u}) \wedge \mathrm{Cng}(\mathbf{u})$, expresses the fact that the degree of $\mathrm{MonCng}(\mathbf{u})$ equals the minimum of the degrees of $\mathrm{Mon}(\mathbf{u})$ and $\mathrm{Cng}(\mathbf{u})$, for any left-continuous t-norm $*$. For more details on the meaning of formulae in $\mathrm{MTL}_\triangle$ and FCT see [7, 8].

**Convention 2.1** For better readability of complex formulae, we shall alternatively use the comma (,) for &; the symbol $\Rightarrow$ for $\to$; and $\Leftrightarrow$ for $\leftrightarrow$. The symbols $\Rightarrow$ and $\Leftrightarrow$ can be chained, with $\varphi_1 \Rightarrow \varphi_2 \Rightarrow \ldots \Rightarrow \varphi_n$ representing the formula $(\varphi_1 \to \varphi_2) \,\&\, (\varphi_2 \to \varphi_3) \,\&\ldots\&\, (\varphi_{n-1} \to \varphi_n)$, and similarly for $\Leftrightarrow$. The sign $\equiv$ will indicate equivalence by definition. By convention, the symbols $\Rightarrow$, $\Leftrightarrow$, and $\equiv$ will have the lowest priority in formulae and the comma the second lowest priority. Of other symbols, $\to$ and

$\leftrightarrow$ will have lower priority than other binary connectives, and quantifiers and unary connectives will have the highest priority.

We shall write $\varphi^n$ for $\varphi \& \ldots \& \varphi$ ($n$ times). In atomic subformulae, the superscript can be attached directly to the predicate, e.g., $\mathrm{Com}^2(\mathbf{c})$ or $\mathbf{c} \approx^3 \mathbf{d}$. Furthermore we shall employ the following defined connectives that express the ordering and equality of truth degrees:

$$\begin{aligned} \varphi \leq \psi &\equiv_{\mathrm{df}} \triangle(\varphi \to \psi) \\ \varphi = \psi &\equiv_{\mathrm{df}} \triangle(\varphi \leftrightarrow \psi) \end{aligned}$$

The priority of these connectives is the same as that of implication.

Fuzzy class theory FCT, or Henkin-style higher-order fuzzy logic $\mathrm{MTL}_\triangle$, is an axiomatic theory over multi-sorted first-order logic $\mathrm{MTL}_\triangle$, with sorts of variables for:

- *Atomic elements,* denoted by lowercase letters $x, y, \ldots$
- *Fuzzy classes*[6] of atomic elements, denoted by uppercase letters $A, B, \ldots$
- *Fuzzy classes of fuzzy classes* of atomic elements, denoted by calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$
- Etc.; in general for *fuzzy classes of the n-th order*, written as $X^{(n)}, Y^{(n)}, \ldots$

The primitive symbols of FCT are:

- The *identity predicates* $=$ on each sort
- The *membership predicates* $\in$ between successive sorts
- The symbols for *tuples* $\langle x_1, \ldots, x_k \rangle$ of individuals $x_1, \ldots, x_k$ of any order and all arities $k \in \mathbb{N}$

The formula $x \in A$ and the term $\langle x_1, \ldots, x_k \rangle$ may be abbreviated, respectively, as $Ax$ and $x_1 \ldots x_k$. FCT has the following axioms, for all formulae $\varphi$ and variables of any order:

- The logical axioms of multi-sorted first-order logic $\mathrm{MTL}_\triangle$
- The *identity axioms:* $x = x$ and $x = y \to (\varphi(x) \leftrightarrow \varphi(y))$
- The *tuple-identity axioms:* $\langle x_1, \ldots, x_k \rangle = \langle y_1, \ldots, y_k \rangle \to x_i = y_i$, for all $k \in \mathbb{N}$ and $1 \leq i \leq k$
- The *comprehension axioms:* $(\exists A)(Ax = \varphi(x))$
- The *extensionality axioms:* $(\forall x)(Ax = Bx) \to A = B$

The axioms for identity entail that the identity predicate $=$ on each sort is crisp (while the membership predicate $\in$ can in general be fuzzy). Notice that due to the logical axioms, theorems of FCT need be proved by the rules of the logic $\mathrm{MTL}_\triangle$ rather than classical logic.

The models of FCT are systems of fuzzy sets and fuzzy relations of all finite arities and orders over a fixed crisp set $X$ (the universe of discourse) that are closed under all FCT-definable operations and whose membership degrees take values in any fixed $\mathrm{MTL}_\triangle$-chain (standardly, the real unit interval $[0, 1]$ equipped with a left-continuous t-norm). All theorems of FCT are therefore valid for Zadeh's $[0, 1]$-valued fuzzy sets and fuzzy relations (of all finite arities and orders).

[6] For certain formal reasons, in FCT we use the term *fuzzy class* besides the more common *fuzzy set* (for details on this distinction, irrelevant to the present paper, see, e.g., [2, Sect. 2.1]).

**Definition 2.2** In FCT, we introduce the following graded properties of and relations between fuzzy classes:

| | |
|---|---|
| $A \subseteq B \equiv_{\mathrm{df}} (\forall x)(Ax \to Bx)$ | inclusion |
| $A \sqsubseteq B \equiv_{\mathrm{df}} \triangle(A \subseteq B)$ | crisp inclusion |
| $A \approx B \equiv_{\mathrm{df}} (\forall x)(Ax \leftrightarrow Bx)$ | weak bi-inclusion |
| $A \approxeq B \equiv_{\mathrm{df}} (A \subseteq B) \mathbin{\&} (B \subseteq A)$ | strong bi-inclusion |
| $\mathrm{Hgt}\, A \equiv_{\mathrm{df}} (\exists x)Ax$ | height |
| $\mathrm{Plt}\, A \equiv_{\mathrm{df}} (\forall x)Ax$ | plinth |
| $\mathrm{Crisp}\, A \equiv_{\mathrm{df}} (\forall x)\triangle(Ax \vee \neg Ax)$ | crispness |

Moreover, we define the following operations on fuzzy classes:

| | |
|---|---|
| $A^k x_1 \ldots x_k \equiv_{\mathrm{df}} Ax_1 \mathbin{\&} \ldots \mathbin{\&} Ax_k$ | Cartesian power |
| $R^{-1}xy \equiv_{\mathrm{df}} Ryx$ | converse relation |

Finite fuzzy classes $\{a_1/\alpha_1, \ldots, a_n/\alpha_n\}$ are defined as follows:

$$x \in \{a_1/\alpha_1, \ldots, a_n/\alpha_n\} \equiv_{\mathrm{df}} ((x = a_1) \mathbin{\&} \alpha_1) \vee \ldots \vee ((x = a_n) \mathbin{\&} \alpha_n).$$

By convention, we can write just $a_i$ instead of $a_i/1$ in the above notation (thus, e.g., $\{a\}$ is the crisp singleton of $a$).

In this paper we shall deal with fuzzy connectives, i.e., algebraic operations on truth degrees. Even though truth degrees are not part of the primitive language of FCT, they can be represented in the theory by subclasses of a crisp singleton. The details of the representation (for which see [9, Sect. 3] and [1]) are not important for our present purposes; we shall thus simply assume that variables $\alpha, \beta, \ldots$ for truth values are at our disposal in FCT, and that the ordering of truth values and the usual propositional connectives and the quantifiers $\forall, \exists$ are definable in FCT. The crisp class of the internal truth values will be denoted by L.

The study of fuzzy connectives in the framework of FCT was initiated in [3, 4]. By *n-ary fuzzy connectives* we understand $n$-ary operations on truth degrees i.e., crisp functions $\mathbf{c}\colon \mathrm{L}^n \to \mathrm{L}$. Being functions into L, they can equivalently be regarded as fuzzy relations on $\mathrm{L}^n$; i.e., $\mathbf{c} \sqsubseteq \mathrm{L}^n$. Thus, e.g., fuzzy inclusion $\mathbf{c} \subseteq \mathbf{d}$ of binary fuzzy connectives $\mathbf{c}, \mathbf{d}$ will be understood as inclusion of fuzzy relations $\mathbf{c}, \mathbf{d}\colon \mathrm{L}^2 \to \mathrm{L}$, i.e., $\mathbf{c} \subseteq \mathbf{d} \equiv (\forall \alpha)(\mathbf{c}\alpha\beta \to \mathbf{d}\alpha\beta)$, rather than as inclusion of crisp functions. Similarly, the converse fuzzy connective is defined as $\mathbf{c}^{-1}\alpha\beta \equiv \mathbf{c}\beta\alpha$ (cf. Definition 2.2).

**Convention 2.3** We shall always use Greek letters for truth values, the letters $\mathbf{u}, \mathbf{v}, \mathbf{w}$ for unary connectives, and the letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ for binary connectives. Infix notation $\alpha \mathbin{\mathbf{c}} \beta$ will usually be employed for binary connectives instead of prefix notation $\mathbf{c}\alpha\beta$. In formulae, infix binary connectives will by convention have the same priority as &: thus, e.g., $\neg\alpha \mathbin{\mathbf{c}} \beta \to \gamma$ will mean $((\neg\alpha) \mathbin{\mathbf{c}} \beta) \to \gamma$.

# 3 Graded properties of unary and binary connectives

Many crisp classes of truth-value operators (e.g., t-norms, uninorms, copulas, negations, etc.) can be defined by formulae of FCT. The apparatus, however, enables also *partial* satisfaction of such conditions. In the following, we therefore give several *fuzzy* conditions on truth-value operators and use them as graded preconditions of theorems which need not be satisfied to the full degree. This yields a completely new *graded* theory of truth-value operators and allows non-trivial generalizations of well-known theorems on such operators, including their consequences for properties of fuzzy relations.

Here we present the basics of the theory needed for the main topic of this paper, the graded notion of dominance relation. For a further elaboration of the theory of graded properties of fuzzy connectives, including illustrative examples, see [1].

## 3.1 Graded properties of unary connectives

Graded properties of unary connectives can be obtained by 'fuzzification' (consisting in replacement of the crisp comparison $\leq, =$ by the graded operations $\rightarrow, \leftrightarrow$ in defining formulae) of usual crisp properties of functions: this method usually yields meaningful graded properties, and therefore will in this paper be employed systematically to obtain graded versions of well-known crisp properties of fuzzy connectives.

**Definition 3.1** *In* FCT*, we define the following graded properties of a unary connective* $\mathbf{u} \sqsubseteq \mathrm{L}$*:*

$$\mathrm{Mon}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \leq \beta) \rightarrow (\mathbf{u}\alpha \rightarrow \mathbf{u}\beta)) \qquad \textit{Graded monotonicity}$$
$$\mathrm{Cng}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \leftrightarrow \beta) \rightarrow (\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)) \qquad \textit{Graded congruence w.r.t.} \leftrightarrow$$

Analogously to monotonicity we could define the property of *antitonicity,* defined as $(\forall\alpha\beta)((\alpha \leq \beta) \rightarrow (\mathbf{u}\beta \rightarrow \mathbf{u}\alpha))$. We shall not study it in this paper due to its little relevance to the properties of dominance. For more details about this property see [1].

The graded property $\mathrm{Cng}(\mathbf{u})$ gives, roughly speaking, the degree to which $\mathbf{u}$ yields close values for close arguments, where closeness is evaluated in the sense of $\leftrightarrow$. In particular, in standard Łukasiewicz models of FCT, where $\leftrightarrow$ corresponds to the Euclidean distance, the property $\triangle\,\mathrm{Cng}(\mathbf{u})$ expresses the 1-Lipschitz property of $\mathbf{u}$. If $\mathbf{u}$ is regarded as a fuzzy class $\mathbf{u} \sqsubseteq \mathrm{L}$ rather than a crisp unary operation $\mathbf{u}\colon \mathrm{L} \rightarrow \mathrm{L}$, then $\mathrm{Cng}(\mathbf{u})$ expresses extensionality (see Definition 2.2) of $\mathbf{u}$ w.r.t. $\leftrightarrow$. The property will play an important role in many graded theorems on fuzzy connectives, as it denotes the largest guaranteed degree of intersubstitutivity of $\mathbf{u}\alpha$ and $\mathbf{u}\beta$ for close (in the sense of $\leftrightarrow$) arguments $\alpha$ and $\beta$.

In models of FCT, the crisp condition

$$\triangle\,\mathrm{Mon}(\mathbf{u}) \equiv (\forall\alpha\beta)((\alpha \leq \beta) \rightarrow (\mathbf{u}\alpha \leq \mathbf{u}\beta))$$

expresses the usual crisp condition of monotonicity of $\mathbf{u}$ w.r.t. the ordering $\leq$ on L. Its graded version $\mathrm{Mon}(\mathbf{u})$ arises from replacing the second occurrences of $\leq$ by $\rightarrow$. Notice that replacing both $\leq$'s in $\triangle\,\mathrm{Mon}(\mathbf{u})$ by $\rightarrow$ would not yield a graded generalization of monotonicity as the resulting notion

$$\mathrm{MonCng}(\mathbf{u}) \equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \rightarrow \beta) \rightarrow (\mathbf{u}\alpha \rightarrow \mathbf{u}\beta))$$

does not coincide with crisp monotony when fully true, but rather are stronger (see the following Theorem 3.2). However, it turns out that this notion is superfluous and can be characterized in terms of Mon and Cng, by the following theorem.

**Theorem 3.2** FCT *proves the following graded theorem:*

(C1) $\mathrm{MonCng}(\mathbf{u}) \;\Leftrightarrow\; \mathrm{Mon}(\mathbf{u}) \wedge \mathrm{Cng}(\mathbf{u})$

**Proof:** From left to right: First observe that trivially $\mathrm{MonCng}(\mathbf{u}) \rightarrow \mathrm{Mon}(\mathbf{u})$, as $\triangle(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$. Second, from $\mathrm{MonCng}(\mathbf{u})$ we get both $(\alpha \leftrightarrow \beta) \Rightarrow (\alpha \rightarrow \beta) \Rightarrow (\mathbf{u}\alpha \rightarrow \mathbf{u}\beta)$ and $(\alpha \leftrightarrow \beta) \Rightarrow (\beta \rightarrow \alpha) \Rightarrow (\mathbf{u}\beta \rightarrow \mathbf{u}\alpha)$. Thus $(\alpha \leftrightarrow \beta) \Rightarrow (\mathbf{u}\alpha \rightarrow \mathbf{u}\beta) \wedge (\mathbf{u}\beta \rightarrow \mathbf{u}\alpha) \Leftrightarrow (\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)$, and the rest is simple.

For the converse direction we take the crisp cases $\alpha \leq \beta$ and $\beta \leq \alpha$, which are exhaustive due to the prelinearity axiom of MTL. For $\alpha \leq \beta$ we obtain $(\alpha \rightarrow \beta) \Rightarrow \triangle(\alpha \rightarrow \beta) \Rightarrow (\mathbf{u}\alpha \rightarrow \mathbf{u}\beta)$ by $\mathrm{Mon}(\mathbf{u})$. For $\beta \leq \alpha$ we get $(\alpha \rightarrow \beta) \Rightarrow (\alpha \leftrightarrow \beta) \Rightarrow (\mathbf{u}\alpha \rightarrow \mathbf{u}\beta)$ by $\mathrm{Cng}(\mathbf{u})$. Thus $\mathrm{Mon}(\mathbf{u}) \wedge \mathrm{Cng}(\mathbf{u}) \rightarrow \mathrm{MonCng}(\mathbf{u})$. □

**Example 3.3** To demonstrate the difference between these properties consider standard Łukasiewicz models of FCT and connectives $\mathbf{u}_1(\alpha) = \min(\alpha, 1-\alpha)$ and $\mathbf{u}_2(\alpha) = \min(2\alpha, 1)$. Clearly $\mathrm{Cng}(\mathbf{u}_1) = \mathrm{Mon}(\mathbf{u}_2) = 1$ (i.e., $\mathbf{u}_1$ is fully 1-Lipschitz and $\mathbf{u}_2$ is fully monotone), but we have only $\mathrm{Cng}(\mathbf{u}_2) = \mathrm{Mon}(\mathbf{u}_1) = \mathrm{MonCng}(\mathbf{u}_{1,2}) = \frac{1}{2}$.

The following easy theorems show how the properties of unary connectives transfer to connectives which are close in the sense of $\approx$ or $\approxeq$. Notice that the results of Theorem 3.4 follow from a general metatheorem [11, Th. 3.5]. Since, however, the proof of the metatheorem is omitted in [11], and also in order for the present paper to be self-contained, we give direct proofs of the particular claims here. (The same remark applies also to further theorems on preservation under $\approx$ or $\approxeq$ in this paper.)

**Theorem 3.4** FCT *proves:*

(C2) $\mathrm{Cng}(\mathbf{u}), \mathbf{u} \approx^2 \mathbf{v} \;\Rightarrow\; \mathrm{Cng}(\mathbf{v})$

(C3) $\mathrm{Mon}(\mathbf{u}), \mathbf{u} \approxeq \mathbf{v} \;\Rightarrow\; \mathrm{Mon}(\mathbf{v})$

**Proof:**

(C2) $\alpha \leftrightarrow \beta$ implies $\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta$ by $\mathrm{Cng}(\mathbf{u})$, whence $\mathbf{v}\alpha \leftrightarrow \mathbf{v}\beta$ by $\mathbf{u} \approx^2 \mathbf{v}$ (as $\mathbf{u}\alpha \leftrightarrow \mathbf{v}\alpha$ by $\mathbf{u} \approx \mathbf{v}$ and $\mathbf{u}\beta \leftrightarrow \mathbf{v}\beta$ by $\mathbf{u} \approx \mathbf{v}$).

(C3) $\alpha \leq \beta$ implies $\mathbf{u}\alpha \rightarrow \mathbf{u}\beta$ by $\mathrm{Mon}(\mathbf{u})$; $\mathbf{v}\alpha \rightarrow \mathbf{u}\alpha$ by $\mathbf{v} \subseteq \mathbf{u}$; and $\mathbf{u}\beta \rightarrow \mathbf{v}\beta$ by $\mathbf{u} \subseteq \mathbf{v}$. The transitivity of implication then completes the proof. □

## 3.2 Graded congruence and monotonicity of binary connectives

We now turn to binary connectives. First we shall discuss the properties of congruence and monotonicity, which are analogous to the unary case. We define a separate variant for each argument as well as for both arguments at once:

**Definition 3.5** *In* FCT*, we define the left-argument properties of congruence and monotonicity for binary connectives as follows:*

$$\begin{aligned}
\mathrm{LCng}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \leftrightarrow \beta) \rightarrow (\alpha\,\mathbf{c}\,\gamma \leftrightarrow \beta\,\mathbf{c}\,\gamma)) \\
\mathrm{RCng}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \leftrightarrow \beta) \rightarrow (\gamma\,\mathbf{c}\,\alpha \leftrightarrow \gamma\,\mathbf{c}\,\beta)) \\
\mathrm{Cng}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha\beta\gamma\delta)((\alpha \leftrightarrow \beta)\,\&\,(\gamma \leftrightarrow \delta) \rightarrow (\alpha\,\mathbf{c}\,\gamma \leftrightarrow \beta\,\mathbf{c}\,\delta)) \\
\mathrm{LMon}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \leq \beta) \rightarrow (\alpha\,\mathbf{c}\,\gamma \rightarrow \beta\,\mathbf{c}\,\gamma)) \\
\mathrm{RMon}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \leq \beta) \rightarrow (\gamma\,\mathbf{c}\,\alpha \rightarrow \gamma\,\mathbf{c}\,\beta)) \\
\mathrm{Mon}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall\alpha\beta\gamma\delta)((\alpha \leq \beta)\,\&\,(\gamma \leq \delta) \rightarrow (\alpha\,\mathbf{c}\,\gamma \rightarrow \beta\,\mathbf{c}\,\delta))
\end{aligned}$$

*For convenience, we also define the abbreviation* $\mathrm{LRCng}(\mathbf{c}) \equiv_{\mathrm{df}} \mathrm{LCng}(\mathbf{c})\ \&\ \mathrm{RCng}(\mathbf{c})$*, and similarly for monotonicity.*

Let us first study the interplay of just defined variants of the notions of congruence and monotonicity:

**Theorem 3.6** FCT *proves:*

(C4) $\mathrm{Mon}^2(\mathbf{c}) \Rightarrow \mathrm{LRMon}(\mathbf{c}) \Rightarrow \mathrm{Mon}(\mathbf{c}) \Rightarrow \mathrm{LMon}(\mathbf{c}) \wedge \mathrm{RMon}(\mathbf{c})$

(C5) $\mathrm{Cng}^2(\mathbf{c}) \Rightarrow \mathrm{LRCng}(\mathbf{c}) \Rightarrow \mathrm{Cng}(\mathbf{c}) \Rightarrow \mathrm{LCng}(\mathbf{c}) \wedge \mathrm{RCng}(\mathbf{c})$

**Proof:** We shall only prove (C4); the proof of (C5) is analogous.

First we shall prove the second implication. By specification,

$$\begin{aligned}
\mathrm{LMon}(\mathbf{c}) &\Rightarrow (\alpha \leq \beta) \rightarrow (\alpha\,\mathbf{c}\,\gamma \rightarrow \beta\,\mathbf{c}\,\gamma) \\
\mathrm{RMon}(\mathbf{c}) &\Rightarrow (\gamma \leq \delta) \rightarrow (\beta\,\mathbf{c}\,\gamma \rightarrow \beta\,\mathbf{c}\,\delta).
\end{aligned}$$

Combining these formulae we obtain $\mathrm{LRMon}(\mathbf{c}) \Rightarrow (\alpha \leq \beta)\&(\gamma \leq \delta) \rightarrow ((\alpha\mathbf{c}\gamma \rightarrow \beta\mathbf{c}\gamma)\&(\beta\mathbf{c}\gamma \rightarrow \beta\mathbf{c}\delta))$. The transitivity of implication (applied to the consequent of the latter formula) then completes the proof.

To prove the third implication just observe that setting $\gamma = \delta$ in the definition of $\mathrm{Mon}(\mathbf{c})$ yields:

$$\mathrm{Mon}(\mathbf{c}) \Rightarrow (\alpha \leq \beta)\,\&\,(\gamma \leq \gamma) \rightarrow (\alpha\,\mathbf{c}\,\gamma \rightarrow \beta\,\mathbf{c}\,\gamma)$$

As clearly $\gamma \leq \gamma$ is a theorem of FCT, the proof is done.

The first implication is a simple corollary of the third one. □

The fact that none of the implications in Theorem 3.6 can in general be reversed has been demonstrated in [1].

The following observation confirms that so-defined coordinate-wise binary properties coincide with the corresponding unary properties in one argument:

**Observation 3.7** *Let* $\mathbf{c}_\gamma(\alpha) = \mathbf{c}(\alpha, \gamma)$ *and* $\mathbf{c}^\gamma(\alpha) = \mathbf{c}(\gamma, \alpha)$ *for all* $\alpha \in \mathrm{L}$. *Then* FCT *obviously proves:*

(C6) $\mathrm{LCng}(\mathbf{c}) \;\Leftrightarrow\; (\forall\gamma)\,\mathrm{Cng}(\mathbf{c}_\gamma)$

(C7) $\mathrm{RCng}(\mathbf{c}) \;\Leftrightarrow\; (\forall\gamma)\,\mathrm{Cng}(\mathbf{c}^\gamma)$

*and analogously for* LMon *and* RMon.

Like in the unary case, the analogue of LCng featuring implication in place of equivalence, i.e.,

$$\mathrm{LMonCng}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha \to \beta) \to (\alpha\,\mathbf{c}\,\gamma \to \beta\,\mathbf{c}\,\gamma)),$$

can again be reduced to min-conjunction of LCng and LMon (and similarly for the right-sided variants):

**Theorem 3.8** FCT *proves:*

(C8) $\mathrm{LMonCng}(\mathbf{c}) \leftrightarrow \mathrm{LMon}(\mathbf{c}) \wedge \mathrm{LCng}(\mathbf{c})$

*and similarly for the right-sided variants.*

**Proof:** Let $\mathbf{c}_\gamma(\alpha) = \mathbf{c}(\alpha, \gamma)$ for all $\alpha \in \mathrm{L}$. Then the following chain of equivalences is provable in FCT:

$$\begin{array}{ll}
(\forall\alpha\beta\gamma)((\alpha \to \beta) \to (\alpha\,\mathbf{c}\,\gamma \to \beta\,\mathbf{c}\,\gamma)) & \\
\Leftrightarrow (\forall\gamma)\,\mathrm{MonCng}(\mathbf{c}_\gamma) & \text{by an analogue to Observation 3.7} \\
\Leftrightarrow (\forall\gamma)(\mathrm{Mon}(\mathbf{c}_\gamma) \wedge \mathrm{Cng}(\mathbf{c}_\gamma)) & \text{by Theorem 3.2} \\
\Leftrightarrow (\forall\gamma)\,\mathrm{Mon}(\mathbf{c}_\gamma) \wedge (\forall\gamma)\,\mathrm{Cng}(\mathbf{c}_\gamma) & \text{by quantifier distribution} \\
\Leftrightarrow \mathrm{LMon}(\mathbf{c}) \wedge \mathrm{LCng}(\mathbf{c}) & \text{by definition,}
\end{array}$$

and similarly for the variants. □

However, the analogous reduction of the both-sided property

$$\mathrm{MonCng}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\gamma\delta)((\alpha \to \beta)\,\&\,(\gamma \to \delta) \to (\alpha\,\mathbf{c}\,\gamma \to \beta\,\mathbf{c}\,\delta))$$

to the both-argument properties Mon(**c**) and Cng(**c**) does not hold; see [1] for a counterexample. The proof of the following theorems are easy (cf. the analogous proof for the unary connectives.)

**Theorem 3.9** FCT *proves:*

(C9) $\mathrm{LCng}(\mathbf{c}), \mathbf{c} \approx^2 \mathbf{d} \;\Rightarrow\; \mathrm{LCng}(\mathbf{d})$, *and analogously for* RCng

(C10) $\mathrm{LMon}(\mathbf{c}), \mathbf{c} \approxeq \mathbf{d} \;\Rightarrow\; \mathrm{LMon}(\mathbf{d})$, *and analogously for* RMon

(C11) $\mathrm{Cng}(\mathbf{c}), \mathbf{c} \approx^2 \mathbf{d} \;\Rightarrow\; \mathrm{Cng}(\mathbf{d})$

(C12) $\mathrm{Mon}(\mathbf{c}), \mathbf{c} \approxeq \mathbf{d} \;\Rightarrow\; \mathrm{Mon}(\mathbf{d})$

## 3.3 Graded null and unit elements

Furthermore we define graded generalizations of unit and null elements, obtained again by replacing = in classical definitions by ↔.

**Definition 3.10** *In* FCT*, we define the following graded properties of a binary connective* $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$*:*

$$\begin{array}{ll}
\mathrm{Unit}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha\beta)(\eta\,\mathbf{c}\,\alpha \leftrightarrow \alpha)\,\&\,(\beta\,\mathbf{c}\,\eta \leftrightarrow \beta) & \textit{Unit element} \\
\mathrm{Null}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall\alpha\beta)(\eta\,\mathbf{c}\,\alpha \leftrightarrow \eta)\,\&\,(\beta\,\mathbf{c}\,\eta \leftrightarrow \eta) & \textit{Null element}
\end{array}$$

It would be possible to study also the left and right variants of the "unitness" and "nullness" degree; however, these variants turn out to play a less significant rôle in theorems on graded dominance, therefore their study has been left for [1].

In the following theorem, the claims (C13) and (C14) show a graded uniqueness of unit and null elements; (C15) shows a graded incompatibility of the properties of being a unit and a null of the same connective; and (C16) is a graded version of the well known fact that if a monotone connective has the unit 1, then it has the null 0.

**Theorem 3.11** FCT *proves:*

(C13) $\mathrm{Null}(\mathbf{c},\eta),\mathrm{Null}(\mathbf{c},\zeta) \;\Rightarrow\; (\eta\leftrightarrow\zeta)^2$

(C14) $\mathrm{Unit}(\mathbf{c},\eta),\mathrm{Unit}(\mathbf{c},\zeta) \;\Rightarrow\; (\eta\leftrightarrow\zeta)^2$

(C15) $\mathrm{Null}(\mathbf{c},\eta),\mathrm{Unit}(\mathbf{c},\eta) \;\Rightarrow\; 0$

(C16) $\mathrm{LRMon}(\mathbf{c}),\mathrm{Unit}(\mathbf{c},1) \;\Rightarrow\; \mathrm{Null}(\mathbf{c},0)$

**Proof:**

(C13) The claim follows from the following implications (and the transitivity of $\leftrightarrow$):

$$\mathrm{Null}(\mathbf{c},\eta)\Rightarrow \eta\,\mathbf{c}\,\zeta\leftrightarrow\eta, \zeta\,\mathbf{c}\,\eta\leftrightarrow\eta$$
$$\mathrm{Null}(\mathbf{c},\zeta)\Rightarrow \eta\,\mathbf{c}\,\zeta\leftrightarrow\zeta, \zeta\,\mathbf{c}\,\eta\leftrightarrow\zeta$$

The proof of (C14) is analogous.

(C15) The claim follows from the following implications, the transitivity of $\leftrightarrow$, and the fact that $(1\leftrightarrow 0)=0$:

$$\mathrm{Unit}(\mathbf{c},\eta)\Rightarrow \eta\,\mathbf{c}\,0\leftrightarrow 0, 1\,\mathbf{c}\,\eta\leftrightarrow 1$$
$$\mathrm{Null}(\mathbf{c},\eta)\Rightarrow \eta\,\mathbf{c}\,0\leftrightarrow \eta, 1\,\mathbf{c}\,\eta\leftrightarrow \eta$$

(C16) Clearly $0\rightarrow 0\,\mathbf{c}\,\alpha$ and $0\rightarrow\alpha\,\mathbf{c}\,0$ are theorems. The converse implications are derived as follows: first we obtain $0\,\mathbf{c}\,\alpha\rightarrow 0\,\mathbf{c}\,1, \alpha\,\mathbf{c}\,0\rightarrow 1\,\mathbf{c}\,0$ by $\mathrm{LRMon}(\mathbf{c})$ and the theorem $\alpha\leq 1$. Then $\mathrm{Unit}(\mathbf{c},1)$ and the transitivity of implication completes the proof. □

The following theorem shows the congruence of the unit and null elements w.r.t. closeness of the connectives and truth degrees (in the sense of $\approx$ or $\approxeq$ and $\leftrightarrow$, resp.).

**Theorem 3.12** FCT *proves:*

(C17) $\mathrm{Unit}(\mathbf{c}),\mathbf{c}\approx^2\mathbf{d} \;\Rightarrow\; \mathrm{Unit}(\mathbf{d})$

(C18) $\mathrm{Null}(\mathbf{c}),\mathbf{c}\approx^2\mathbf{d} \;\Rightarrow\; \mathrm{Null}(\mathbf{d})$

(C19) $\mathrm{Unit}(\mathbf{c},\eta),\mathrm{LRCng}(\mathbf{c}),(\eta\leftrightarrow\zeta)^2 \;\Rightarrow\; \mathrm{Unit}(\mathbf{c},\zeta)$

(C20) $\mathrm{Null}(\mathbf{c},\eta),\mathrm{LRCng}(\mathbf{c}),(\eta\leftrightarrow\zeta)^4 \;\Rightarrow\; \mathrm{Null}(\mathbf{c},\zeta)$

*and analogously for* RUnit *and* RNull.

**Proof:** The proof of the first two claims is straightforward.

(C19) By $\mathrm{LRCng}(\mathbf{c})$ and $(\eta\leftrightarrow\zeta)^2$ we obtain $\eta\,\mathbf{c}\,\alpha\leftrightarrow\zeta\,\mathbf{c}\,\alpha, \beta\,\mathbf{c}\,\eta\leftrightarrow\beta\,\mathbf{c}\,\zeta$. An application of $\mathrm{Unit}(\mathbf{c},\eta)$ then completes the proof.

(C20) By $\mathrm{LRCng}(\mathbf{c})$ and $(\eta\leftrightarrow\zeta)^2$ we obtain $\eta\,\mathbf{c}\,\alpha\leftrightarrow\zeta\,\mathbf{c}\,\alpha, \beta\,\mathbf{c}\,\eta\leftrightarrow\beta\,\mathbf{c}\,\zeta$. Then by $\mathrm{Null}(\mathbf{c},\eta)$ we obtain $\eta\leftrightarrow\zeta\,\mathbf{c}\,\alpha, \eta\leftrightarrow\beta\,\mathbf{c}\,\zeta$ and the proof is concluded by using $(\eta\leftrightarrow\zeta)^2$. □

Observe that the double occurrence of $\eta$ in the defining formula of null elements causes in (C20) a greater sensitivity of the "nullness degree" to small (in the sense of $\leftrightarrow$) changes in $\eta$, compared to the sensitivity of the "unitness degree" to the same changes in $\eta$.

## 3.4 Graded idempotence, commutativity, and associativity

In this section we shall investigate graded versions of the properties of idempotence, commutativity, and associativity of binary connectives.

**Definition 3.13** *In* FCT*, we define the following graded properties of a binary connective* $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$*:*

$$\mathrm{Idem}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha)(\alpha\,\mathbf{c}\,\alpha \leftrightarrow \alpha) \qquad \textit{Idempotence}$$

$$\mathrm{Com}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta \leftrightarrow \beta\,\mathbf{c}\,\alpha) \qquad \textit{Commutativity}$$

$$\mathrm{Ass}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall\alpha\beta\gamma)((\alpha\,\mathbf{c}\,\beta)\,\mathbf{c}\,\gamma) \leftrightarrow (\alpha\,\mathbf{c}\,(\beta\,\mathbf{c}\,\gamma)) \qquad \textit{Associativity}$$

These graded properties arise by replacing the (crisp) = in the defining formulae of the corresponding crisp properties by the (graded) equivalence connective of fuzzy logic. Notice that by (C21) of Theorem 3.14, it is immaterial whether we define graded commutativity with implication or equivalence. Theorem (C22) gives a similar result for the associativity of sufficiently commutative connectives. Observe also that commutativity of $\mathbf{c}\colon \mathrm{L}^2 \to \mathrm{L}$ (i.e., $\mathbf{c}$ regarded as a crisp binary operation on L) is in fact symmetry of $\mathbf{c} \sqsubseteq \mathrm{L}^2$ (i.e., $\mathbf{c}$ regarded as a binary fuzzy relation on L).

Theorems (C24) and (C25) show that partial commutativity and associativity is abundant: in particular, all connectives with less than full "difference" (in the sense of $\to$) between their height and plinth are at least partially commutative and associative: thus, e.g., all subnormal connectives in Łukasiewicz models have non-zero degrees of commutativity and associativity. Theorem (C26) generalizes the basic fact that the minimum is the only idempotent t-norm [15].

**Theorem 3.14** FCT *proves:*

(C21) $\mathrm{Com}(\mathbf{c}) \;\Leftrightarrow\; (\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta \to \beta\,\mathbf{c}\,\alpha)$

(C22) $\mathrm{Com}^2(\mathbf{c}) \;\Rightarrow\; \mathrm{Ass}(\mathbf{c}) \leftrightarrow (\forall\alpha\beta\gamma)((\alpha\,\mathbf{c}\,\beta)\,\mathbf{c}\,\gamma) \to (\alpha\,\mathbf{c}\,(\beta\,\mathbf{c}\,\gamma))$

(C23) $\mathrm{Com}(\mathbf{c}) \;\Leftrightarrow\; \mathbf{c} \approx \mathbf{c}^{-1} \;\Leftrightarrow\; \mathbf{c} \subseteq \mathbf{c}^{-1}$

(C24) $\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c}) \;\Rightarrow\; \mathrm{Com}(\mathbf{c})$

(C25) $\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c}) \;\Rightarrow\; \mathrm{Ass}(\mathbf{c})$

(C26) $\mathrm{Idem}(\mathbf{c}), \mathrm{LMon}(\mathbf{c}) \wedge \mathrm{RMon}(\mathbf{c}) \;\Rightarrow\; \wedge \subseteq \mathbf{c}$

**Proof:**

(C21) The left-to-right direction holds trivially (a fortiori). Conversely, $(\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta \to \beta\,\mathbf{c}\,\alpha)$ implies by specification $\alpha\,\mathbf{c}\,\beta \to \beta\,\mathbf{c}\,\alpha$ as well as $\beta\,\mathbf{c}\,\alpha \to \alpha\,\mathbf{c}\,\beta$, and so it implies $(\alpha\,\mathbf{c}\,\beta \to \beta\,\mathbf{c}\,\alpha) \wedge (\beta\,\mathbf{c}\,\alpha \to \alpha\,\mathbf{c}\,\beta)$, i.e., $(\alpha\,\mathbf{c}\,\beta \leftrightarrow \beta\,\mathbf{c}\,\alpha)$, whence $(\forall\alpha\beta)(\alpha\,\mathbf{c}\,\beta \to \beta\,\mathbf{c}\,\alpha) \to \mathrm{Com}(\mathbf{c})$ is obtained by generalization.

(C22) One direction of the claim holds a fortiori. To obtain the converse direction, use commutativity twice to rearrange the arguments on both sides of implication.

(C23) The first equivalence follows directly from the definition and the second from (C21).

(C24) First observe that $\mathrm{Hgt}(\mathbf{c}) \to \mathrm{Plt}(\mathbf{c})$, i.e., $(\exists\alpha\beta)(\alpha\,\mathbf{c}\,\beta) \to (\forall\alpha'\beta')(\alpha'\,\mathbf{c}\,\beta')$, is equivalent to $(\forall\alpha\beta\alpha'\beta')(\alpha\,\mathbf{c}\,\beta \to \alpha'\,\mathbf{c}\,\beta')$ by quantifier shifts valid in MTL. Then specify $\beta$ for $\alpha'$ and $\alpha$ for $\beta'$ in the latter formula to obtain $\mathrm{Com}(\mathbf{c})$ by (C21).

(C25) Proceed as in (C24), only specify $\alpha\,\mathbf{c}\,\beta$ for $\alpha$, $\gamma$ for $\beta$, $\alpha$ for $\alpha'$, and $\beta\,\mathbf{c}\,\gamma$ for $\beta'$.

(C26) By prelinearity, we can take two crisp cases, $\alpha \leq \beta$ and $\beta \leq \alpha$. If $\alpha \leq \beta$, we have:

$$\begin{aligned} \alpha \wedge \beta \Rightarrow \alpha \Rightarrow \alpha\,\mathbf{c}\,\alpha & \quad \text{by Idem}(\mathbf{c}) \\ \Rightarrow \alpha\,\mathbf{c}\,\beta & \quad \text{by RMon}(\mathbf{c}) \end{aligned}$$

Analogously we have $\alpha \wedge \beta \to \alpha\,\mathbf{c}\,\beta$ by $\mathrm{Idem}(\mathbf{c})$ and $\mathrm{LMon}(\mathbf{c})$ if $\beta \leq \alpha$. □

As expected, commutativity converts left-properties to right-properties and vice versa, and swaps the indices in both-sided properties. The following theorem indicates the multiplicities of commutativity needed for these conversions:

**Theorem 3.15** FCT *proves:*

(C27) $\mathrm{Com}^2(\mathbf{c}) \;\Rightarrow\; \mathrm{LCng}(\mathbf{c}) \leftrightarrow \mathrm{RCng}(\mathbf{c})$

(C28) $\mathrm{Com}^2(\mathbf{c}) \;\Rightarrow\; \mathrm{LMon}(\mathbf{c}) \leftrightarrow \mathrm{RMon}(\mathbf{c})$

**Proof:**

(C27) It is sufficient to derive $\mathrm{LCng}(\mathbf{c}) \to \mathrm{RCng}(\mathbf{c})$, as the converse follows by symmetry and the equivalence is obtained by min-conjunction. The proof follows easily from the fact that $\alpha\,\mathbf{c}\,\gamma \leftrightarrow \beta\,\mathbf{c}\,\gamma$ implies $\gamma\,\mathbf{c}\,\alpha \leftrightarrow \gamma\,\mathbf{c}\,\beta$ by applying $\mathrm{Com}(\mathbf{c})$ to both sides of the former equivalence.

The proof of (C28) is analogous. □

The following theorem shows the transmission of these properties to connectives that are close in the sense of $\approx$ or $\approxeq$:

**Theorem 3.16** FCT *proves:*

(C29) $\mathrm{Idem}(\mathbf{c}), \mathbf{c} \approx \mathbf{d} \;\Rightarrow\; \mathrm{Idem}(\mathbf{d})$

(C30) $\mathrm{Com}(\mathbf{c}), \mathbf{c} \approxeq \mathbf{d} \;\Rightarrow\; \mathrm{Com}(\mathbf{d})$

(C31) $\mathrm{Ass}(\mathbf{c}_1), \mathbf{c}_1 \approx^4 \mathbf{c}_2, \mathrm{LCng}(\mathbf{c}_i), \mathrm{RCng}(\mathbf{c}_j) \;\Rightarrow\; \mathrm{Ass}(\mathbf{c}_2),$ *for* $i, j \in \{1, 2\}$

**Proof:**

(C29) By respectively using $\mathbf{c} \approx \mathbf{d}$ and $\mathrm{Idem}(\mathbf{c})$ we obtain: $\alpha\,\mathbf{d}\,\alpha \Leftrightarrow \alpha\,\mathbf{c}\,\alpha \Leftrightarrow \alpha$.

(C30) By $\mathbf{d} \subseteq \mathbf{c}$, $\mathrm{Com}(\mathbf{c})$, and $\mathbf{c} \subseteq \mathbf{d}$, respectively, we obtain the following chain of implications:

$$\alpha\,\mathbf{d}\,\beta \Rightarrow \alpha\,\mathbf{c}\,\beta \Rightarrow \beta\,\mathbf{c}\,\alpha \Rightarrow \beta\,\mathbf{d}\,\alpha$$

Observe that $\mathrm{Com}(\mathbf{c})$ in the form of (C21) has to be used, as we would only obtain $\approx^2$ rather than $\approxeq$ in (C30) from the form of Definition 3.13.

(C31) We shall only prove the case $i = j = 1$, by the following chain of equivalences:

$$\begin{aligned}
(\alpha\,\mathbf{c}_2\,\beta)\,\mathbf{c}_2\,\gamma &\Leftrightarrow (\alpha\,\mathbf{c}_2\,\beta)\,\mathbf{c}_1\,\gamma && \text{by } \mathbf{c}_1 \approx \mathbf{c}_2\\
&\Leftrightarrow (\alpha\,\mathbf{c}_1\,\beta)\,\mathbf{c}_1\,\gamma && \text{by } \mathbf{c}_1 \approx \mathbf{c}_2 \text{ and } \mathrm{LCng}(\mathbf{c}_1)\\
&\Leftrightarrow \alpha\,\mathbf{c}_1\,(\beta\,\mathbf{c}_1\,\gamma) && \text{by } \mathrm{Ass}(\mathbf{c}_1)\\
&\Leftrightarrow \alpha\,\mathbf{c}_1\,(\beta\,\mathbf{c}_2\,\gamma) && \text{by } \mathbf{c}_1 \approx \mathbf{c}_2 \text{ and } \mathrm{RCng}(\mathbf{c}_1)\\
&\Leftrightarrow \alpha\,\mathbf{c}_2\,(\beta\,\mathbf{c}_2\,\gamma) && \text{by } \mathbf{c}_1 \approx \mathbf{c}_2
\end{aligned}$$

The other cases only differ in the order of replacing $\mathbf{c}_1$ and $\mathbf{c}_2$, which determines whether the left and right congruence of $\mathbf{c}_1$ or $\mathbf{c}_2$ is used. □

The assertions of the following Theorem 3.17 are generalizations of well-known basic properties of t-norms. Theorem (C32) corresponds to the fact that the minimum is the greatest (so-called strongest) t-norm [15], while (C33) provides a graded characterization of the idempotents of $\mathbf{c}$ [15].

**Theorem 3.17** FCT *proves:*

(C32) $\mathrm{Mon}(\mathbf{c}), \mathrm{Unit}(\mathbf{c}, 1) \;\Rightarrow\; \mathbf{c} \subseteq \wedge$

(C33) $\mathrm{Mon}(\mathbf{c}), \mathrm{Unit}(\mathbf{c}, 1) \;\Rightarrow\; (\alpha\,\mathbf{c}\,\alpha \leftrightarrow \alpha) \leftrightarrow (\forall \beta)((\alpha\,\mathbf{c}\,\beta) \leftrightarrow (\alpha \wedge \beta))$

**Proof:**

(C32) First we derive $\alpha\,\mathbf{c}\,\beta \to \beta$ by $\mathrm{Mon}(\mathbf{c})$ and $\mathrm{Unit}(\mathbf{c}, 1)$:

$$\begin{aligned}
\alpha\,\mathbf{c}\,\beta &\Rightarrow 1\,\mathbf{c}\,\beta && \text{by } \mathrm{Mon}(\mathbf{c}) \text{ and the theorem } \alpha \le 1\\
&\Rightarrow \beta && \text{by } \mathrm{Unit}(\mathbf{c}, 1).
\end{aligned}$$

Analogously we derive $\alpha\,\mathbf{c}\,\beta \to \alpha$. Thus $\alpha\,\mathbf{c}\,\beta \to \alpha \wedge \beta$ by the theorem's premises.

(C33) The right-to-left direction of the equivalence in the conclusion is trivial by specification $\beta = \alpha$. The converse consists of proving two implications: using (C32) we get the left-to right direction for free. For the right-to-left implication inspect the proof of (C26) and instead of Idem($\mathbf{c}$) use just $(\alpha \,\mathbf{c}\, \alpha \leftrightarrow \alpha)$. Finally observe that by (C4), the premise LMon($\mathbf{c}$) $\wedge$ RMon($\mathbf{c}$) used in (C26) is weaker than our premise Mon($\mathbf{c}$). □

Note that the premises of both claims in the previous theorem could be weakened to (LMon($\mathbf{c}$) & LUnit($\mathbf{c}, 1$)) $\wedge$ (RMon($\mathbf{c}$) & RUnit($\mathbf{c}, 1$)), where LUnit and RUnit would be the corresponding obvious definitions of left and right 'uniteness' degrees.

**Remark 3.18** It can be observed that the traditional non-graded classes of truth-value operators can be defined by requiring the full satisfaction of some of the properties defined in Definition 3.13 and 3.10. In particular, a connective $\mathbf{c}$ is a (non-graded)

$$\begin{array}{rcl} \textit{t-norm} & \text{iff} & \triangle\,\mathrm{Com}(\mathbf{c}), \triangle\,\mathrm{Ass}(\mathbf{c}), \triangle\,\mathrm{LMon}(\mathbf{c}), \triangle\,\mathrm{Unit}(\mathbf{c}, 1) \\ \textit{uninorm} & \text{iff} & \triangle\,\mathrm{Com}(\mathbf{c}), \triangle\,\mathrm{Ass}(\mathbf{c}), \triangle\,\mathrm{LMon}(\mathbf{c}), (\exists \eta) \triangle\,\mathrm{Unit}(\mathbf{c}, \eta) \\ \textit{binary aggregation operator} & \text{iff} & \triangle\,\mathrm{Mon}(\mathbf{c}), 1 \,\mathbf{c}\, 1 = 1, 0 \,\mathbf{c}\, 0 = 0 \end{array}$$

Furthermore, in standard Łukasiewicz logic, $\mathbf{c}$ is a (non-graded)

$$\textit{quasicopula} \quad \text{iff} \quad \triangle\,\mathrm{Unit}(\mathbf{c}, 1),\ \triangle\,\mathrm{Null}(\mathbf{c}, 0), \triangle\,\mathrm{MonCng}(\mathbf{c}).$$

Idempotent binary aggregation operators are those which additionally satisfy $\triangle\,\mathrm{Idem}(\mathbf{c})$; commutative quasicopulas those which also satisfy $\triangle\,\mathrm{Com}(\mathbf{c})$; etc. Observe that quasicopulas can in our setting be generalized not just in a graded manner, but also to analogous operators that satisfy MonCng or Cng wrt an equivalence $\leftrightarrow$ other than standard Łukasiewicz as a measure of distance. We shall, however, not pursue this direction here and shall now turn to the graded notion of dominance relation.

# 4 Graded dominance

By replacing the crisp $\leq$ with $\rightarrow$ (i.e., by deleting the $\triangle$ hidden by Convention 2.1 in $\leq$ that appears in the non-graded definition), we obtain the following notion of graded dominance between binary fuzzy connectives. As usually, the traditional notion of dominance is expressible as the graded notion satisfied to degree 1, i.e., prepended with $\triangle$.

**Definition 4.1** *The graded relation* $\ll$ *of* dominance *between binary connectives is defined as follows:*

$$\mathbf{c} \ll \mathbf{d} \equiv_{\mathrm{df}} (\forall \alpha\beta\gamma\delta)((\alpha \,\mathbf{d}\, \gamma) \,\mathbf{c}\, (\beta \,\mathbf{d}\, \delta) \rightarrow (\alpha \,\mathbf{c}\, \beta) \,\mathbf{d}\, (\gamma \,\mathbf{c}\, \delta))$$

## 4.1 General properties of graded dominance

The following theorem shows how graded dominance is transmitted to $\approx$-close connectives:

**Theorem 4.2** FCT *proves:*

(D1) $\mathbf{c} \ll \mathbf{d}, \mathbf{c} \approx^3 \mathbf{c}', \mathrm{Cng}(\mathbf{d}) \;\Rightarrow\; \mathbf{c}' \ll \mathbf{d}$

(D2) $\mathbf{c} \ll \mathbf{d}, \mathbf{d} \approx^3 \mathbf{d}', \mathrm{Cng}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \mathbf{d}'$

**Proof:** Claim (D1) is proved by the following chain of implications:

$$\begin{array}{ll} (\alpha \,\mathbf{d}\, \gamma) \,\mathbf{c}'\, (\beta \,\mathbf{d}\, \delta) \Leftrightarrow (\alpha \,\mathbf{d}\, \gamma) \,\mathbf{c}\, (\beta \,\mathbf{d}\, \delta) & \text{by } \mathbf{c} \approx \mathbf{c}' \\ \Rightarrow (\alpha \,\mathbf{c}\, \beta) \,\mathbf{d}\, (\gamma \,\mathbf{c}\, \delta) & \text{by } \mathbf{c} \ll \mathbf{d} \\ \Rightarrow (\alpha \,\mathbf{c}'\, \beta) \,\mathbf{d}\, (\gamma \,\mathbf{c}'\, \delta) & \text{by } \mathbf{c} \approx^2 \mathbf{c}',\ \mathrm{Cng}(\mathbf{d}). \end{array}$$

The proof of (D2) is analogous. □

The following theorem shows the behavior of graded dominance under (graded) commutativity and associativity:

**Theorem 4.3** FCT *proves, for any* $i \in \{1,2\}$*:*

(D3) $\triangle\,\mathrm{Com}(\mathbf{c}), \triangle\,\mathrm{Ass}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \mathbf{c}$

(D4) $\triangle\,\mathrm{Com}(\mathbf{c}_i), \triangle\,\mathrm{Ass}(\mathbf{c}_i), \mathrm{Cng}(\mathbf{c}_i), \mathbf{c}_1 \approx^3 \mathbf{c}_2 \;\Rightarrow\; \mathbf{c}_1 \ll \mathbf{c}_2$

(D5) $\triangle\,\mathrm{Com}(\mathbf{c}_i), \triangle\,\mathrm{Ass}(\mathbf{c}_i), \mathrm{Mon}(\mathbf{c}_i), \mathbf{c}_1 \subseteq \mathbf{c}_2, \mathbf{c}_2 \sqsubseteq \mathbf{c}_1 \;\Rightarrow\; \mathbf{c}_1 \ll \mathbf{c}_2$

(D6) $\mathrm{Com}(\mathbf{c}), \mathrm{Ass}^4(\mathbf{c}), \mathrm{LRCng}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \mathbf{c}$

(D7) $\mathrm{Com}(\mathbf{c}_i), \mathrm{Ass}^4(\mathbf{c}_i), \mathrm{Cng}^2(\mathbf{c}_i), \mathbf{c}_1 \approx^3 \mathbf{c}_2 \;\Rightarrow\; \mathbf{c}_1 \ll \mathbf{c}_2$

(D8) $\mathrm{Com}(\mathbf{c}_i), \mathrm{Ass}^4(\mathbf{c}_i), \mathrm{LRCng}(\mathbf{c}_i), \mathrm{Mon}(\mathbf{c}_i), \mathbf{c}_1 \subseteq \mathbf{c}_2, \mathbf{c}_2 \sqsubseteq \mathbf{c}_1 \;\Rightarrow\; \mathbf{c}_1 \ll \mathbf{c}_2$

**Proof:**

(D3) The proof is in fact classical (non-graded), but shall be adapted in (D6) to obtain a graded version.

From $\triangle\,\mathrm{Com}(\mathbf{c})$ we obtain $\gamma\mathbf{c}\beta = \beta\mathbf{c}\gamma$, so (by the axioms of identity) also $(\gamma\mathbf{c}\beta)\mathbf{c}\delta = (\beta\mathbf{c}\gamma)\mathbf{c}\delta$. Thence by $\triangle\,\mathrm{Ass}(\mathbf{c})$ (used twice) we obtain $\gamma\,\mathbf{c}\,(\beta\,\mathbf{c}\,\delta) = \beta\,\mathbf{c}\,(\gamma\,\mathbf{c}\,\delta)$, and so also (by the axioms of identity) $\alpha\,\mathbf{c}\,(\gamma\,\mathbf{c}\,(\beta\,\mathbf{c}\,\delta)) = \alpha\,\mathbf{c}\,(\beta\,\mathbf{c}\,(\gamma\,\mathbf{c}\,\delta))$. Applying $\triangle\,\mathrm{Ass}(\mathbf{c})$ twice then completes the proof.

(D4) We shall prove the claim for $i = 1$; the proof for $i = 2$ just uses (D2) instead of (D1).

By (D3) we obtain that $\triangle\,\mathrm{Com}(\mathbf{c}_1)$ and $\triangle\,\mathrm{Ass}(\mathbf{c}_1)$ imply $\mathbf{c}_1 \ll \mathbf{c}_1$, and by (D1) we further obtain that $\mathbf{c}_1 \ll \mathbf{c}_1$, $\mathbf{c}_1 \approx^3 \mathbf{c}_2$, and $\mathrm{Cng}(\mathbf{c}_1)$ imply $\mathbf{c}_1 \ll \mathbf{c}_2$.

(D5) We shall prove the theorem for $i = 1$; the proof for $i = 2$ is analogous. From $\mathbf{c}_2 \sqsubseteq \mathbf{c}_1$ we get $(\alpha\,\mathbf{c}_2\,\gamma) \le (\alpha\,\mathbf{c}_1\,\gamma)$ and $(\beta\,\mathbf{c}_2\,\delta) \le (\beta\,\mathbf{c}_1\,\delta)$. Then we have the following chain of implications:

$$
\begin{aligned}
(\alpha\,\mathbf{c}_2\,\gamma)\,\mathbf{c}_1\,(\beta\,\mathbf{c}_2\,\delta) &\Rightarrow (\alpha\,\mathbf{c}_1\,\gamma)\,\mathbf{c}_1\,(\beta\,\mathbf{c}_1\,\delta) && \text{by } \mathrm{Mon}(\mathbf{c}_1)\\
&\Rightarrow (\alpha\,\mathbf{c}_1\,\beta)\,\mathbf{c}_1\,(\gamma\,\mathbf{c}_1\,\delta) && \text{by } \triangle\,\mathrm{Com}(\mathbf{c}_1) \text{ and } \triangle\,\mathrm{Ass}(\mathbf{c}_1)\text{, using (D3)}\\
&\Rightarrow (\alpha\,\mathbf{c}_1\,\beta)\,\mathbf{c}_2\,(\gamma\,\mathbf{c}_2\,\delta) && \text{by } \mathbf{c}_1 \subseteq \mathbf{c}_2.
\end{aligned}
$$

(D6) The proof is analogous to that of (D3), just using the graded assumptions and $\mathrm{LCng}(\mathbf{c})$ instead of the first use of the identity axiom and $\mathrm{RCng}(\mathbf{c})$ instead of the second.

The proofs of (D7) and (D8) are analogous to those of (D4) and (D5), respectively, only using (D6) instead of (D3). □

Theorem (D6) can be informally explained as saying that self-domination (or Aczél's property of bisymmetry) holds not only for t-norms (or any commutative and associative connective, see (D3)), but to a fair degree also for all connectives that are fairly commutative, very associative, and fairly left-right congruent. Theorems (D4) and (D7) generalize the result for a pair of connectives that may not be identical, but are still very close to each other (i.e., when $\mathbf{c}_1 \approx^3 \mathbf{c}_2$ holds to a fairly large degree). Theorems (D5) and (D8) have no counterparts among known results; they provide us with bounds for the degree to which $(\mathbf{c}_1 \ll \mathbf{c}_2)$ holds, where the assumption $(\mathbf{c}_1 \subseteq \mathbf{c}_2)$ & $(\mathbf{c}_2 \sqsubseteq \mathbf{c}_1)$ would be obviously useless in the crisp non-graded framework (as it necessitates that $\mathbf{c}_1$ and $\mathbf{c}_2$ coincide anyway). Notice that theorems (D6)–(D8) show that the crisp preconditions of $\triangle\,\mathrm{Com}$ and $\triangle\,\mathrm{Ass}$ can be relaxed by replacing them with graded ones, if some of the connectives in question are sufficiently congruent with equivalence. (Note, however, the heavy dependence of the estimated degree of dominance on the degree of associativity of one of the connectives, due to $\mathrm{Ass}^4$ in these theorems.)

The following theorems deal with interactions between dominance and units. Read contrapositively, they provide bounds to the degree of dominance from the (usually known or at least more easily calculable) degrees of subsethood of the connectives and their units. (generalizing the known fact that dominance implies inclusion / pointwise order.)

**Theorem 4.4** FCT *proves the following graded properties of dominance:*

(D9) $\triangle\,\mathrm{Unit}(\mathbf{c},\eta), \triangle\,\mathrm{Unit}(\mathbf{d},\eta)), \mathbf{c} \ll \mathbf{d} \;\Rightarrow\; \mathbf{c} \subseteq \mathbf{d}$

(D10) $\mathrm{Unit}(\mathbf{c},\eta), \mathrm{Unit}(\mathbf{d},\eta), \mathrm{Cng}(\mathbf{c}), \mathrm{Cng}(\mathbf{d}), \mathbf{c} \ll \mathbf{d} \;\Rightarrow\; \mathbf{c} \subseteq \mathbf{d}$

(D11) $\mathrm{Unit}(\mathbf{c},\eta), \mathrm{Unit}(\mathbf{d},\zeta), \mathrm{Cng}(\mathbf{c}), \mathrm{Cng}^3(\mathbf{d}), \mathbf{c} \ll \mathbf{d}, (\eta \leftrightarrow \zeta)^2 \;\Rightarrow\; \mathbf{c} \subseteq \mathbf{d}$

(D12) $\mathrm{Unit}(\mathbf{c},\eta), \mathrm{Unit}(\mathbf{d},\zeta), \mathrm{Cng}(\mathbf{c}), \mathrm{Cng}(\mathbf{d}), \mathbf{c} \ll \mathbf{d}, \eta\,\mathbf{c}\,\eta \leftrightarrow \eta, \zeta\,\mathbf{d}\,\zeta \leftrightarrow \zeta \;\Rightarrow\; \eta \to \zeta$

(D13) $\mathrm{Unit}^2(\mathbf{c},\eta), \mathrm{Unit}^2(\mathbf{d},\zeta)), \mathrm{Cng}(\mathbf{c}), \mathrm{Cng}(\mathbf{d}), \mathbf{c} \ll \mathbf{d} \;\Rightarrow\; \eta \to \zeta$

**Proof:**

(D9) By $\mathbf{c} \ll \mathbf{d}$ we have $(\alpha\,\mathbf{d}\,\eta)\,\mathbf{c}\,(\eta\,\mathbf{d}\,\delta) \to (\alpha\,\mathbf{c}\,\eta)\,\mathbf{d}\,(\eta\,\mathbf{c}\,\delta)$. From the assumptions $\alpha\,\mathbf{d}\,\eta = \alpha\,\mathbf{c}\,\eta = \alpha$ and $\eta\,\mathbf{d}\,\delta = \eta\,\mathbf{c}\,\delta = \delta$ we get the required $\alpha\,\mathbf{c}\,\delta \to \alpha\,\mathbf{d}\,\delta$.

(D10) Analogously to the previous proof, we derive:

| | |
|---|---|
| $(\alpha\,\mathbf{d}\,\eta)\,\mathbf{c}\,(\eta\,\mathbf{d}\,\delta) \to (\alpha\,\mathbf{c}\,\eta)\,\mathbf{d}\,(\eta\,\mathbf{c}\,\delta)$ | by $\mathbf{c} \ll \mathbf{d}$ |
| $\Rightarrow (\alpha\,\mathbf{d}\,\eta)\,\mathbf{c}\,(\eta\,\mathbf{d}\,\delta) \to \alpha\,\mathbf{d}\,\delta$ | by $\mathrm{Unit}(\mathbf{c},\eta)$ (whence $\alpha\,\mathbf{c}\,\eta \leftrightarrow \alpha, \eta\,\mathbf{c}\,\delta \leftrightarrow \delta$) and $\mathrm{Cng}(\mathbf{d})$ |
| $\Rightarrow \alpha\,\mathbf{c}\,\delta \to \alpha\,\mathbf{d}\,\delta$ | analogously by $\mathrm{Unit}(\mathbf{d},\eta)$ and $\mathrm{Cng}(\mathbf{c})$ |

(D11) The claim follows from (D10) by (C19). (Note that even the weaker assumption $\mathrm{Cng}(\mathbf{d})$ & $\mathrm{LRCng}(\mathbf{d})$ could be used instead of $\mathrm{Cng}^3(\mathbf{d})$.)

(D12) The claim is proved by the following chain of implications and equivalences:

| | |
|---|---|
| $\eta \Leftrightarrow \eta\,\mathbf{c}\,\eta$ | by the assumption |
| $\Leftrightarrow (\zeta\,\mathbf{d}\,\eta)\,\mathbf{c}\,(\eta\,\mathbf{d}\,\zeta)$ | by $\mathrm{Unit}(\mathbf{d},\zeta)$ and $\mathrm{Cng}(\mathbf{c})$ |
| $\Rightarrow (\zeta\,\mathbf{c}\,\eta)\,\mathbf{d}\,(\eta\,\mathbf{c}\,\zeta)$ | by $\mathbf{c} \ll \mathbf{d}$ |
| $\Leftrightarrow \zeta\,\mathbf{d}\,\zeta$ | by $\mathrm{Unit}(\mathbf{c},\eta)$ and $\mathrm{Cng}(\mathbf{d})$ |
| $\Leftrightarrow \zeta$ | by the assumption |

(D13) This claim is just a weaker (but simpler to formulate) corollary of (D12). □

The following theorem shows preservation of dominance under compositions:

**Theorem 4.5** *Let* $\alpha\,\mathbf{e}\,\beta \equiv (\alpha\,\mathbf{a}\,\beta)\,\mathbf{c}\,(\alpha\,\mathbf{b}\,\beta)$ *and* $\alpha\,\mathbf{f}\,\beta \equiv (\alpha\,\mathbf{a}\,\alpha)\,\mathbf{c}\,(\beta\,\mathbf{b}\,\beta)$ *for all* $\alpha, \beta \in \mathrm{L}$. *Then* FCT *proves:*

(D14) $\mathbf{d} \ll \mathbf{c}, \triangle(\mathbf{d} \ll \mathbf{a}), \triangle(\mathbf{d} \ll \mathbf{b}), \mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; \mathbf{d} \ll \mathbf{e}$

(D15) $\mathbf{d} \ll \mathbf{c}, \triangle(\mathbf{d} \ll \mathbf{a}), \triangle(\mathbf{d} \ll \mathbf{b}), \mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; \mathbf{d} \ll \mathbf{f}$

(D16) $\mathbf{d} \ll \mathbf{c}, \mathbf{d} \ll \mathbf{a}, \mathbf{d} \ll \mathbf{b}, \mathrm{MonCng}(\mathbf{c}) \;\Rightarrow\; \mathbf{d} \ll \mathbf{e}$

(D17) $\mathbf{d} \ll \mathbf{c}, \mathbf{d} \ll \mathbf{a}, \mathbf{d} \ll \mathbf{b}, \mathrm{MonCng}(\mathbf{c}) \;\Rightarrow\; \mathbf{d} \ll \mathbf{f}$

**Proof:**

(D14) The claim is proved by the following chain of implications:

| | |
|---|---|
| $(\alpha\,\mathbf{e}\,\gamma)\,\mathbf{d}\,(\beta\,\mathbf{e}\,\delta)$ | |
| $\Leftrightarrow ((\alpha\,\mathbf{a}\,\gamma)\,\mathbf{c}\,(\alpha\,\mathbf{b}\,\gamma))\,\mathbf{d}\,((\beta\,\mathbf{a}\,\delta)\,\mathbf{c}\,(\beta\,\mathbf{b}\,\delta))$ | by the definition of $\mathbf{e}$ |
| $\Rightarrow ((\alpha\,\mathbf{a}\,\gamma)\,\mathbf{d}\,(\beta\,\mathbf{a}\,\delta))\,\mathbf{c}\,((\alpha\,\mathbf{b}\,\gamma)\,\mathbf{d}\,(\beta\,\mathbf{b}\,\delta))$ | by $\mathbf{d} \ll \mathbf{c}$ |
| $\Rightarrow ((\alpha\,\mathbf{d}\,\beta)\,\mathbf{a}\,(\gamma\,\mathbf{d}\,\delta))\,\mathbf{c}\,((\alpha\,\mathbf{d}\,\beta)\,\mathbf{b}\,(\gamma\,\mathbf{d}\,\delta))$ | by $\triangle(\mathbf{d} \ll \mathbf{a}), \triangle(\mathbf{d} \ll \mathbf{b}), \mathrm{Mon}(\mathbf{c})$ |
| $\Leftrightarrow (\alpha\,\mathbf{d}\,\beta)\,\mathbf{e}\,(\gamma\,\mathbf{d}\,\delta)$ | by the definition of $\mathbf{e}$ |

(D15) Analogously to (D14), the claim is proved by the following chain of implications:

$$
\begin{array}{ll}
(\alpha\,\mathbf{e}\,\gamma)\,\mathbf{f}\,(\beta\,\mathbf{e}\,\delta) & \\
\Leftrightarrow ((\alpha\,\mathbf{a}\,\alpha)\,\mathbf{c}\,(\gamma\,\mathbf{b}\,\gamma))\,\mathbf{d}\,((\beta\,\mathbf{a}\,\beta)\,\mathbf{c}\,(\delta\,\mathbf{b}\,\delta)) & \text{by the definition of } \mathbf{f} \\
\Rightarrow ((\alpha\,\mathbf{a}\,\alpha)\,\mathbf{d}\,(\beta\,\mathbf{a}\,\beta))\,\mathbf{c}\,((\gamma\,\mathbf{b}\,\gamma)\,\mathbf{d}\,(\delta\,\mathbf{b}\,\delta)) & \text{by } \mathbf{d} \ll \mathbf{c} \\
\Rightarrow ((\alpha\,\mathbf{d}\,\beta)\,\mathbf{a}\,(\alpha\,\mathbf{d}\,\beta))\,\mathbf{c}\,((\gamma\,\mathbf{d}\,\delta)\,\mathbf{b}\,(\gamma\,\mathbf{d}\,\delta)) & \text{by } \triangle(\mathbf{d} \ll \mathbf{a}), \triangle(\mathbf{d} \ll \mathbf{b}), \mathrm{Mon}(\mathbf{c}) \\
\Leftrightarrow (\alpha\,\mathbf{d}\,\beta)\,\mathbf{f}\,(\gamma\,\mathbf{d}\,\delta) & \text{by the definition of } \mathbf{e}
\end{array}
$$

The proofs of (D16) and (D17) are analogous, only the use of $\mathbf{d} \ll \mathbf{a}, \mathbf{d} \ll \mathbf{b}$ enforces the use of MonCng instead of just Mon. □

**Theorem 4.6** FCT *proves the following graded properties of dominance:*

(D18) $\& \ll \mathbf{c}, \mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \rightarrow$

(D19) $\& \ll \mathbf{c}, \mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \leftrightarrow$

(D20) $\& \ll \mathbf{c}, \mathrm{Mon}(\mathbf{c}), \mathrm{Unit}(\mathbf{c}, 1) \;\Rightarrow\; \mathrm{LCng}(\mathbf{c})$, *and analogously for* $\mathrm{RCng}(\mathbf{c})$

(D21) $\& \ll \mathbf{c}, \mathrm{Mon}(\mathbf{c}), \& \subseteq \mathbf{c} \;\Rightarrow\; \mathrm{Cng}(\mathbf{c})$

(D22) $\& \ll^2 \mathbf{c}, \mathrm{Mon}(\mathbf{c}), \triangle\,\mathrm{Unit}(\mathbf{c}, 1) \;\Rightarrow\; \mathrm{Cng}(\mathbf{c})$

**Proof:**

(D18) $((\alpha \rightarrow \beta) \,\&\, \alpha) \leq \beta$ and $((\gamma \rightarrow \delta) \,\&\, \gamma) \leq \delta$. Thus by $\mathrm{Mon}(\mathbf{c})$ we obtain $((\alpha \rightarrow \beta) \,\&\, \alpha)\,\mathbf{c}\,((\gamma \rightarrow \delta) \,\&\, \gamma) \rightarrow \beta\,\mathbf{c}\,\delta$. As $\& \ll \mathbf{c}$, we obtain $((\alpha \rightarrow \beta)\,\mathbf{c}\,(\gamma \rightarrow \delta)) \,\&\, (\alpha\,\mathbf{c}\,\gamma) \rightarrow \beta\,\mathbf{c}\,\delta$, whence by residuation $(\alpha \rightarrow \beta)\,\mathbf{c}\,(\gamma \rightarrow \delta) \rightarrow (\alpha\,\mathbf{c}\,\gamma \rightarrow \beta\,\mathbf{c}\,\delta)$.

(D19) The proof is analogous to (D18).

(D20) By $\mathrm{Unit}(\mathbf{c}, 1)$ we obtain $\alpha \leftrightarrow \beta \Leftrightarrow (\alpha \leftrightarrow \beta)\,\mathbf{c}\,1 \Leftrightarrow (\alpha \leftrightarrow \beta)\,\mathbf{c}\,(\gamma \leftrightarrow \gamma)$. Furthermore, by (D19) we obtain $(\alpha \leftrightarrow \beta)\,\mathbf{c}\,(\gamma \leftrightarrow \gamma) \Rightarrow (\alpha\,\mathbf{c}\,\gamma \leftrightarrow \beta\,\mathbf{c}\,\gamma)$ from $\& \ll \mathbf{c}$ and $\mathrm{Mon}(\mathbf{c})$.

(D21) By (D19) we obtain $(\alpha \leftrightarrow \beta)\,\mathbf{c}\,(\gamma \leftrightarrow \gamma) \Rightarrow (\alpha\,\mathbf{c}\,\gamma \leftrightarrow \beta\,\mathbf{c}\,\gamma)$ from $\& \ll \mathbf{c}$ and $\mathrm{Mon}(\mathbf{c})$. The assumption $\& \subseteq \mathbf{c}$ completes the proof.

(D22) The proof follows from the previous claim and Theorem (D9). □

**Theorem 4.7** FCT *proves the following graded properties of dominance w.r.t.* $\wedge$*:*

(D23) $\mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \wedge$

(D24) $\triangle\,\mathrm{Unit}(\mathbf{c}, 1) \;\Rightarrow\; (\wedge \ll \mathbf{c}) \leq (\wedge \subseteq \mathbf{c})$

(D25) $\mathrm{Unit}(\mathbf{c}, 1), \mathrm{Cng}(\mathbf{c}), \wedge \ll \mathbf{c} \;\Rightarrow\; \wedge \subseteq \mathbf{c}$

(D26) $\triangle\,\mathrm{Mon}(\mathbf{c}), \triangle\,\mathrm{Unit}(\mathbf{c}, 1) \;\Rightarrow\; (\wedge \subseteq \mathbf{c}) = (\wedge \ll \mathbf{c})$

(D27) $\mathrm{LMon}(\mathbf{c}) \wedge \mathrm{RMon}(\mathbf{c}) \;\Rightarrow\; (\wedge \ll \mathbf{c}) \leftrightarrow (\forall \alpha\beta)((\alpha\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,\beta) \rightarrow \alpha\,\mathbf{c}\,\beta) \,\&\, (\forall \alpha\beta)(\alpha\,\mathbf{c}\,\beta \rightarrow (\alpha\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,\beta))$

(D28) $\triangle\,\mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; \triangle(\wedge \ll \mathbf{c}) \leftrightarrow (\forall \alpha\beta)((\alpha\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,\beta) = \alpha\,\mathbf{c}\,\beta)$

**Proof:**

(D23) From $\alpha \wedge \gamma \leq \alpha$ and $\beta \wedge \delta \leq \beta$ we obtain $(\alpha \wedge \gamma)\,\mathbf{c}\,(\beta \wedge \delta) \rightarrow (\alpha\,\mathbf{c}\,\beta)$ by $\mathrm{Mon}(\mathbf{c})$. Analogously we obtain $(\alpha \wedge \gamma)\,\mathbf{c}\,(\beta \wedge \delta) \rightarrow (\gamma\,\mathbf{c}\,\delta)$ and the proof is done.

(D24) The claim follows from (D9) by $\triangle\,\mathrm{Unit}(\wedge, 1)$.

(D25) The claim follows from (D10) by $\triangle\,\mathrm{Unit}(\wedge, 1)$ and $\triangle\,\mathrm{Cng}(\wedge)$.

(D26) We only need to derive $(\wedge \subseteq \mathbf{c}) \leq (\wedge \ll \mathbf{c})$, as the converse inequality follows from (D24). First observe that from $\triangle\,\mathrm{Mon}(\mathbf{c})$ and $\triangle\,\mathrm{Unit}(\mathbf{c}, 1)$ we obtain $\mathbf{c} \sqsubseteq \wedge$ by (C32). Since furthermore $\triangle\,\mathrm{Com}(\wedge)$, $\triangle\,\mathrm{Ass}(\wedge)$, and $\triangle\,\mathrm{Mon}(\wedge)$, we can use (D5) to complete the proof.

(D27) Left to right: From $\wedge \ll \mathbf{c}$ we obtain $(\alpha\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,\beta) \Rightarrow (\alpha \wedge 1)\,\mathbf{c}\,(1 \wedge \beta) \Leftrightarrow \alpha\,\mathbf{c}\,\beta$. Since $\alpha\mathbf{c}\beta \to \alpha\mathbf{c}1$ by RMon($\mathbf{c}$) and $\alpha\mathbf{c}\beta \to 1\mathbf{c}\beta$ by LMon($\mathbf{c}$), we have derived $\alpha\mathbf{c}\beta \to (\alpha\mathbf{c}1) \wedge (1\mathbf{c}\beta)$ from the premises of the theorem.

Right to left: Starting from $\alpha \leq \beta$, which implies $\alpha \leq \alpha \wedge \beta$, by LMon($\mathbf{c}$) we obtain $\alpha\,\mathbf{c}\,1 \to (\alpha \wedge \beta)\,\mathbf{c}\,1$, and so $(\alpha\,\mathbf{c}\,1) \wedge (\beta\,\mathbf{c}\,1) \to (\alpha \wedge \beta)\,\mathbf{c}\,1$. The same claim can be proved starting from $\beta \leq \alpha$, thus by prelinearity we have: $\mathrm{LMon}(\mathbf{c}) \;\Rightarrow\; (\alpha\,\mathbf{c}\,1) \wedge (\beta\,\mathbf{c}\,1) \to (\alpha \wedge \beta)\,\mathbf{c}\,1$. Analogously we can prove $\mathrm{RMon}(\mathbf{c}) \;\Rightarrow\; (1\,\mathbf{c}\,\gamma) \wedge (1\,\mathbf{c}\,\delta) \to 1\,\mathbf{c}\,(\gamma \wedge \delta)$. Thus together (with a little rearranging) we have:

$$\mathrm{LMon}(\mathbf{c}) \wedge \mathrm{RMon}(\mathbf{c}) \;\Rightarrow\; (\alpha\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,\gamma) \wedge (\beta\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,\delta) \to ((\alpha \wedge \beta)\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,(\gamma \wedge \delta)) \quad (4.1)$$

The claim then follows from the following chain of implications:

$$\begin{array}{rcll} (\alpha\,\mathbf{c}\,\gamma) \wedge (\beta\,\mathbf{c}\,\delta) & \Rightarrow & (\alpha\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,\gamma) \wedge (\beta\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,\delta) & \text{by the second part of the assumption} \\ & \Rightarrow & ((\alpha \wedge \beta)\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,(\gamma \wedge \delta)) & \text{by (4.1)} \\ & \Rightarrow & (\alpha \wedge \beta)\,\mathbf{c}\,(\gamma \wedge \delta) & \text{by the first part of the assumption} \end{array}$$

(D28) The claim is a direct corollary (by $\triangle$-necessitation) of (D27). □

Note that (D26) is illustrated by Example 4.8. Theorem (D23) is a graded generalization of the well-known fact that the minimum dominates any aggregation operator [16]. Theorem (D26) demonstrates a rather surprising fact: that the degree to which a monotonic binary operation with neutral element 1 dominates the minimum is nothing else but the degree to which it is larger. Theorem (D27) is an alternative characterization of operators dominating the minimum; for its non-graded version (D28) see [16, Prop. 5.1].

**Example 4.8** Theorem (D26) can be utilized to easily compute degrees to which standard t-norms on the unit interval dominate the minimum. It can be shown easily that

$$(\wedge \subseteq \mathbf{c}) = \inf_{x \in [0,1]} (x \Rightarrow \mathbf{c}(x, x))$$

holds, i.e. the largest "difference" of a t-norm $\mathbf{c}$ from the minimum can always be found on the diagonal. In standard Łukasiewicz logic, this is, for instance, 0.75 for the product t-norm and 0.5 for the Łukasiewicz t-norm itself. So we can infer that the product t-norm dominates the minimum with a degree of 0.75 (assuming that the underlying logic is standard Łukasiewicz!); with the same assumption, the Łukasiewicz t-norm dominates the minimum to a degree of 0.5.

# 5 Application to graded properties of fuzzy relations

In this section we shall apply graded dominance to graded properties of fuzzy relations.

**Definition 5.1** *In* FCT*, we define the following graded properties of binary fuzzy relations:*

$$\begin{array}{rcll} \mathrm{Refl}(R) & \equiv_{\mathrm{df}} & (\forall x)Rxx & \textit{reflexivity} \\ \mathrm{Sym}(R) & \equiv_{\mathrm{df}} & (\forall xy)(Rxy \to Ryx) & \textit{symmetry} \\ \mathrm{Trans}_{\mathbf{c}}(R) & \equiv_{\mathrm{df}} & (\forall xyz)(Rxy\,\mathbf{c}\,Ryz \to Rxz) & \mathbf{c}\textit{-transitivity} \\ \mathrm{Ext}_{\mathbf{c}}(A, R) & \equiv_{\mathrm{df}} & (\forall xy)(Ax\,\mathbf{c}\,Rxy \to Ay) & \mathbf{c}\textit{-extensionality of } A \textit{ w.r.t. } R \\ \mathrm{AntiSym}_{\mathbf{c}}(R, E) & \equiv_{\mathrm{df}} & (\forall xy)(Rxy\,\mathbf{c}\,Ryx \to Exy) & \mathbf{c}\textit{-antisymmetry of } R \textit{ w.r.t. } E \end{array}$$

*Furthermore we define the class operation* $\underline{\mathbf{c}}$ *given by the connective* $\mathbf{c}$ *as follows:*

$$(P\,\underline{\mathbf{c}}\,Q)\vec{x} \;\equiv_{\mathrm{df}}\; P\vec{x}\,\mathbf{c}\,Q\vec{x},$$

*for tuples* $\vec{x}$ *of an arbitrary arity. (Thus, e.g.,* $\underline{\mathbf{c}}$ *is strong intersection if* $\mathbf{c} = \&$*, weak union if* $\mathbf{c} = \vee$*, etc., of fuzzy classes or fuzzy relations.)*

The following theorems show the importance of graded dominance for graded properties of fuzzy relations. Theorem 5.2 is a graded generalization of the well-known theorem by De Baets and Mesiar that uses dominance to characterize preservation of transitivity by aggregation [12, Th. 2]. By (R4), in monotone operators with the null element 0 (e.g., t-norms), the degree of graded dominance $\mathbf{c} \ll \mathbf{d}$ is exactly the degree to which $\mathbf{c}$-transitivity is preserved by $\mathbf{d}$-intersections.

**Theorem 5.2** FCT *proves:*

(R1) $\mathbf{c} \ll \mathbf{d}, \mathrm{Mon}(\mathbf{d}), \triangle\,\mathrm{Trans}_\mathbf{c}(R), \triangle\,\mathrm{Trans}_\mathbf{c}(S) \;\Rightarrow\; \mathrm{Trans}_\mathbf{c}(R\,\underline{\mathbf{d}}\,S)$

(R2) $\mathbf{c} \ll \mathbf{d}, \mathrm{Mon}(\mathbf{d}), \mathrm{Cng}(\mathbf{d}), \mathrm{Trans}_\mathbf{c}(R), \mathrm{Trans}_\mathbf{c}(S) \;\Rightarrow\; \mathrm{Trans}_\mathbf{c}(R\,\underline{\mathbf{d}}\,S)$

(R3) $\triangle\,\mathrm{Null}(\mathbf{c}, 0), (\forall RS)(\triangle\,\mathrm{Trans}_\mathbf{c}(R) \;\&\; \triangle\,\mathrm{Trans}_\mathbf{c}(S) \to \mathrm{Trans}_\mathbf{c}(R\,\underline{\mathbf{d}}\,S)) \;\Rightarrow\; \mathbf{c} \ll \mathbf{d}$

(R4) $\mathrm{Mon}(\mathbf{d}), \triangle\,\mathrm{Null}(\mathbf{c}, 0) \;\Rightarrow\; (\mathbf{c} \ll \mathbf{d}) \leftrightarrow (\forall RS)(\triangle\,\mathrm{Trans}_\mathbf{c}(R) \;\&\; \triangle\,\mathrm{Trans}_\mathbf{c}(S) \to \mathrm{Trans}_\mathbf{c}(R\,\underline{\mathbf{d}}\,S))$

**Proof:**

(R1) We need to derive $Rxz \,\mathbf{d}\, Sxz$ from $(Rxy \,\mathbf{d}\, Sxy)\,\mathbf{c}\,(Ryz \,\mathbf{d}\, Syz)$. Now the latter implies $(Rxy\,\mathbf{c}\,Ryz)\,\mathbf{d}\,(Sxy\,\mathbf{c}\,Syz)$ by $\mathbf{c} \ll \mathbf{d}$, which in turn implies the required $Rxz\,\mathbf{d}\,Sxz$ by $\mathrm{Mon}(\mathbf{d})$, since we have $Rxy\,\mathbf{c}\,Ryz \leq Rxz$ and $Sxy\,\mathbf{c}\,Syz \leq Sxz$ by $\triangle\,\mathrm{Trans}_\mathbf{c}(R)$ and $\triangle\,\mathrm{Trans}_\mathbf{c}(S)$, respectively.

(R2) Analogously as in the proof of (R1) we derive:

$$\begin{aligned}
&(Rxy\,\mathbf{d}\,Sxy)\,\mathbf{c}\,(Ryz\,\mathbf{d}\,Syz) && \\
&\Rightarrow (Rxy\,\mathbf{c}\,Ryz)\,\mathbf{d}\,(Sxy\,\mathbf{c}\,Syz) && \text{by } \mathbf{c} \ll \mathbf{d} \\
&\Rightarrow Rxz\,\mathbf{d}\,Sxz && \text{by } \mathrm{Trans}_\mathbf{c}(R), \mathrm{Trans}_\mathbf{c}(S), \mathrm{Mon}(\mathbf{d}), \mathrm{Cng}(\mathbf{d})
\end{aligned}$$

(R3) Fix three elements $a \neq b \neq c \neq a$ and define two relations (see Definition 2.2 for the notation used):

$$\begin{aligned}
R &=_{\mathrm{df}} \{ab/\alpha, bc/\beta, ac/\alpha\,\mathbf{c}\,\beta\} \\
S &=_{\mathrm{df}} \{ab/\gamma, bc/\delta, ac/\gamma\,\mathbf{c}\,\delta\}
\end{aligned}$$

Since $\triangle\,\mathrm{Null}(\mathbf{c}, 0)$, obviously $\triangle\,\mathrm{Trans}_\mathbf{c}(R)$ and $\triangle\,\mathrm{Trans}_\mathbf{c}(S)$. Thus we can infer $\mathrm{Trans}_\mathbf{c}(R\underline{\mathbf{d}}S)$, whence by specification we obtain $(Rab\,\mathbf{d}\,Sab)\,\mathbf{c}\,(Rbc\,\mathbf{d}\,Sbc) \to (Rac\,\mathbf{d}\,Sac)$. To complete the proof, use the definitions of $R$ and $S$ (by which $Rab = \alpha$, $Sab = \beta$, etc.).

(R4) The claim is a corollary of (R1) and (R3). □

The similitude between the defining formulae of $\mathrm{Trans}_\mathbf{c}(R)$ and $\mathrm{Ext}_\mathbf{c}(A, R)$ makes it possible to transfer the results of Theorem 5.2 to graded extensionality:

**Theorem 5.3** FCT *proves:*

(R5) $\mathbf{c} \ll \mathbf{d}, \mathrm{Mon}(\mathbf{d}), \triangle\,\mathrm{Ext}_\mathbf{c}(A, R), \triangle\,\mathrm{Ext}_\mathbf{c}(B, S) \;\Rightarrow\; \mathrm{Ext}_\mathbf{c}(A\,\underline{\mathbf{d}}\,B, R\,\underline{\mathbf{d}}\,S)$

(R6) $\mathbf{c} \ll \mathbf{d}, \mathrm{Mon}(\mathbf{d}), \mathrm{Cng}(\mathbf{d}), \mathrm{Ext}_\mathbf{c}(A, R), \mathrm{Ext}_\mathbf{c}(B, S) \;\Rightarrow\; \mathrm{Ext}_\mathbf{c}(A\,\underline{\mathbf{d}}\,B, R\,\underline{\mathbf{d}}\,S)$

(R7) $\triangle\,\mathrm{Null}(\mathbf{c}, 0), (\forall ABRS)(\triangle\,\mathrm{Ext}_\mathbf{c}(A, R) \;\&\; \triangle\,\mathrm{Ext}_\mathbf{c}(B, S) \to \mathrm{Ext}_\mathbf{c}(A\,\underline{\mathbf{d}}\,B, R\,\underline{\mathbf{d}}\,S)) \;\Rightarrow\; \mathbf{c} \ll \mathbf{d}$

(R8) $\mathrm{Mon}(\mathbf{d}), \triangle\,\mathrm{Null}(\mathbf{c}, 0) \;\Rightarrow\; (\mathbf{c} \ll \mathbf{d}) \leftrightarrow (\forall ABRS)(\triangle\,\mathrm{Ext}_\mathbf{c}(A, R) \;\&\; \triangle\,\mathrm{Ext}_\mathbf{c}(B, S) \to \mathrm{Ext}_\mathbf{c}(A\,\underline{\mathbf{d}}\,B, R\,\underline{\mathbf{d}}\,S))$

**Proof:** The proofs are analogous to those of Theorem 5.2. As an example we shall show the proof of (R6):

$$\begin{aligned}
(Ax\,\mathbf{d}\,Bx)\,\mathbf{c}\,(Rxy\,\mathbf{d}\,Sxy) &\Rightarrow (Ax\,\mathbf{c}\,Rxy)\,\mathbf{d}\,(Bx\,\mathbf{c}\,Sxy) && \text{by } \mathbf{c} \ll \mathbf{d} \\
&\Rightarrow Ay\,\mathbf{d}\,By && \text{by } \mathrm{Ext}_\mathbf{c}(A, R), \mathrm{Ext}_\mathbf{c}(B, S), \mathrm{Mon}(\mathbf{d}), \mathrm{Cng}(\mathbf{d})
\end{aligned}$$

The definitions of $R, S$ in the proof of (R4) need to be changed as follows for the proof of (R8). We only fix two elements $b \neq c$ and define:

$$A =_{\mathrm{df}} \{b/\alpha, c/\alpha\,\mathbf{c}\,\beta\}, \quad R =_{\mathrm{df}} \{bc/\beta\}$$
$$B =_{\mathrm{df}} \{b/\gamma, c/\gamma\,\mathbf{c}\,\delta\}, \quad S =_{\mathrm{df}} \{bc/\delta\}$$

The rest of the proof is analogous to (R4). □

**Remark 5.4** Theorems 5.2 and 5.3 can be proved jointly if we define $R \preceq_{\mathbf{c}} S \equiv_{\mathrm{df}} R \circ_{\mathbf{c}} S \subseteq S$, where $(R \circ_{\mathbf{c}} S)xz \equiv_{\mathrm{df}} (\forall y)(Rxz\,\mathbf{c}\,Syz)$. Then by definition, $\mathrm{Trans}_{\mathbf{c}}(R) \equiv R \preceq_{\mathbf{c}} R$ and $\mathrm{Ext}_{\mathbf{c}}(A, R) \equiv R \preceq_{\mathbf{c}} R_A$, where $R_Axy \equiv_{\mathrm{df}} (x = a)\ \&\ Ay$ for an arbitrary fixed element $a$ (serving as a dummy argument to increase the arity of $A$, cf. [9]). Consequently, e.g., both (R2) and (R6) are instances of the more general theorem $\mathbf{c} \ll \mathbf{d}, \mathrm{Mon}(\mathbf{d}), \mathrm{Cng}(\mathbf{d}), R_1 \preceq_{\mathbf{c}} S_1, R_2 \preceq_{\mathbf{c}} S_2 \;\Rightarrow\; (R_1\,\underline{\mathbf{d}}\,R_2) \preceq_{\mathbf{c}} (S_1\,\underline{\mathbf{d}}\,S_2)$, proved in the similar manner as above.

The following theorem provides us with results on the preservation of various properties by symmetrizations of fuzzy relations.

**Theorem 5.5** *FCT proves:*

(R9) $\mathrm{Com}(\mathbf{c}) \;\Rightarrow\; \mathrm{Sym}(R\,\underline{\mathbf{c}}\,R^{-1})$

(R10) $\mathrm{Refl}^2(R), \& \subseteq \mathbf{c} \;\Rightarrow\; \mathrm{Refl}(R\,\underline{\mathbf{c}}\,R^{-1})$

(R11) $\mathbf{d} \subseteq \mathbf{c} \;\Rightarrow\; \mathrm{AntiSym}_{\mathbf{d}}(R, R\,\underline{\mathbf{c}}\,R^{-1})$

(R12) $\triangle\,\mathrm{Trans}_{\mathbf{d}}(R), \mathbf{d} \ll \mathbf{c}, \mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; \mathrm{Trans}_{\mathbf{d}}(R\,\underline{\mathbf{c}}\,R^{-1})$

**Proof:** Claims (R9)–(R11) follow easily from the definitions. Claim (R12) is a simple corollary of (R1). □

In the crisp case, the commutativity of an operator trivially implies the symmetry of symmetrizations by this operator. In the graded case, (R9) above states that the degree to which a symmetrization is actually symmetric is bounded below by the degree to which the aggregation operator $\mathbf{c}$ is commutative. Theorems (R10)–(R12) are also well-known in the non-graded case [10, 12, 17].

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