Notes on Condensed Detachment

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2011

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Field of Study:
Logic

This work was supported by GAČR projects P202/10/1826 and 401/09/H007.

Abstract

We study some basic properties of Hilbert-style propositional calculi with the rule of condensed detachment instead of modus pones and substitution. The rule of condensed detachment, proposed by Carew A. Meredith, can be seen as a version of modus ponens with the "minimal" amount of substitution.

1. Introduction

Hilbert-style calculi for various propositional logics has been studied by prominent logicians, including Łukasiewicz and Tarski, constituting historically a well-established branch of mathematical logic. These calculi are usually equipped with the rules of detachment, we shall prefer call it modus (ponendo) ponens, and substitution.[1] One of the logicians who significantly contributed to the study of such calculi was Carew A. Meredith. In the 1950's, he proposed, cf. [1], the rule of condensed detachment as a rule which combines modus ponens with a "minimal" amount of substitution, cf. [2].

The general idea behind the rule of condensed detachment is that from two formulae $\varphi \to \psi$ and χ, such that there is a most general unifier σ of φ and χ, derive $\sigma(\psi)$. However, this brief version does not contain some important technical details which will be discussed later in the paper, see Definition 2.1.

The use of unification in the definition of condensed detachment suggests its connection with binary resolution, cf. [3]. However, the original formulation did not use unification, which was proposed by Robinson [4] in the 1960's. There is also a very tight connection with combinatory logic, cf. [2].

It is usually claimed that one of the main advantages of condensed detachment over the rules of modus ponens and substitution is an economic presentation of proofs. The reason is that the result of application of condensed detachment is unique (up to variable renaming) and a proof can be presented as a sequence of axioms, there is no need to write substitutions. In this paper we try to discuss some interesting questions which arise if we replace the rules of modus ponens and substitution in Hilbert-style propositional calculi solely by the rule of condensed detachment. Although condensed detachment may seem as a toy tool, there are some rather interesting applications e.g. in proof complexity [5], see Section 3.2.

The paper is organised as follows. In Section 2 we define some basic notions including the rule of condensed detachment. In Section 3 we prove Theorem 3.1 which connects proofs using the rule of condensed detachment and proofs using the rules of modus pones and substitution. Also the uniqueness of application of condensed detachment concerning the number of different formulae provable from a finite set of axioms by proofs of some maximal given length is discussed in Section 3.1. In Section 4 the notion of **D**-completeness of a set of axioms A, which means that the very same formulae are provable by condensed detachment as by modus ponens and substitution in A, is studied and some basic properties are proved.

We would like to note that the most of the results in this paper, although mainly (re)discovered independently, are implicitly or explicitly discussed in several papers on condensed detachment, cf. [2,3,6]. These papers also influenced the presentation given here.

[1] Since axiom schemata are sometimes used instead of axioms, the rule of substitution is in these cases only implicitly presented.

2. Preliminaries

We fix a countably infinite set of variables $Var = \{p, q, r, \dots\}$. The set of formulae Fml is defined in the standard way: any variable from Var is an element of Fml, if $\varphi, \psi \in Fml$ then also $(\varphi \to \psi) \in Fml$ and nothing other is a member of Fml. Hence the only connective we are interested in is the implication. The reason for this is that all the things we want to discuss become apparent already in implication fragments. We usually denote formulae by φ, ψ, and χ. The outermost brackets are mostly omitted.

A *substitution* σ is a function $\sigma\colon Var \to Fml$. We say that a substitution σ is a *renaming* if $\sigma\colon Var \to Var$ is a bijection. The result of an application of a substitution σ on a formula φ, denoted $\sigma(\varphi)$, is the formula obtained by replacing variables in φ according to σ simultaneously. A composition of substitutions $\sigma\colon Var \to Fml$ and $\delta\colon Var \to Fml$ is a substitution $\sigma \circ \delta = \{\, \langle p, \psi\rangle \mid (\exists \psi')(\langle p, \psi'\rangle \in \sigma \text{ and } \psi = \delta(\psi'))\,\}$. The empty substitution is denoted $\epsilon = \{\, \langle p, p\rangle \mid p \in Var\,\}$. In this paper substitutions are denoted σ, δ, θ, η, and ζ. Instead of using ordered pairs we write a substitution as a set of pairs p/ψ, usually writing only the important one, meaning the substitution is defined as the empty substitution on the other variables.

A formula ψ is a *variant of a formula* φ, abbreviated by $\psi \sim \varphi$, if there is a renaming σ such that $\psi = \sigma(\varphi)$, i.e. $\varphi = \sigma^{-1}(\psi)$. Moreover, we say that a substitution σ is a *variant of a substitution* δ if there is a renaming θ such that $\sigma = \delta \circ \theta$, i.e. $\delta = \sigma \circ \theta^{-1}$.

A *unification* of a set of formulae $F = \{\varphi_1, \dots, \varphi_n\}$ is such a substitution σ that $\sigma(\varphi_1) = \dots = \sigma(\varphi_n)$. If such a substitution exists we say that F is unifiable. Due to the Unification Theorem of Robinson [4], for any unifiable set of formulae F there exists a most general unifier of F. A *most general unifier* (m.g.u.) σ of F is such a unification that for any other unification δ of F, there is a substitution θ such that $\sigma \circ \theta = \delta$. All the most general unifiers, if they exist, are the same up to renaming, they are variants of each other. Since this difference will be unimportant for us we shall write the m.g.u. instead of a m.g.u.

2.1. Hilbert-style calculi

In this paper we study Hilbert-style propositional calculi. A Hilbert-style calculus consists of a set of axioms A, which is just a set of formulae, and deduction rules. The following axioms are discussed in the paper:

(B) $(p \to q) \to ((r \to p) \to (r \to q))$,

(B′) $(p \to q) \to ((q \to r) \to (p \to r))$,

(C) $(p \to (q \to r)) \to (q \to (p \to r))$,

(I) $p \to p$,

(K) $p \to (q \to p)$,

(W) $(p \to (p \to q)) \to (p \to q)$,

(P) $((p \to q) \to p) \to p$.

The names of axioms are based on corresponding combinators in combinatory logic, with the exception of (P) which stands for Peirce's law. We can present a set of axioms listing the axioms it contains, e.g. BCK denotes the set containing (B), (C), and (K).

We shall use only three deduction rules: modus ponens, substitution, and condensed detachment. The rule of modus ponens (or detachment) derives ψ from $\varphi \to \psi$ and φ. The rule of substitution derives $\sigma(\varphi)$ from φ for any substitution σ.

Definition 2.1 (Condensed Detachment) *Let us have two formulae $\varphi \to \psi$ and χ. We produce a variant of χ called χ', which does not have a common variable with $\varphi \to \psi$. If there is the m.g.u. σ of φ and χ', then produce a variant σ' of σ such that no new variable in $\sigma'(\varphi)$ occurs in ψ. The condensed detachment of $\varphi \to \psi$ and χ, denoted $D(\varphi \to \psi)\,\chi$, is $\sigma'(\psi)$. Otherwise, the condensed detachment of $\varphi \to \psi$ and χ is not defined.*

Note. For technical reasons it is sometimes useful to define condensed detachment not only for formulae containing implication but also for variables. In this case, the condensed detachment of φ, which is a variable, and χ, is defined as φ, cf. [2].

Remark. It is evident that the condensed detachment of φ and ψ is defined uniquely up to variants (renaming). Thus we shall write that $D\varphi\,\psi \sim \chi$. When the rule of condensed detachment is the only rule we shall also sometimes write $\varphi\,\psi \sim \chi$.

As Definition 2.1 is quite technical, we discuss the whole process of an application of condensed detachment in details. First, we produce a variant χ' of χ with no common variable with $\varphi \to \psi$. To see why, consider $\varphi = p \to p$ and $\chi = p$: there would be no unification of $p \to p$ and p. Moreover, if we had $\varphi = p \to p$, $\psi = q \to q$ and $\chi = q$ the condensed detachment of $\varphi \to \psi$ and χ would be $(p \to p) \to (p \to p)$.

Another important technical aspect is that the definition requires to produce a variant σ' of σ (note that

σ' is also the m.g.u. of φ and ψ') which satisfies $(Var(\sigma'(\varphi)) \setminus Var(\varphi)) \cap Var(\psi) = \emptyset$. If this condition was not satisfied we would get a result that would not be the most general one.

A *proof* of φ in A is a finite sequence of formulae $\psi_1, \ldots, \psi_n$, where $\psi_n = \varphi$, with the following properties. Every element is a member of A or is derived from the preceding elements of the sequence by a deduction rule. In this paper we study **MP**-*proofs* which have modus ponens and substitution as their only deduction rules, and **D**-*proofs* which have condensed detachment as the only deduction rule.

If there is a **D**-proof (**MP**-proof) of φ in A we say that φ is **D**-*provable* (**MP**-*provable*) in A. Since we already pointed out that the result of an application of condensed detachment is unique up to variants we mostly do not mention that if φ is **D**-provable in A then also all the variants of φ are **D**-provable in A etc.

It is worth to point out that all the **MP**-provable formulae in BCI, BCK, BCKW, and BCKWP correspond to logics BCI, BCK, the implicational fragment of intuitionistic propositional logic, and the implicational fragment of classical propositional logic, respectively.

Example 2.1 *We prove I in CK by condensed detachment. The proof can be described as (CK)K, which means that we use condensed detachment on C and K and then on the result and K.*

Since C $= (p \to (q \to r)) \to (q \to (p \to r))$ *and K* $= p \to (q \to p)$*, we produce a variant of K e.g.* $s \to (t \to s)$*. There is the m.g.u.* $\sigma = \{r/p, s/p, t/q\}$ *of* $p \to (q \to r)$ *and* $s \to (t \to s)$*, which satisfies that no new variable in* $\sigma(p \to (q \to r))$ *occurs in* $q \to (p \to r)$*. It follows that CK* $\sim \sigma(q \to (p \to r)) = q \to (p \to p)$*.*

Now we can use $q \to (p \to p)$ *and any provable formula, e.g. K, to prove I. We produce a variant of K e.g. again* $s \to (t \to s)$*. There is the m.g.u.* $\tau = \{q/s \to (t \to s)\}$ *of* q *and* $s \to (t \to s)$*. Moreover,* s *and* t *does not occur in* $p \to p$*. It follows that (CK)K* $\sim \tau(p \to p) = p \to p$*.*

3. Condensed detachment

It is obvious that condensed detachment can be simply simulated by modus ponens and substitution. As the idea behind the rule of condensed detachment is to be a version of modus ponens equipped with the "minimal" amount of substitution, we would expect that there is also some connection in the other direction. This connection was probably first explicitly showed in [3] by Kalman.

Theorem 3.1 *Let A be a set of axioms and* $\mathcal{P}$ *be an* **MP**-*proof in A. Then there is a* **D**-*proof* $\mathcal{P}'$ *in A such that every step in* $\mathcal{P}$ *is a substitution instance of a step in* $\mathcal{P}'$*. Moreover,* $\mathcal{P}'$ *is not longer than* $\mathcal{P}$*.*

Proof: By induction on the length of the proof $\mathcal{P}$. If $\mathcal{P} = \psi_1$ then $\psi_1 \in A$ and hence $\mathcal{P}' = \psi_1$. Assume that the claim holds for n and we shall prove it for $n+1$. It means we have an **MP**-proof $\mathcal{P} = \psi_1, \ldots, \psi_n, \psi_{n+1}$ and **D**-proof $\mathcal{P}'' = \psi'_1, \ldots, \psi'_m$, where $m \leq n$, corresponding to the **MP**-proof $\mathcal{P}^\star = \psi_1, \ldots, \psi_n$ as the theorem says. If $\psi_{n+1} \in A$ then $\mathcal{P}' = \psi_1,', \ldots, \psi'_m, \psi_{n+1}$, or $\mathcal{P}' = \mathcal{P}''$ if ψ_{n+1} already occurs in $\mathcal{P}''$, and the claim holds trivially. Otherwise ψ_{n+1} is derived by some deduction rule from $\mathcal{P}^\star$. Both deduction rules are discussed separately.

First, ψ_{n+1} is derived by the rule of substitution from ψ_i, $1 \leq i \leq n$. It means that there is a substitution σ s.t. $\psi_{n+1} = \sigma(\psi_i)$. There is a formula $\psi'_j \in \mathcal{P}''$, $1 \leq j \leq i$, and substitution θ s.t. $\psi_i = \theta(\psi'_j)$. It means that $\psi_{n+1} = \theta \circ \sigma(\psi'_j)$ and $\mathcal{P}' = \mathcal{P}''$.

Second, ψ_{n+1} is derived by the rule of modus ponens from ψ_i and ψ_j, $1 \leq i < j \leq n$. For the sake of generality $\psi_i = \psi_j \to \psi_{n+1}$. There are formulae $\psi'_k, \psi'_l \in \mathcal{P}''$, $1 \leq k, l \leq j$, formulae φ, ψ, and substitutions θ and η s.t. $\psi_i = \theta(\psi'_k) = \theta(\varphi) \to \theta(\psi)$ and $\psi_j = \eta(\psi'_l)$. We produce a variant ψ''_l of ψ'_l, which does not have a common variable with φ and ψ. Since $\theta(\varphi) = \eta(\psi'_l)$ there is the m.g.u. ζ of φ and ψ''_l. We produce a variant ζ' of ζ s.t. $(Var(\zeta'(\varphi)) \setminus Var(\varphi)) \cap Var(\psi) = \emptyset$. Thus $\mathcal{P}' = \psi'_1, \ldots, \psi'_m, \zeta'(\psi)$ and there is τ s.t. $\psi_{n+1} = \theta(\psi) = \zeta' \circ \tau(\psi) = \tau(\zeta'(\psi))$. ■

Corollary 3.2 *Let* φ *be a formula and A be a set of axioms. Then* φ *is* **MP**-*provable in A iff there is a formula* ψ *and substitution* σ *s.t.* ψ *is* **D**-*provable in A and* $\sigma(\psi) = \varphi$*.*

Note. It is easy to transform any **MP**-proof $\mathcal{P}$ to another **MP**-proof $\mathcal{P}'$ such that all the substitutions occur before any application of modus ponens. Theorem 3.1 can be from a certain point of view understood as an attempt to produce an **MP**-proof $\mathcal{P}''$ where modus ponens occurs before substitution as much as possible.

3.1. Proofs with a given length

In Hilbert-style calculi with only finitely many axioms it is hard to enumerate explicitly all the formulae provable in a given number of steps, because there are in general infinitely many substitution instances. Our situation is completely different, there are only finitely many such provable formulae (up to variants) if we use only condensed detachment, namely:

Observation 3.3 *Let $|A| = m$ be a set of axioms and Γ_n^A be the set of all formulae* **D***-provable in A by proofs with at most n steps, then $\left|\Gamma_n^A\right|$ is $\mathcal{O}(m^{2^{n-1}})$ up to variants.*

This means that for a finite set of axioms A we can iteratively generate all formulae provable in it. Thus if there is an **MP**-proof $\mathcal{P}$ of φ in a finite A with at most n steps then there is by Theorem 3.1 a **D**-proof $\mathcal{P}'$ of ψ in A with at most n steps such that there is a substitution σ such that $\sigma(\psi) = \varphi$. Since there is a finite upper bound on the number of all possible ψ, see Observation 3.3, and we can easily test whether there is such a substitution σ for given ψ and φ, we can produce a proof $\mathcal{P}'$ in finite time. Moreover, we can find all such ψ, there are only finitely many up to variants, and all **D**-proofs $\mathcal{P}'$ of ψ in A not longer than n. Among them, there is also some ψ' and its **D**-proof $\mathcal{P}''$ in A, from which we can construct an **MP**-proof $\mathcal{P}'''$ of φ in A with at most n steps. This way we can show that there is no **MP**-proof of φ in a finite A with at most a given number of steps.

3.2. An application of condensed detachment in proof complexity

Urquhart in [5] proves a lower bound on the length of the proofs in Hilbert-style calculi for classical propositional logic with the rules of modus ponens and substitution, called *substitution Frege systems* in proof complexity. There are tautologies of length $\mathcal{O}(n)$, for sufficiently large n, which require proofs with $\Omega(\frac{n}{\log n})$ steps. The proof is based on the connection between **MP**-proofs and **D**-proofs via Theorem 3.1.

4. D-completeness

Although we know that there is a tight connection for a given set of axioms A between **MP**-provable formulae and substitution instances of **D**-provable formulae, it does not mean that any **MP**-provable formula is also **D**-provable (up to variants) without the use of substitution. On the other hand, it does not either mean that there is a **MP**-provable formula which is not **D**-provable. To elaborate this problem we define a notion of **D**-complete set of axioms.

Definition 4.1 *Let A be a set of axioms and Γ be the set of all formulae* **MP***-provable in A. We say that A is* **D**-complete *if all the formulae in Γ are* **D***-provable in A.*

Theorem 3.1 says how the sets which are not **D**-complete look like:

Observation 4.1 *Let A be a set of axioms then A is not* **D***-complete iff there is a formula φ and substitution σ s.t. φ is* **D***-provable in A, but $\sigma(\varphi)$ is not.*

The essential question is whether such a bit strange notion of **D**-completeness makes sense at all. However, in [2] Hindley and D. Meredith show that BCI and BCK are not **D**-complete, but BCKW and BCKWP are **D**-complete.

Definition 4.2 *Let φ be a formula* **MP***-provable in a set of axioms A. We say that a formula φ is* basic w.r.t. *A if there is no formula ψ* **MP***-provable in A and non-renaming substitution σ s.t. $\varphi = \sigma(\psi)$. We say that a set of formulae Γ is* basic w.r.t. *A if all $\varphi \in \Gamma$ are basic w.r.t. A. Moreover, we say that a set of axioms A is* basic *if A is basic w.r.t. A.*

Note. For any formula φ **MP**-provable in A, there is a formula ψ basic w.r.t. A and a substitution σ s.t. $\varphi = \sigma(\psi)$. However, such a formula need not be unique: formula $((p \to p) \to p) \to p$ is a substitution instance of $((q \to r) \to q) \to q$ or $((q \to q) \to r) \to r$. Both these formulae are basic w.r.t. any set of axioms complete for classical propositional logic.

Lemma 4.2 *Let A be a set of axioms and φ be a formula basic w.r.t. A. Then φ is* **D***-provable in A.*

Proof: From Theorem 3.1 it follows that there is a formula ψ **D**-provable in A and substitution σ such that $\sigma(\psi) = \varphi$. Since φ is basic in A, σ is renaming and consequently $\psi \sim \varphi$. ■

We say that two sets of axioms A_1 and A_2 are **MP**-*equivalent* if they have the same sets of **MP**-provable formulae.

Theorem 4.3 *Let sets of axioms A_1 and A_2 be* **MP***-equivalent. If A_1 is* **D***-complete and basic, then A_2 is also* **D***-complete.*

Proof: Let φ be a formula **MP**-provable in A_2. Then φ is **MP**-provable in A_1, and consequently also **D**-provable in A_1, by the **D**-completeness of A_1. Since A_1 is basic w.r.t. A_1, and thus it is basic w.r.t. A_2 as well, all the formulae in A_1 are **D**-provable in A_2, by Lemma 4.2. Therefore we can transform any **D**-proof of φ in A_1 into a **D**-proof of φ in A_2. ∎

Note. In [6], three **MP**-equivalent sets of three axioms are presented, BB′I among them, but only one of them is **D**-complete. Hence BB′I is not **D**-complete by Theorem 4.3, because BB′I is basic. Moreover, the two remaining sets differ only in one axiom, and the one from the **D**-complete set is a substitution instance of the other one from the set which is not **D**-complete. Although it may look a bit surprising it holds generally.

Corollary 4.4 *If a set of axioms A is not* **D***-complete then there is no set of axioms A'* **MP***-equivalent to A,* **D***-complete, and basic.*

As we already know about BCI, BCK, and BB′I that these sets are not **D**-complete, we know that there are no **D**-complete and basic sets of axioms **MP**-equivalent to them.

On the other hand, Theorem 4.3 has mainly a positive meaning. We can easily check that BCKW and BCKWP are basic. It means that any set of axioms which is together with modus ponens and substitution complete for the implicational fragment of intuitionistic logic or classical logic, respectively, is also **D**-complete.

The following lemmata, especially the second one, are very useful to prove that some set of axioms is **D**-complete. They say that not even all the instances of axioms are **D**-provable in sets of axioms which are not **D**-complete.

Lemma 4.5 *Let A be a set of axioms. All the substitution instances of axioms in A are* **D***-provable iff A is* **D***-complete.*

Proof: Any **MP**-proof $\mathcal{P}$ can be transformed to an **MP**-proof $\mathcal{P}'$ where all the substitutions occur before any application of modus ponens, and modus ponens can be easily simulated by condensed detachment. The converse direction follows from the definition of **D**-completeness. ∎

Lemma 4.6 ([6]) *Let A be a set of axioms and $\varphi \to \varphi$ be* **D***-provable in A for any formula φ. Then A is* **D***-complete.*

Proof: For any φ **MP**-provable in A, there exists ψ s.t. ψ is **D**-provable in A and φ is a substitution instance of ψ. From the provability of ψ and $\varphi \to \varphi$ we immediately obtain that φ is provable by condensed detachment. ∎

Note. The fact that A contains I and all the instances of other axioms are provable does not mean that A is **D**-complete. Let $A = \{\, ((\varphi \to \varphi) \to \varphi) \to \varphi \mid \varphi \text{ is a formula} \,\} \cup \{p \to p\}$. Then A is not **D**-complete since only formulae in A are provable.

It is evident that for any set A there exists its superset $A' = \{\, \varphi \mid \varphi \text{ is } \mathbf{MP}\text{-provable in } A \,\}$ which is **D**-complete and have the same **MP**-provable formulae as A. However, such a set is infinite even for a finite A, if $A \neq \emptyset$. Moreover, there is a finite set A, namely $A = \mathrm{I}$, which does not have a finite superset **MP**-equivalent to A.

Theorem 4.7 *There is no finite set of axioms A which is* **D***-complete and* **MP***-equivalent to I.*

Proof: Assume that such a set $A = \{\varphi_1, \ldots, \varphi_n\}$, consisting only of substitution instances of $p \to p$, exists. Since our setting is very special, we show that any **D**-proof in A can be transformed to an equivalent **D**-proof in A, proves the same formula, with very special properties.

The condensed detachment of $\varphi \to \varphi$ and ψ is $\sigma(\varphi) = \sigma(\psi')$, for the m.g.u. σ of φ and ψ', which is a suitable variant of ψ. The key point is that a formula which is the result of unification of φ and ψ' is itself the result of condensed detachment. Let $\psi \colon \chi_1, \ldots, \chi_m$ mean $D(\ldots(D(D\psi\,\chi_1)\,\chi_2)\ldots)\,\chi_m$. Such a notation represents a formula by presenting its proof. The following three statements hold. All of them can be proved by checking the properties of most general unifiers and how the rule of condensed detachment behaves in our very special setting.

1. $\psi \colon \chi_1, \ldots, \chi_m$ is a variant of $\psi \colon \chi'_1, \ldots, \chi'_k$, where $\chi'_1, \ldots, \chi'_k$, for $k \leq m$, contains exactly once all the members of $\chi_1, \ldots, \chi_m$ in any order.

2. All the following formulae are variants of each other:

$$\psi_1\colon \chi_1,\ldots,\chi_k,(\psi_2\colon \chi_{k+1},\ldots,\chi_m), \quad (1)$$
$$\psi_1\colon (\psi_2\colon \chi_1,\ldots,\chi_m), \quad (2)$$
$$\psi_1\colon (\psi_2\colon \chi'_1,\ldots,\chi'_l), \quad (3)$$

where $\chi'_1,\ldots,\chi'_l$, for $l \leq m$, contains exactly once all the members of $\chi_1,\ldots,\chi_m$ in any order.

3. $\psi_1\colon (\psi_2\colon \cdots(\psi_k\colon \chi_1,\ldots,\chi_m)\cdots)$ is a variant of $\psi'_1\colon (\psi'_2\colon \cdots(\psi'_l\colon \chi_1,\ldots,\chi_m)\cdots)$, where $\psi'_1,\ldots,\psi'_l$, for $l \leq k$, contains exactly once all the members of $\psi_1,\ldots,\psi_k$ in any order.

Consequently, any **D**-proof in A can be transformed to a **D**-proof $\psi_1\colon (\psi_2\colon \cdots(\psi_k\colon \chi_1,\ldots,\chi_m)\cdots)$, where $k,m \leq n$; if $i<j$, $\psi_i=\varphi_{i'}$, and $\psi_j=\varphi_{j'}$ then $i'<j'$; and if $i<j$, $\chi_i=\varphi_{i'}$, and $\chi_j=\varphi_{j'}$ then $i'<j'$. Therefore there are only finitely many **D**-provable formulae in A up to variants. ■

5. Conclusion

We presented the rule of condensed detachment and studied Hilbert-style propositional calculi in which it is the only deduction rule. We showed a connection between such calculi and more standard calculi with the rules of modus ponens and substitution. Although generally not all the substitution instances of axioms are provable by condensed detachment, there are sets of axioms in which this is true and we provided some observations on such calculi.

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