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Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:rohn@cs.cas.cz

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Raena Farhadsefat[1], Taher Lotfi[2] and Jiří Rohn[3]

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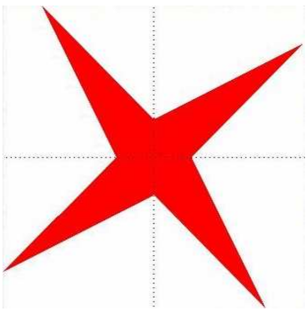
Abstract:

We present a sufficient regularity condition for interval matrices which generalizes two previously known ones. It is formulated in terms of positive definiteness of a certain point matrix, and can also be used for checking positive definiteness of interval matrices. Comparing it with Beeck's strong regularity condition, we show by counterexamples that none of the two conditions is more general than the other one.



Keywords:
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[1]Department of Applied Mathematics and Computer Sciences, Hamedan Branch, Islamic Azad University, Hamedan, Iran, e-mail: rftmath@gmail.com.

[2]Department of Applied Mathematics and Computer Sciences, Hamedan Branch, Islamic Azad University, Hamedan, Iran, e-mail: lotfi@iauh.ac.ir.

[3]Institute of Computer Science, Czech Academy of Sciences, Prague, Czech Republic, e-mail: rohn@cs.cas.cz. This author's work was supported by the Institutional Research Plan AV0Z10300504.

[4]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction and notation

A square interval matrix

$$\mathbf{A} = [A_c - \Delta, A_c + \Delta] = \{ A \mid A_c - \Delta \le A \le A_c + \Delta \}$$

is called *regular* if each $A \in \mathbf{A}$ is nonsingular, and is said to be *singular* otherwise (i.e., if it contains a singular matrix). The problem of checking regularity of interval matrices is known to be NP-hard [6], which, roughly said, means that existence of a polynomial-time algorithm for its solution is very unlikely because it would imply existence of polynomial-time algorithms for thousands of so-called NP-complete problems [4] for none of which such a polynomial-time algorithm has been found so far despite immense efforts of thousands of computer sciencists over the last 40 years. And indeed, forty necessary and sufficient regularity conditions have been found so far [9], all of which exhibit, in some form of another, exponential behavior. This underlines the importance of studying *sufficient* regularity conditions.

In view of what has been said above, one could expect existence of many sufficient regularity conditions. But, surprisingly, the converse is true: only three of them, listed below, are known, at least to these authors.

Theorem 1. *Each of the three conditions implies regularity of* $[A_c - \Delta, A_c + \Delta]$*:*

(i) $\varrho(|A_c^{-1}|\Delta) < 1$,

(ii) $\|\Delta\|_2 < \sigma_{\min}(A_c)$,

(iii) the matrix $A_c^T A_c - \|\Delta^T \Delta\| I$ *is positive definite for some consistent matrix norm* $\|\cdot\|$.

The condition (i) is due to Beeck [2], (ii) is due to Rump [10, Thm. 1.8], and (iii) is due to Rex and Rohn [7, Thm. 5.1]. In (i), ϱ denotes the spectral radius, in (ii) $\sigma_{\min}$ denotes the minimum singular value, and

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^T A)},$$

where $\lambda_{\max}$, $\sigma_{\max}$ denote the maximum eigenvalue and maximum singular value, respectively. Under a consistent matrix norm in (iii) we understand a matrix norm satisfying $\|AB\| \le \|A\|\|B\|$ for each A, B; I denotes the identity matrix of the respective size.

In fact the condition (iii) represents infinitely many conditions depending on the choice of the consistent norm. It is our goal to show that (iii) can be specified in such a way that the resulting condition generalizes not only all the former conditions (iii), but also the condition (ii).

2 Auxiliary results

In this section we mention briefly some results that will be used in the proofs of the main theorems to follow. These results are intended to fix some notions for the subsequent sections and not to furnish a complete treatment of the subject.

Theorem 2. *For each rectangular matrix* A *and each consistent matrix norm* $\|\cdot\|$ *there holds*

$$\|A\|_2^2 \le \|A^T A\|.$$

Proof. As we know,

$$\|A\|_2^2 = \varrho(A^T A).$$

On the other hand, $\|A^T A\|$ is a consistent norm. So we have

$$\varrho(A^T A) \leq \|A^T A\|,$$

which was to be proved. □

Definition. Let $\mathbf{A}$ be an $n \times n$ interval matrix. The matrix $\mathbf{A}$ is said to be regular if each $A \in \mathbf{A}$ is nonsingular, and it is said to be singular otherwise.

The following important characterization is a consequence of the Oettli-Prager theorem [5]; the currently used version can be found e.g. in [3, Thm. 2.9].

Theorem 3. *An interval matrix* $[A_c - \Delta, A_c + \Delta]$ *is singular if and only if the inequality*

$$|A_c x| \leq \Delta |x| \tag{2.1}$$

has a nontrivial solution.

Corollary 4 *If an interval matrix* $[A_c - \Delta, A_c + \Delta]$ *is singular, then there exists a vector* $x_0 \neq 0$ *satisfying*

$$\|A_c x_0\|_2 \leq \|\Delta |x_0|\|_2. \tag{2.2}$$

Proof. In view of Theorem 3, singularity of the interval matrix $[A_c - \Delta, A_c + \Delta]$ implies existence of a vector $x_0 \neq 0$ satisfying

$$|A_c x_0| \leq \Delta |x_0|.$$

Now we have

$$\|A_c x_0\|_2^2 = (A_c x_0)^T (A_c x_0) \leq |A_c x_0|^T |A_c x_0| \leq (\Delta |x_0|)^T (\Delta |x_0|) = \|\Delta |x_0|\|_2^2$$

and finally

$$\|A_c x_0\|_2 \leq \|\Delta |x_0|\|_2,$$

which proves the result. □

We limit ourselves to those notions which are strictly necessary for the sequel. An organic treatment of the subject will be found in the following sections.

3 New sufficient regularity condition

In this section we present the main result of this paper. The following theorem shows that regularity of $\mathbf{A}$ can be described in terms of positive definiteness of the matrix (3.1).

Theorem 5. *Let the matrix*

$$A_c^T A_c - \|\Delta\|_2^2 I \tag{3.1}$$

be positive definite. Then $[A_c - \Delta, A_c + \Delta]$ *is regular.*

Proof. Assume to the contrary that $[A_c - \Delta, A_c + \Delta]$ is singular. Then Corollary 4 implies existence of some $x_0 \neq 0$ such that

$$x_0^T A_c^T A_c x_0 = \|A_c x_0\|_2^2 \leq \|\Delta|x_0|\|_2^2 \leq \|\Delta\|_2^2 (x_0^T x_0)$$

hence we have

$$x_0^T (A_c^T A_c - \|\Delta\|_2^2 I) x_0 \leq 0$$

which means that the matrix (3.1) is not positive definite, a contradiction. □

The formulation of Theorem 5 is advantageous in that it leads us to a clue about the relation between our present sufficient regularity condition and the two older ones. We shall explain this relation in the next section.

4 New condition as a generalization of two older ones

In this section we show that Theorem 5 offers a unified view of two earlier published results. It will be shown that it generalizes not only the regularity condition due to Rump [10], but also all the former regularity conditions due to Rex and Rohn [7].

Theorem 6. *If*

$$\|\Delta\|_2 < \sigma_{\min}(A_c),$$

holds, then the matrix

$$A_c^T A_c - \|\Delta\|_2^2 I$$

is positive definite.

Proof. Assume to the contrary that the matrix $A_c^T A_c - \|\Delta\|_2^2 I$ is not positive definite, then there exists an x_0 satisfying

$$x_0^T (A_c^T A_c - \|\Delta\|_2^2 I) x_0 \leq 0,$$

such that $\|x_0\|_2 = 1$. Consequently,

$$\sigma_{\min}^2(A_c) = \lambda_n(A_c^T A_c) = \min_{\|x\|_2=1} x^T A_c^T A_c x \leq x_0^T A_c^T A_c x_0 \leq \|\Delta\|_2^2$$

hence

$$\sigma_{\min}(A_c) \leq \|\Delta\|_2,$$

which is a contradiction, and the proof is complete. □

In other words, if $[A_c - \Delta, A_c + \Delta]$ satisfies the regularity condition due to Rump, then it also satisfies the regularity condition of Theorem 5, hence the new regularity condition is a generalization of the old one.

Theorem 7. *If the matrix*

$$A_c^T A_c - \|\Delta^T \Delta\| I \tag{4.1}$$

is positive definite for some consistent matrix norm $\|\cdot\|$, then the matrix

$$A_c^T A_c - \|\Delta\|_2^2 I$$

is positive definite.

Proof. Let (4.1) be positive definite for some consistent matrix norm. Now using Theorem 2, for each $x \neq 0$ we have

$$\begin{aligned} x^T(A_c^T A_c - \|\Delta\|_2^2 I)x &= x^T A_c^T A_c x - \|\Delta\|_2^2 \|x\|_2^2 \\ &\geq x^T A_c^T A_c x - \|\Delta^T \Delta\| \|x\|_2^2 \\ &\geq x^T(A_c^T A_c - \|\Delta^T \Delta\| I)x > 0. \end{aligned}$$

So the proof is completed. □

Thus, reasoning as before, we get that the new regularity condition is a generalization of that ones due to Rex and Rohn. We employ the sufficient regularity condition of Theorem 5 for checking positive definiteness of interval matrices in the next section.

5 Positive definiteness of interval matrices

Definition. A square interval matrix $\mathbf{A}$ is called symmetric if $\mathbf{A}^T = \mathbf{A}$, where

$$\mathbf{A}^T = \{A^T \mid A \in \mathbf{A}\}.$$

It can be easily seen that $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ is symmetric if and only if both A_c and Δ are symmetric. But, generally, a symmetric interval matrix may contain nonsymmetric point matrices as well.

Definition. A symmetric interval matrix is said to be positive definite if each symmetric $A \in \mathbf{A}$ is positive definite.

Now we have this characterization.

Theorem 8. *A symmetric interval matrix $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$ is positive definite if and only if*

$$x^T A_c x - |x|^T \Delta |x| > 0 \tag{5.1}$$

holds for each $x \neq 0$.

Proof. First we prove that if (5.1) holds, then each symmetric $A \in \mathbf{A}$ is positive definite. We show that for each $A \in \mathbf{A}$ and each $x \neq 0$ there holds

$$x^T A x \geq x^T A_c x - |x|^T \Delta |x|.$$

Assume to the contrary that

$$x_0^T A_0 x_0 < x_0^T A_c x_0 - |x_0|^T \Delta |x_0|$$

for some $A_0 \in \mathbf{A}$ and $x_0 \neq 0$. This implies

$$|x_0|^T \Delta |x_0| < x_0^T (A_c - A_0) x_0 \leq |x_0|^T |A_c - A_0| |x_0| \leq |x_0|^T \Delta |x_0|,$$

a contradiction. Hence the matrix $\mathbf{A}$ is positive definite.

Conversely, we are to prove that positive definiteness of all symmetric matrices $A \in \mathbf{A}$ implies that (5.1) holds for each $x \neq 0$. So let $x \neq 0$ and define a diagonal matrix T as follows: $T_{ii} = 1$ if $x_i \geq 0$, and $T_{ii} = -1$ otherwise ($i = 1, \ldots, n$), then $Tx = |x|$, and let $A^* = A_c - T\Delta T$. Then A^* is symmetric because A_c, Δ, and T are symmetric, and

$$|A^* - A_c| = |T\Delta T| = \Delta,$$

which means that $A^* \in \mathbf{A}$, so that A^* is positive definite. Now we have

$$0 < x^T A^* x = x^T(A_c - T\Delta T)x = x^T A_c x - x^T T\Delta T x = x^T A_c x - |x|^T \Delta |x|,$$

which was to be proved. □

The proof also yields the following result.

Theorem 9. *If a symmetric interval matrix* $\mathbf{A}$ *is positive definite, then*

$$x^T A x > 0$$

holds for each nonsymmetric $A \in \mathbf{A}$ *and each* $x \neq 0$.

Proof. If $\mathbf{A}$ is positive definite, according to Theorem 8, (5.1) holds, and it was shown in the first part of its proof that this implies $x^T A x > 0$ for each $A \in \mathbf{A}$ and each $x \neq 0$. Let us emphasize that symmetry of A was not assumed in the first part of the proof. □

Hence, nonsymmetric matrices are also "positive definite" except that the term does not apply to them. Now, as soon as we have a tool for checking regularity we can use it for checking positive definiteness of interval matrices. The following link between positive definiteness and regularity of interval matrices was established in [8, Thm. 3].

Theorem 10. *A symmetric interval matrix* $[A_c - \Delta, A_c + \Delta]$ *is positive definite if and only if it is regular and* A_c *is positive definite.*

Using this link, we can turn our sufficient regularity condition into a sufficient positive definiteness condition.

Theorem 11. *Let* $[A_c - \Delta, A_c + \Delta]$ *be symmetric and let both the matrices* A_c *and* $A_c^T A_c - \|\Delta\|_2^2 I$ *be positive definite. Then* $[A_c - \Delta, A_c + \Delta]$ *is positive definite.*

Proof. According to Theorem 5 positive definiteness of $A_c^T A_c - \|\Delta\|_2^2 I$ guarantees that $[A_c - \Delta, A_c + \Delta]$ is regular. Also A_c is positive definite. Now using Theorem 10 gives that $[A_c - \Delta, A_c + \Delta]$ is also positive definite. Which was to be proved. □

That means, checking the two mentioned point matrices for positive definiteness suffices to verify positive definiteness of the whole interval matrix. So far, we have studied regularity and positive definiteness of interval matrices. These results will now be utilized to prove the last result of this paper in the next section.

6 Relation between $[A_c^T A_c - \Delta^T\Delta, A_c^T A_c + \Delta^T\Delta]$ and $[A_c - \Delta, A_c + \Delta]$

In this section we prove an interesting, but rather theoretical result. First we prepare the stage by proving this corollary.

Corollary 12 *An interval matrix of the form $[A_c^T A_c - \Delta^T\Delta, A_c^T A_c + \Delta^T\Delta]$ is positive definite if and only if*

$$\|A_c x\|_2 > \|\Delta|x|\|_2 \tag{6.1}$$

holds for each $x \neq 0$.

Proof. Applying Theorem 8 to the interval matrix $[A_c^T A_c - \Delta^T\Delta, A_c^T A_c + \Delta^T\Delta]$ we obtain that

$$x^T A_c^T A_c x - |x|^T \Delta^T \Delta |x| > 0$$

holds for each $x \neq 0$. Hence

$$\|A_c x\|_2^2 - \|\Delta|x|\|_2^2 = (A_c x)^T (A_c x) - (\Delta|x|)^T (\Delta|x|) > 0,$$

and consequently

$$\|A_c x\|_2 > \|\Delta|x|\|_2$$

for each $x \neq 0$. So the proof is completed. □

The result is clear: relations (2.2) and (6.1) contradict each other. This contradiction leads us to our last result.

Theorem 13. *If $\mathbf{A} = [A_c^T A_c - \Delta^T\Delta, A_c^T A_c + \Delta^T\Delta]$ is regular, then $[A_c - \Delta, A_c + \Delta]$ is regular.*

Proof. Regularity of $\mathbf{A}$ implies that each $A \in \mathbf{A}$ is nonsingular. So $A_c^T A_c$ is nonsingular. Also it is obvious that $A_c^T A_c$ is positive definite. Thus Theorem 10 gives that $\mathbf{A}$ is positive definite, hence (6.1) holds by Corollary 12 for each $x \neq 0$. Now assume to the contrary that $[A_c - \Delta, A_c + \Delta]$ is singular. Then (2.2) holds by Corollary 4 for some $x_0 \neq 0$, a contradiction. This contradiction shows that $[A_c - \Delta, A_c + \Delta]$ is regular as well. □

7 Comparison with the strong regularity condition

In Section 4 we proved that our new sufficient condition of Theorem 5 generalizes the earlier sufficient conditions (ii) and (iii) of Theorem 1. Finally we compare it with Beeck's condition (i) (also called the *strong regularity condition*) and we show by two counterexamples computed in MATLAB that neither of the two conditions is a generalization of the other one. In both examples we use `rand('state',i)` (with $i = 21$ in the first one and $i = 72$ in the second one), so that the data may be reproduced in full precision.

```
n=3; rand('state',21); Ac=2*rand(n,n)-1; Delta=(1/n)*rand(n,n);
A=Ac'*Ac-norm(Delta,2)^2*eye(size(Ac,1)); midrad(Ac,Delta)
rho=max(abs(eig(abs(inv(Ac))*Delta))), eiv=min(eig(A))

intval ans =
[    0.5247,    0.6063] [    0.5343,    0.5599] [   -0.6093,   -0.5652]
[    0.6003,    1.2387] [    0.4443,    0.5948] [    0.0391,    0.2357]
[   -0.7952,   -0.5000] [   -0.1003,    0.1598] [    0.1859,    0.6221]
rho =
    0.9711
eiv =
   -0.0273
```

Here $\varrho(|A_c^{-1}|\Delta) = 0.9711 < 1$ and $\lambda_{\min}(A_c^T A_c - \|\Delta\|_2^2 I) = -0.0273 < 0$, hence the strong regularity condition is satisfied whereas the matrix $A_c^T A_c - \|\Delta\|_2^2 I$ is not positive definite.

```
n=3; rand('state',72); Ac=2*rand(n,n)-1; Delta=(1/n)*rand(n,n);
A=Ac'*Ac-norm(Delta,2)^2*eye(size(Ac,1)); midrad(Ac,Delta)
rho=max(abs(eig(abs(inv(Ac))*Delta))), eiv=min(eig(A))

intval ans =
[   -0.6089,   -0.2581] [   -1.2267,   -0.7475] [   -0.5973,   -0.2492]
[   -0.0397,    0.1292] [   -0.6346,   -0.0022] [    0.3064,    0.8378]
[   -0.9808,   -0.5854] [    0.6140,    1.1957] [    0.5602,    0.6420]
rho =
    1.0254
eiv =
    0.0321
```

Here $\varrho(|A_c^{-1}|\Delta) = 1.0254 > 1$ and $\lambda_{\min}(A_c^T A_c - \|\Delta\|_2^2 I) = 0.0321 > 0$, hence the strong regularity condition is violated whereas the matrix $A_c^T A_c - \|\Delta\|_2^2 I$ is positive definite.

These results finally show that neither of the two conditions can be replaced by the other one, so that we recommend them to be used in conjunction.

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