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## The Equation - $x$ - - $-A x$ - $=b$

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## The Equation $|x|-|A x|=b$

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Technical report No. V-1277
22.02.2020

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## Abstract:

We formulate conditions on $A$ and $b$ under which the double absolute value equation $|x|-|A x|=b$ possesses in each orthant a unique solution which, moreover, belongs to the interior of that orthant. ${ }^{2}$


Keywords:
Absolute value equation, double absolute value equation, orthantwise solvability, theorem of the alternatives.

[^0]
### 0.1 Notation

In this report we consider an equation of the form

$$
\begin{equation*}
|x|-|A x|=b \tag{1}
\end{equation*}
$$

(with $A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^{n}$ ) which we call a double absolute value equation. The absolute value of a vector as well as vector inequalities $\geq,>$ are understood entrywise. For each $y \in\{-1,1\}^{n}$ (i.e., a $\pm 1$-vector in $\mathbb{R}^{n}$ ) we denote by $T_{y}$ the diagonal matrix with diagonal vector $y$. Then

$$
\mathbb{R}_{y}^{n}=\left\{x \mid T_{y} x \geq 0\right\}
$$

is the orthant prescribed by the sign vector $y$, and

$$
\left(\mathbb{R}_{y}^{n}\right)^{0}=\left\{x \mid T_{y} x>0\right\}
$$

is its interior. The equation (1) is said to be orthantwise solvable if in each orthant of $\mathbb{R}^{n}$ it possesses a unique solution which, moreover, belongs to the interior of that orthant. Hence an orthantwise solvable equation (1) possesses exactly $2^{n}$ solutions. A square interval matrix is called regular if all matrices contained therein are nonsingular, and it is said to be singular otherwise. $I$ denotes the $n \times n$ identity matrix.

### 0.2 The result

The following theorem shows a particular property of the double absolute value equation.
Theorem 1. Let $A$ be nonsingular and let the interval matrix

$$
\begin{equation*}
\left[A^{-1}-I, A^{-1}+I\right] \tag{2}
\end{equation*}
$$

be regular. Then for each $b>0$ the equation (1) is orthantwise solvable.
Proof. Take a $y \in\{-1,1\}^{n}$ and consider the absolute value equation

$$
\begin{equation*}
A^{-1} x^{\prime}-T_{y}\left|x^{\prime}\right|=T_{y} b \tag{3}
\end{equation*}
$$

Because the interval matrix $\left[A^{-1}-\left|T_{y}\right|, A^{-1}+\left|T_{y}\right|\right]=\left[A^{-1}-I, A^{-1}+I\right]$ is regular by assumption, by [3, Thm. 1] the equation (3) possesses a unique solution $x^{\prime}$. Put

$$
\begin{equation*}
x_{y}=A^{-1} x^{\prime} \tag{4}
\end{equation*}
$$

then (3) can be rewritten as

$$
x_{y}-T_{y}\left|A x_{y}\right|=T_{y} b
$$

and

$$
T_{y} x_{y}-\left|A x_{y}\right|=b
$$

where

$$
T_{y} x_{y}=\left|A x_{y}\right|+b \geq b>0,
$$

hence

$$
T_{y} x_{y}=\left|x_{y}\right|
$$

so that $x_{y}$ solves (1), belongs to $\left(\mathbb{R}_{y}^{n}\right)^{0}$ and by [3, Thm. 1] it is a unique such a solution. As $y \in\{-1,1\}^{n}$ was arbitrary, the property holds for each orthant of $\mathbb{R}^{n}$.

Thus, under (2) regular and $b>0$, to compute the unique solution $x_{y}$ of $(1)$ in $\left(\mathbb{R}_{y}^{n}\right)^{0}$, we must first solve the absolute value equation (3) and then rearrange its solution by (4). Performing this process for each $y \in\{-1,1\}^{n}$, we can find all solutions of (1).

Solving the absolute value equation (3) may be performed using MATLAB file absvaleqn.m freely downloadable from http://uivtx.cs.cas.cz/~rohn/other/absvaleqn.m.

As regards regularity of (2), for moderate values of $n$ (say, $n \leq 20$ ), it may be checked using a necessary and sufficient condition [6, Thm. 1, (iv)]: (2) is regular if and only if the numbers

$$
\operatorname{det}\left(A^{-1}-T_{y}\right), \quad y \in\{-1,1\}^{n}
$$

are either all negative, or all positive. For larger values of $n$, one may try a sufficient regularity condition [5, Thm. 4]: if $A$ is nonsingular and

$$
\min \left\{\varrho(|A|), \varrho\left(\left|A A^{T}\right|\right)<1\right.
$$

holds, then the interval matrix (2) is regular. Here $\varrho$ stands for the spectral radius of a matrix.

### 0.3 Example

Consider a double absolute value equation with randomly generated data

```
A =
    -0.1825 0.0111 0.4944
    -0.1642 -0.4793 0.3795
    -0.2134 0.4314 -0.3814
b =
        0.1757
        0.2089
        0.9052
X =
\begin{tabular}{rrrrrr}
1.0000 & 1.0000 & 1.0000 & 0.7487 & 0.4239 & 1.4260 \\
-1.0000 & 1.0000 & 1.0000 & -0.7777 & 0.4636 & 0.9201 \\
-1.0000 & -1.0000 & 1.0000 & -2.0065 & -3.2464 & 3.0352 \\
1.0000 & -1.0000 & 1.0000 & 1.7962 & -2.7524 & 4.0028 \\
1.0000 & -1.0000 & -1.0000 & 0.7777 & -0.4636 & -0.9201 \\
-1.0000 & -1.0000 & -1.0000 & -0.7487 & -0.4239 & -1.4260 \\
-1.0000 & 1.0000 & -1.0000 & -1.7962 & 2.7524 & -4.0028 \\
1.0000 & 1.0000 & -1.0000 & 2.0065 & 3.2464 & -3.0352
\end{tabular}
```

Each row of the output matrix $X$ is of the form $\left(y^{T} x_{y}^{T}\right)$ where $x_{y}$ is the unique solution of (1) in $\left(\mathbb{R}_{y}^{n}\right)^{0}$. Observe that indeed $T_{y} x_{y}>0$ for each $y \in\{-1,1\}^{n}$ and that $x_{-y}=-x_{y}$ for each such a $y$ as it can be easily proved from (3), (4).

The example was solved using the following MATLAB code.

```
function [X]=dblabsvaleqn(A,b)
% Orthantwise solvability of the double absolute value equation
% abs(x)-abs(A*x)=b.
X=[];
if ~(b>0), error('vector not positive'), end
n=size(A,1); I=eye(n);
if rank(A)<n, error('singular matrix'), end
B=inv(A);
S=regising(B,I);
% download: http://uivtx.cs.cas.cz/~rohn/other/regising.m
if `isempty(S), error('interval matrix not regular'), end
z=zeros(1,n); y=ones(1,n);
x=absvaleqn(B,-diag(y),diag(y)*b);
% download: http://uivtx.cs.cas.cz/~rohn/other/absvaleqn.m
x=B*x; X=[y x'];
while any(z~}=0\mathrm{ ones(1,n))
    k=find(z==0,1);
    z(1:(k-1))=zeros(1,k-1);
    z(k)=1; y(k)=-y(k);
    x=absvaleqn(B,-diag(y),diag(y)*b);
    x=B*x; X=[X; [y x']];
end
```


### 0.4 Related results

Theorem 1 asserts [unique] solvability in the interior of each orthant. There are some results related to this property. We have the following theorem of the alternatives.

Theorem 2. For each nonsingular A exactly one of the following two alternatives holds:
(i) the inequality

$$
|A x| \geq|x|
$$

has a solution $x \neq 0$,
(ii) the inequality

$$
|A x|<|x|
$$

has a solution in the interior of each orthant.

Proof. The proof proceeds by showing using [2, Lemma 2.1] and [4, Thm. 3.2, (v)] (with obvious details omitted here) that the alternative (i) is equivalent to singularity of the interval matrix (2), and (ii) is equivalent to its regularity. Hence at least one of the two alternatives always holds and they exclude each other, which completes the proof.

The result can also be formulated in a normwise form.
Theorem 3. For each nonsingular A exactly one of the following two alternatives holds:
(i) the inequality

$$
\|A x\|_{1} \geq\|x\|_{\infty}
$$

has a solution $x \neq 0$,
(ii) the inequality

$$
\|A x\|_{1}<\min _{i}\left|x_{i}\right|
$$

has a solution in the interior of each orthant.
Proof. The proof runs in parallel to the previous one, with the interval matrix (2) being replaced by $\left[A^{-1}-E, A^{-1}+E\right], E$ being the matrix of all ones.

## Bibliography

[1] W. Barth and E. Nuding, Optimale Lösung von Intervallgleichungssystemen, Computing, 12 (1974), pp. 117-125.
[2] J. Rohn, Interval matrices: Singularity and real eigenvalues, SIAM Journal on Matrix Analysis and Applications, 14 (1993), pp. 82-91.
[3] J. Rohn, A theorem of the alternatives for the equation $A x+B|x|=b$, Linear and Multilinear Algebra, 52 (2004), pp. 421-426.
[4] J. Rohn, Regularity of interval matrices and theorems of the alternatives, Reliable Computing, 12 (2006), pp. 99-105.
[5] J. Rohn, A sufficient condition for an interval matrix to have full column rank, Journal of Computational Technologies, 22 (2017), pp. 59-66.
[6] J. Rohn, Diagonally singularizable matrices, Linear Algebra and Its Applications, 555 (2018), pp. 84-91.


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    ${ }^{2}$ Above: Logo of interval computations and related areas (depiction of the solution set of the system $[2,4] x_{1}+[-2,1] x_{2}=[-2,2],[-1,2] x_{1}+[2,4] x_{2}=[-2,2]$ (Barth and Nuding [1])).

