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Institute of Computer Science
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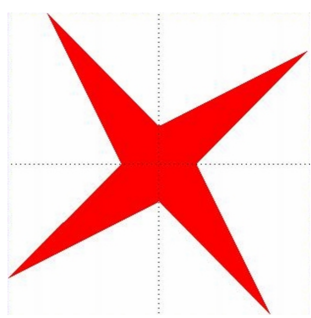
Abstract:

We formulate conditions on A and b under which the double absolute value equation $|x|-|Ax| = b$ possesses in each orthant a unique solution which, moreover, belongs to the interior of that orthant. [2]



Keywords:
Absolute value equation, double absolute value equation, orthantwise solvability, theorem of the alternatives.

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[2] Above: Logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

0.1 Notation

In this report we consider an equation of the form

$$|x| - |Ax| = b \tag{1}$$

(with $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$) which we call a *double absolute value equation.* The absolute value of a vector as well as vector inequalities $\geq$, $>$ are understood entrywise. For each $y \in \{-1, 1\}^n$ (i.e., a ± 1-vector in $\mathbb{R}^n$) we denote by T_y the diagonal matrix with diagonal vector y. Then

$$\mathbb{R}^n_y = \{\, x \mid T_y x \geq 0 \,\}$$

is the orthant prescribed by the sign vector y, and

$$(\mathbb{R}^n_y)^0 = \{\, x \mid T_y x > 0 \,\}$$

is its interior. The equation (1) is said to be *orthantwise solvable* if in each orthant of $\mathbb{R}^n$ it possesses a unique solution which, moreover, belongs to the interior of that orthant. Hence an orthantwise solvable equation (1) possesses exactly 2^n solutions. A square interval matrix is called regular if all matrices contained therein are nonsingular, and it is said to be singular otherwise. I denotes the $n \times n$ identity matrix.

0.2 The result

The following theorem shows a particular property of the double absolute value equation.

Theorem 1. *Let A be nonsingular and let the interval matrix*

$$[A^{-1} - I,\, A^{-1} + I] \tag{2}$$

be regular. Then for each $b > 0$ the equation (1) is orthantwise solvable.

Proof. Take a $y \in \{-1, 1\}^n$ and consider the absolute value equation

$$A^{-1}x' - T_y|x'| = T_y b. \tag{3}$$

Because the interval matrix $[A^{-1} - |T_y|,\, A^{-1} + |T_y|] = [A^{-1} - I,\, A^{-1} + I]$ is regular by assumption, by [3, Thm. 1] the equation (3) possesses a unique solution x'. Put

$$x_y = A^{-1}x', \tag{4}$$

then (3) can be rewritten as

$$x_y - T_y|Ax_y| = T_y b$$

and

$$T_y x_y - |Ax_y| = b,$$

where

$$T_y x_y = |Ax_y| + b \geq b > 0,$$

hence

$$T_y x_y = |x_y|$$

so that x_y solves (1), belongs to $(\mathbb{R}^n_y)^0$ and by [3, Thm. 1] it is a unique such a solution. As $y \in \{-1,1\}^n$ was arbitrary, the property holds for each orthant of $\mathbb{R}^n$. □

Thus, under (2) regular and $b > 0$, to compute the unique solution x_y of (1) in $(\mathbb{R}^n_y)^0$, we must first solve the absolute value equation (3) and then rearrange its solution by (4). Performing this process for each $y \in \{-1,1\}^n$, we can find all solutions of (1).

Solving the absolute value equation (3) may be performed using MATLAB file **absvaleqn.m** freely downloadable from `http://uivtx.cs.cas.cz/~rohn/other/absvaleqn.m`.

As regards regularity of (2), for moderate values of n (say, $n \leq 20$), it may be checked using a necessary and sufficient condition [6, Thm. 1, (iv)]: *(2) is regular if and only if the numbers*

$$\det(A^{-1} - T_y), \quad y \in \{-1,1\}^n$$

are either all negative, or all positive. For larger values of n, one may try a sufficient regularity condition [5, Thm. 4]: *if A is nonsingular and*

$$\min\{\varrho(|A|),\, \varrho(|AA^T|) < 1$$

holds, then the interval matrix (2) is regular. Here ϱ stands for the spectral radius of a matrix.

0.3 Example

Consider a double absolute value equation with randomly generated data

```
A =

   -0.1825    0.0111    0.4944
   -0.1642   -0.4793    0.3795
   -0.2134    0.4314   -0.3814


b =

    0.1757
    0.2089
    0.9052

X =

    1.0000    1.0000    1.0000    0.7487    0.4239    1.4260
   -1.0000    1.0000    1.0000   -0.7777    0.4636    0.9201
   -1.0000   -1.0000    1.0000   -2.0065   -3.2464    3.0352
    1.0000   -1.0000    1.0000    1.7962   -2.7524    4.0028
    1.0000   -1.0000   -1.0000    0.7777   -0.4636   -0.9201
   -1.0000   -1.0000   -1.0000   -0.7487   -0.4239   -1.4260
   -1.0000    1.0000   -1.0000   -1.7962    2.7524   -4.0028
    1.0000    1.0000   -1.0000    2.0065    3.2464   -3.0352
```

Each row of the output matrix X is of the form $(y^T x_y^T)$ where x_y is the unique solution of (1) in $(\mathbb{R}^n_y)^0$. Observe that indeed $T_y x_y > 0$ for each $y \in \{-1,1\}^n$ and that $x_{-y} = -x_y$ for each such a y as it can be easily proved from (3), (4).

The example was solved using the following MATLAB code.

```
function [X]=dblabsvaleqn(A,b)
% Orthantwise solvability of the double absolute value equation
% abs(x)-abs(A*x)=b.
X=[];
if ~(b>0), error('vector not positive'), end
n=size(A,1); I=eye(n);
if rank(A)<n, error('singular matrix'), end
B=inv(A);
S=regising(B,I);
% download: http://uivtx.cs.cas.cz/~rohn/other/regising.m
if ~isempty(S), error('interval matrix not regular'), end
z=zeros(1,n); y=ones(1,n);
x=absvaleqn(B,-diag(y),diag(y)*b);
% download: http://uivtx.cs.cas.cz/~rohn/other/absvaleqn.m
x=B*x; X=[y x'];
while any(z~=ones(1,n))
    k=find(z==0,1);
    z(1:(k-1))=zeros(1,k-1);
    z(k)=1; y(k)=-y(k);
    x=absvaleqn(B,-diag(y),diag(y)*b);
    x=B*x; X=[X; [y x']];
end
```

0.4 Related results

Theorem 1 asserts *[unique] solvability in the interior of each orthant.* There are some results related to this property. We have the following theorem of the alternatives.

Theorem 2. *For each nonsingular A exactly one of the following two alternatives holds:*

(i) the inequality

$$|Ax| \geq |x|$$

has a solution $x \neq 0$,

(ii) the inequality

$$|Ax| < |x|$$

has a solution in the interior of each orthant.

Proof. The proof proceeds by showing using [2, Lemma 2.1] and [4, Thm. 3.2, (v)] (with obvious details omitted here) that the alternative (i) is equivalent to singularity of the interval matrix (2), and (ii) is equivalent to its regularity. Hence at least one of the two alternatives always holds and they exclude each other, which completes the proof. □

The result can also be formulated in a normwise form.

Theorem 3. *For each nonsingular A exactly one of the following two alternatives holds:*

(i) the inequality

$$\|Ax\|_1 \geq \|x\|_\infty$$

has a solution $x \neq 0$,

(ii) the inequality

$$\|Ax\|_1 < \min_i |x_i|$$

has a solution in the interior of each orthant.

Proof. The proof runs in parallel to the previous one, with the interval matrix (2) being replaced by $[A^{-1} - E, A^{-1} + E]$, E being the matrix of all ones. □

Bibliography

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