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# Two limited-memory optimization methods with minimum violation of the previous quasi-Newton equations 

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# Two limited-memory optimization methods with minimum violation of the previous quasi-Newton equations ${ }^{1}$ 

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#### Abstract

: Limited-memory variable metric methods based on the well-known BFGS update are widely used for large scale optimization. The block version of the BFGS update, derived by Schnabel (1983), Hu and Storey (1991) and Vlček and Lukšan (2019), satisfies the quasi-Newton equations with all used difference vectors and for quadratic objective functions gives the best improvement of convergence in some sense, but the corresponding direction vectors are not descent directions generally. To guarantee the descent property of direction vectors and simultaneously violate the quasi-Newton equations as little as possible in some sense, two methods based on the block BFGS update are proposed. They can be advantageously combined with methods based on vector corrections for conjugacy (Vlček and Lukšan, 2015). Global convergence of the proposed algorithm is established for convex and sufficiently smooth functions. Numerical experiments demonstrate the efficiency of the new methods.


Keywords:
Unconstrained minimization, variable metric methods, limited-memory methods, variationally derived methods, global convergence, numerical results

[^0]
## 1 Introduction

In this report we assume that the problem function $f: \mathcal{R}^{N} \rightarrow \mathcal{R}$ is differentiable and propose two new limited-memory variable metric (VM) or quasi-Newton (QN) methods for large scale unconstrained optimization

$$
\min f(x): x \in \mathcal{R}^{N}
$$

which are based on a block version of the widely used BFGS update, see below.
The VM methods [7, 14] are iterative. They start with an initial point $x_{0} \in \mathcal{R}^{N}$ and generate iterations $x_{k+1} \in \mathcal{R}^{N}$ by the process $x_{k+1}=x_{k}+s_{k}, s_{k}=t_{k} d_{k}, k \geq 0$, where $d_{k}$ is the direction vector, $t_{k}>0$ the stepsize, chosen regularly in such a way that

$$
\begin{equation*}
f\left(x_{+}\right)-f(x) \leq \varepsilon_{1} t g^{T} d, \quad g_{+}^{T} d \geq \varepsilon_{2} g^{T} d \tag{1.1}
\end{equation*}
$$

(the Wolfe line search conditions [17]), $0<\varepsilon_{1}<1 / 2, \varepsilon_{1}<\varepsilon_{2}<1$ and $g_{k}=\nabla f\left(x_{k}\right)$. Usually $d_{k}=-H_{k} g_{k}$ with a symmetric positive VM matrix $H_{k} \triangleq B_{k}^{-1}$; typically $H_{0}$ is a multiple of $I$ and $H_{k+1}$ is obtained from $H_{k}$ to satisfy the QN (secant) equation (condition)

$$
\begin{equation*}
H_{k+1} y_{k}=s_{k}, \tag{1.2}
\end{equation*}
$$

with the difference vectors $s_{k}, y_{k}=g_{k+1}-g_{k}$. Among VM methods, the BFGS method $[7,14,17]$ belongs to the most efficient ones and can be easily modified for large scale optimization; the L-BFGS $[10,16]$ and BNS [3] methods represent its well-known limitedmemory adaptations. We refer to Section 2 for a brief description of these methods.

To incorporate more past information to the update formula, the block version of the BFGS update was derived in [18] for symmetric positive definite VM matrices, using a variational approach, in [8] for quadratic functions, using corrections for the exact line search, and recently in [21] for general functions, using a block variant of the approach in [4]. This update satisfies the QN equations with all used difference vectors and brings the best improvement of convergence in some sense [20, 21] for quadratic objective functions. Nevertheless, it does not guarantee that the corresponding direction vectors are descent directions for general functions.

Using the block BFGS update in a generalized form [21], in Section 3 we derive two new updates variationally with direction vectors that satisfy the descent property. Naturally, some QN equations with the used difference vectors can be violated. To utilize the advantageous properties of the block BFGS updates, we look for a solution to the following constrained optimization problem: to find such an update which satisfies the QN equations with several latest difference vectors and violates the QN equations from previous iterations as little as possible in some sense. In Section 3.1 we show that this problem can be converted equivalently to a problem without equality constraints and with a smaller number of variables. We also derive an expression for the solution of this simpler problem. Our first new update is based on this expression.

Besides, in such optimization problems we can use further constraints, derived from properties of the block BNS method [21]. Then the resultant second new update has a simple implementation and some interesting properties, see Section 3.2.

Our experiments indicate that the efficiency of the methods based on these two updates is similar. Moreover, it can be significantly improved by combination with methods based on vector corrections for conjugacy which use previous difference vectors and store
the corrected difference vectors [19, 20, 22]. A main reason for this improvement probably consists in the fact that these vectors commonly contain cumulative information from previous iterations.

In Section 4 we combine these corrections with the corrections that use the subsequent difference vectors (see Section 4.1) to satisfy the QN equations with both the corrected and original, uncorrected latest difference vectors, see Section 4.2. For our new updates, these special corrections appear to give slightly better numerical results than the others.

The application to the limited-memory VM methods and the corresponding algorithm are described in Section 5. Global convergence of the algorithm is established in Section 6 and numerical results are reported in Section 7.

To simplify notation we frequently omit the index $k$ and replace the index $k+1$ by the symbol + and the index $k-1$ by the symbol - .

We will denote by $\|\cdot\|_{F}$ the Frobenius matrix norm, by $\|\cdot\|$ the spectral matrix norm and by $|\cdot|$ the size of both scalars and vectors (the Euclidean vector norm).

## 2 The L-BFGS and BNS methods

In this section we briefly describe the limited-memory VM methods L-BFGS [10, 16], implemented as subroutine PLIS in [11], and BNS [3]. These methods are based on the BFGS update formula, mentioned in Section 1, which preserves the positive definiteness of $H$ and can be written in the following quasi-product form

$$
\begin{equation*}
H_{+}=(1 / b) s s^{T}+\left(I-(1 / b) s y^{T}\right) H\left(I-(1 / b) y s^{T}\right), \quad b=s^{T} y \tag{2.1}
\end{equation*}
$$

( $b>0$ for $g \neq 0$ by (1.1)). To modify the BFGS method for large scale optimization, we choose $H_{k}^{I} \in \mathcal{R}^{N \times N}$ in every iteration (usually $H_{k}^{I}=\zeta_{k} I, \zeta_{k}>0$ ) and recurrently update $H_{k}^{I}$ (without forming an approximation of the inverse Hessian matrix explicitly) by the BFGS formula, using $m$ pairs of vectors $\left(s_{k-\tilde{m}}, y_{k-\tilde{m}}\right), \ldots,\left(s_{k}, y_{k}\right)$ successively, where

$$
\begin{equation*}
\tilde{m}=\min [k, \hat{m}-1], \quad m=\tilde{m}+1, \quad k \geq 0 \tag{2.2}
\end{equation*}
$$

and $\hat{m}>1$ is a given parameter. We use the BNS update to guarantee global convergence of Algorithm 5.1 if some conditions for our new methods are not satisfied, see Section 5. Instead of the famous compact form [3], we use it in the form (also given in [3])

$$
\begin{equation*}
H_{+}=S R^{-T} D R^{-1} S^{T}+\left(I-S R^{-T} Y^{T}\right) H^{I}\left(I-Y R^{-1} S^{T}\right) \tag{2.3}
\end{equation*}
$$

where $S_{k}=\left[s_{k-\tilde{m}}, \ldots, s_{k}\right], Y_{k}=\left[y_{k-\tilde{m}}, \ldots, y_{k}\right], D_{k}=\operatorname{diag}\left[b_{k-\tilde{m}}, \ldots, b_{k}\right],\left(R_{k}\right)_{i, j}=\left(S_{k}^{T} Y_{k}\right)_{i, j}$ for $i \leq j,\left(R_{k}\right)_{i, j}=0$ otherwise (an upper triangular matrix), $k \geq 0$. We can see that for $H^{I}=\zeta I$ the direction vector $-H_{+} g_{+}$(and subsequently an auxiliary vector $Y^{T} H_{+} g_{+}$, see Section 5) can be calculated efficiently (without computing of $H_{+}$explicitly) by

$$
\begin{align*}
-H_{+} g_{+} & =-\zeta g_{+}-S\left[R^{-T}\left(\left(D+\zeta Y^{T} Y\right) R^{-1} S^{T} g_{+}-\zeta Y^{T} g_{+}\right)\right]+Y\left[\zeta R^{-1} S^{T} g_{+}\right]  \tag{2.4}\\
Y^{T} H_{+} g_{+} & =\zeta Y^{T} g_{+}+Y^{T} S\left[R^{-T}\left(\left(D+\zeta Y^{T} Y\right) R^{-1} S^{T} g_{+}-\zeta Y^{T} g_{+}\right)\right]-Y^{T} Y\left[\zeta R^{-1} S^{T} g_{+}\right] \tag{2.5}
\end{align*}
$$

where in the square brackets we multiply by low-order matrices.

## 3 Two variationally derived VM updates

The basic variant of the block BFGS update derived in [21] for general functions is

$$
\begin{equation*}
H_{+}=S A^{-1} S^{T}+\left(I-S A^{-T} Y^{T}\right) H^{I}\left(I-Y A^{-1} S^{T}\right), \quad A=S^{T} Y \tag{3.1}
\end{equation*}
$$

where $H^{I} \in \mathcal{R}^{N \times N}$ is symmetric and $A \in \mathcal{R}^{m \times m}$ arbitrary nonsingular. The matrix $H_{+}$is the nearest matrix to $H^{I}$ in some sense that satisfies the QN equation $H_{+} Y=S$, however the direction vector $d_{+}=-H_{+} g_{+}$is not a descent direction generally.

Replacing $S$ by $S \Theta, \Theta \in \mathcal{R}^{m \times m}$ nonsingular, in the same way as (3.1) we can derive the generalized form

$$
\begin{equation*}
H_{+}=S X S^{T}+\left(I-S A^{-T} Y^{T}\right) H^{I}\left(I-Y A^{-1} S^{T}\right), \quad X=\Theta A^{-1} \tag{3.2}
\end{equation*}
$$

which satisfies the violated (for $\Theta \neq I$ ) QN equations $H_{+} Y=S(X A)=S \Theta$. Note that the infinitely many times repeated BNS update, investigated in [22], has the same form (with a special symmetric positive definite matrix $X$ ).

To express the violation of the QN equations $H_{+} Y=S$ for $H_{+}$nonsingular, we will use the matrix

$$
\begin{equation*}
\Delta=\left(H_{+} Y-S\right)^{T} B_{+}\left(H_{+} Y-S\right)=\left(B_{+} S-Y\right)^{T} H_{+}\left(B_{+} S-Y\right), \quad B_{+}=H_{+}^{-1} \tag{3.3}
\end{equation*}
$$

Obviously, for $H_{+}$symmetric positive definite we can write $\operatorname{Tr} \Delta=\left\|B_{+}^{1 / 2}\left(H_{+} Y-S\right)\right\|_{F}^{2}$. The following lemma gives another expression of $\Delta$, which uses only low-order matrices.

Lemma 3.1. Let $H_{+}$be given by (3.2) with $X$ symmetric and $X A$ nonsingular. Then

$$
\begin{equation*}
\Delta=A^{T} X A+X^{-1}-A-A^{T} \tag{3.4}
\end{equation*}
$$

Proof. Since (3.2) implies $H_{+} Y=S(X A)$ and $Y(X A)^{-1}=B_{+} S$, the desired conclusion follows from $\Delta=Y^{T} H_{+} Y+S^{T} B_{+} S-A-A^{T}$.

Besides (1.2), more latest QN equations can be satisfied, if some lower-right-corner principal submatrix of $A$ is symmetric (see Lemmas 3.2 and 3.4), which can be achieved e.g. by means of vector corrections for conjugacy (see Section 4).

According to our intention, see Section 1, first we will look for such a symmetric positive definite matrix $X$, which minimizes $\operatorname{Tr} \Delta$ in some sense, see below, and for which update (3.2) satisfies the QN equations with several latest difference vectors.

### 3.1 The first new update

To derive our first update of the form (3.2), we will solve the problem

$$
\begin{equation*}
\min _{X \in \mathcal{S}_{m}} \operatorname{Tr} \Delta \quad \text { s.t. } \quad H_{+} Y_{(2)}=S_{(2)}, \tag{3.5}
\end{equation*}
$$

where we split $S, Y$ in such a way that $S=\left[S_{(1)}, S_{(2)}\right], Y=\left[Y_{(1)}, Y_{(2)}\right], S_{(2)}, Y_{(2)} \in \mathcal{R}^{N \times \mu}$, $0<\mu<m$, and by $\mathcal{S}_{i}$ we denote the set of symmetric positive definite matrices of order $i$.

The following lemma and theorem show that the QN equations in (3.5) can be replaced by some conditions for structure of $X$ and that it will be sufficient to look for some submatrix of $X$, which minimizes the trace of some submatrix of $\Delta$.

Lemma 3.2. Let $S=\left[S_{(1)}, S_{(2)}\right]$, $Y=\left[Y_{(1)}, Y_{(2)}\right], S_{(2)}, Y_{(2)} \in \mathcal{R}^{N \times \mu}, 0<\mu<m$, let

$$
A=\left[\begin{array}{ll}
A_{11} & A_{12}  \tag{3.6}\\
A_{21} & A_{22}
\end{array}\right]=\left[\begin{array}{ll}
S_{(1)}^{T} Y_{(1)} & S_{(1)}^{T} Y_{(2)} \\
S_{(2)}^{T} Y_{(1)} & S_{(2)}^{T} Y_{(2)}
\end{array}\right]
$$

be nonsingular and a symmetric matrix $X \in \mathcal{R}^{m \times m}$ be partitioned in the same way. Let $A_{22}$ be symmetric nonsingular and $H_{+}$be given by (3.2). Then $H_{+} Y_{(2)}=S_{(2)}$ if and only if

$$
\begin{equation*}
X_{12}=X_{21}^{T}=-X_{11} A_{12} A_{22}^{-1}, \quad X_{22}=A_{22}^{-1}+A_{22}^{-1} A_{12}^{T} X_{11} A_{12} A_{22}^{-1} \tag{3.7}
\end{equation*}
$$

Proof. In view of (3.6) we have

$$
X A=\left[\begin{array}{ll}
X_{11} A_{11}+X_{12} A_{21} & X_{11} A_{12}+X_{12} A_{22}  \tag{3.8}\\
X_{12}^{T} A_{11}+X_{22} A_{21} & X_{12}^{T} A_{12}+X_{22} A_{22}
\end{array}\right] \triangleq\left[\begin{array}{ll}
(X A)_{11} & (X A)_{12} \\
(X A)_{21} & (X A)_{22}
\end{array}\right],
$$

thus (3.7) is equivalent to $(X A)_{12}=0,(X A)_{22}=I$ and further to $H_{+} Y_{(2)}=S_{(2)}$ by $H_{+} Y=S(X A)$ and linear independence of columns of $S$, which follows from $\operatorname{det} A \neq 0$.

Theorem 3.1. Let the assumptions of Lemma 3.2 be satisfied, $C=A_{11}-A_{12} A_{22}^{-1} A_{21}$, $X_{11}$ be nonsingular and (3.7) hold. Then $X$ and $C$ are nonsingular, $H_{+} Y_{(2)}=S_{(2)}$ and

$$
\begin{align*}
X A & =\left[\begin{array}{cc}
X_{11} C & 0 \\
A_{22}^{-1}\left(A_{21}-A_{12}^{T} X_{11} C\right) & I
\end{array}\right], \quad X^{-1}=\left[\begin{array}{cc}
X_{11}^{-1}+A_{12} A_{22}^{-1} A_{12}^{T} & A_{12} \\
A_{12}^{T} & A_{22}
\end{array}\right],  \tag{3.9}\\
\Delta & =\left[\begin{array}{cc}
C^{T} X_{11} C+X_{11}^{-1}-C-C^{T}+\left(A_{12}-A_{21}^{T}\right) A_{22}^{-1}\left(A_{12}^{T}-A_{21}\right) & 0 \\
0 & 0
\end{array}\right] . \tag{3.10}
\end{align*}
$$

Moreover, if $A_{22} \in \mathcal{S}_{\mu}$ then we get $X \in \mathcal{S}_{m}$ if and only if $X_{11} \in \mathcal{S}_{m-\mu}$.
Proof. Theorem 1.4.2 in [6] yields $\operatorname{det} A=\operatorname{det} C$. $\operatorname{det} A_{22}$, therefore $\operatorname{det} C \neq 0$. As in the proof of Lemma 3.2 we get (3.8), $H_{+} Y_{(2)}=S_{(2)},(X A)_{12}=0$ and $(X A)_{22}=I$. Since

$$
X_{22}-X_{21} X_{11}^{-1} X_{12}=A_{22}^{-1}+A_{22}^{-1} A_{12}^{T} X_{11} A_{12} A_{22}^{-1}-A_{22}^{-1} A_{12}^{T} X_{11} A_{12} A_{22}^{-1}=A_{22}^{-1}
$$

by (3.7), we similarly obtain $\operatorname{det} X=\operatorname{det} A_{22}^{-1} \cdot \operatorname{det} X_{11}$, thus $X$ is nonsingular and all assumptions of Lemma 3.1 are satisfied. From (3.8) and (3.7) we derive

$$
\begin{aligned}
(X A)_{11} & =X_{11}\left(A_{11}-A_{12} A_{22}^{-1} A_{21}\right)=X_{11} C \\
(X A)_{21} & =-A_{22}^{-1} A_{12}^{T} X_{11} A_{11}+\left(A_{22}^{-1}+A_{22}^{-1} A_{12}^{T} X_{11} A_{12} A_{22}^{-1}\right) A_{21} \\
& =A_{22}^{-1}\left(A_{21}-A_{12}^{T} X_{11}\left(A_{11}-A_{12} A_{22}^{-1} A_{21}\right)\right)
\end{aligned}
$$

This gives the first relation in (3.9) and further

$$
A^{T} X A-A^{T}=A^{T}(X A-I)=\left[\begin{array}{ll}
A_{11}^{T} & A_{21}^{T} \\
A_{12}^{T} & A_{22}
\end{array}\right]\left[\begin{array}{cc}
X_{11} C-I & 0 \\
A_{22}^{-1}\left(A_{21}-A_{12}^{T} X_{11} C\right) & 0
\end{array}\right],
$$

which implies

$$
\begin{align*}
\left(A^{T} X A-A^{T}\right)_{12} & =0, \quad\left(A^{T} X A-A^{T}\right)_{22}=0, \quad\left(A^{T} X A-A^{T}\right)_{21}=A_{21}-A_{12}^{T},  \tag{3.11}\\
\left(A^{T} X A-A^{T}\right)_{11} & =A_{11}^{T} X_{11} C-A_{11}^{T}+A_{21}^{T} A_{22}^{-1} A_{21}-A_{21}^{T} A_{22}^{-1} A_{12}^{T} X_{11} C \\
& =C^{T} X_{11}^{T} C-A_{11}^{T}+A_{21}^{T} A_{22}^{-1} A_{21} . \tag{3.12}
\end{align*}
$$

Similarly, we can express

$$
X^{-1}-A=A\left((X A)^{-1}-I\right)=\left[\begin{array}{ll}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array}\right]\left[\begin{array}{cc}
C^{-1} X_{11}^{-1}-I & 0 \\
-A_{22}^{-1}\left(A_{21}-A_{12}^{T} X_{11} C\right) C^{-1} X_{11}^{-1} & 0
\end{array}\right]
$$

by the first relation in (3.9), which yields

$$
\begin{align*}
\left(X^{-1}-A\right)_{12} & =0, \quad\left(X^{-1}-A\right)_{22}=0, \quad\left(X^{-1}-A\right)_{21}=-A_{21}+A_{12}^{T}  \tag{3.13}\\
\left(X^{-1}-A\right)_{11} & =A_{11} C^{-1} X_{11}^{-1}-A_{11}-A_{12} A_{22}^{-1} A_{21} C^{-1} X_{11}^{-1}+A_{12} A_{22}^{-1} A_{12}^{T} \\
& =C\left(C^{-1} X_{11}^{-1}\right)-A_{11}+A_{12} A_{22}^{-1} A_{12}^{T} \tag{3.14}
\end{align*}
$$

i.e. the second relation in (3.9). Considering that

$$
\begin{aligned}
\left(A_{12}-A_{21}^{T}\right) A_{22}^{-1}\left(A_{12}^{T}-A_{21}\right) & =A_{12} A_{22}^{-1} A_{12}^{T}-A_{12} A_{22}^{-1} A_{21}-A_{21}^{T} A_{22}^{-1} A_{12}^{T}+A_{21}^{T} A_{22}^{-1} A_{21} \\
& =A_{12} A_{22}^{-1} A_{12}^{T}+A_{21}^{T} A_{22}^{-1} A_{21}-A_{11}-A_{11}^{T}+C+C^{T}
\end{aligned}
$$

we obtain (3.10) from Lemma 3.1, (3.12) and (3.14). Moreover, for $A_{22} \in \mathcal{S}_{\mu}$ we have $X_{11} \in \mathcal{S}_{m-\mu}$ if and only if $X \in \mathcal{S}_{m}$ by the second relation in (3.9), since the Schur complement of $A_{22}$ in $X^{-1}$ is $X_{11}^{-1}$.

For $A_{22} \in \mathcal{S}_{\mu}$ and $A$ nonsingular, by Theorem 3.1 we can equivalently solve the problem

$$
\begin{equation*}
\min _{X_{11} \in \mathcal{S}_{m-\mu}} \operatorname{Tr}\left(C^{T} X_{11} C+X_{11}^{-1}-C-C^{T}\right) \tag{3.15}
\end{equation*}
$$

instead of (3.5). A solution to this problem is given by the following theorem.
Theorem 3.2. Let $C$ be nonsingular. Then $X_{11}=\left(C C^{T}\right)^{-1 / 2}$ is the unique solution to (3.15) and for this $X_{11}$, the matrices $X_{11}^{-1}, C^{T} X_{11} C$ are similar.

Proof. For $X_{11}=\left(C C^{T}\right)^{-1 / 2}$, the matrices $X_{11}^{-1}, C^{T} X_{11} C=C^{-1}\left(C C^{T}\right) X_{11} C=C^{-1} X_{11}^{-1} C$ are similar. Let $J=\left(C C^{T}\right)^{1 / 2}$. Obviously, $J \in \mathcal{S}_{m-\mu}$. Since the trace of a product of two square matrices is independent of the order of multiplication, for any $X_{11} \in \mathcal{S}_{m-\mu}$ we have

$$
\begin{aligned}
\operatorname{Tr}\left(C^{T} X_{11} C+X_{11}^{-1}\right) & =\operatorname{Tr}\left(X_{11} C C^{T}+X_{11}^{-1}\right)=\operatorname{Tr}\left(X_{11}^{1 / 2}\left(J^{2}+X_{11}^{-2}\right) X_{11}^{1 / 2}\right) \\
& =\operatorname{Tr}\left(X_{11}^{1 / 2}\left(J^{2}-J X_{11}^{-1}-X_{11}^{-1} J+X_{11}^{-2}\right) X_{11}^{1 / 2}+2 J\right) \\
& =\operatorname{Tr}\left(X_{11}^{1 / 2}\left(J-X_{11}^{-1}\right)^{2} X_{11}^{1 / 2}+2 J\right),
\end{aligned}
$$

which completes the proof by $X_{11}^{1 / 2} \in \mathcal{S}_{m-\mu}$ and symmetry of $J-X_{11}^{-1}$.
Note that for $X_{11}=\left(C C^{T}\right)^{-1 / 2}$, the matrix $X_{11} C$ is orthogonal, as we can see from

$$
\left(X_{11} C\right)^{-1}=C^{-1} X_{11}^{-2} X_{11}=C^{-1}\left(C C^{T}\right) X_{11}=\left(X_{11} C\right)^{T} .
$$

### 3.2 The second new update

Our second new update of the form (3.2) utilizes some of properties of the block BNS update [21] to minimize not only $\operatorname{Tr} \Delta$, but also $\|\Delta\|_{F}$. We consider the update

$$
\begin{equation*}
H_{+}^{R}=S_{R} X_{R} S_{R}^{T}+\left(I-S_{R} A_{R}^{-T} Y_{R}^{T}\right) H^{I}\left(I-Y_{R} A_{R}^{-1} S_{R}^{T}\right) \tag{3.16}
\end{equation*}
$$

where $S_{R}, Y_{R} \in \mathcal{R}^{N \times m-1}$ are the matrices $S, Y$ without the first column, $A_{R}=S_{R}^{T} Y_{R}$ is nonsingular and $X_{R}$ is symmetric. The block BNS update of $H^{I}$, where one of the diagonal blocks is $S_{R}^{T} Y_{R}$ (of order $m-1$ ), can have the form

$$
H_{+}^{B B N S}=S_{R} X_{R} S_{R}^{T}+\left(I-S_{R} A_{R}^{-T} Y_{R}^{T}\right) H^{A}\left(I-Y_{R} A_{R}^{-1} S_{R}^{T}\right)
$$

with some suitable $H^{A} \in \mathcal{R}^{N \times N}$ (see Lemma 6 in [21]). The matrix $H_{+}^{B B N S}$ has the property $H_{+}^{B B N S} Y_{R}=S_{R}\left(X_{R} A_{R}\right)=H_{+}^{R} Y_{R}$. By analogy with this, we will look for $X \in \mathcal{S}_{m}$ which minimizes both $\operatorname{Tr} \Delta$ and $\|\Delta\|_{F}$ subject to $H_{+} Y_{R}=H_{+}^{R} Y_{R}$. For such $X$ we will show that $X A \in \mathcal{T}_{m}$, where $\mathcal{T}_{i}$ is the set of lower triangular matrices of order $i$.

Theorem 3.3. Let $H^{I} \in \mathcal{R}^{N \times N}$ be a symmetric positive definite matrix, $m>1$ and $S_{R}, Y_{R} \in \mathcal{R}^{N \times m-1}$ the matrices $S, Y$ without the first column. Let $A_{R}=S_{R}^{T} Y_{R}$ be nonsingular, let some $X_{R} \in \mathcal{S}_{m-1}$ satisfying $X_{R} A_{R} \in \mathcal{T}_{m-1}$ be firmly given, $H_{+}^{R}$ be given by (3.16) and $H_{+}$by (3.2) with $A$ nonsingular and $X$ symmetric. Then for

$$
A=\left[\begin{array}{cc}
\alpha & u^{T}  \tag{3.17}\\
v & A_{R}
\end{array}\right], \quad X=\left[\begin{array}{cc}
\xi & \chi^{T} \\
\chi & X_{P}
\end{array}\right], \quad p=A_{R}^{-T} u, \quad c=\alpha-p^{T} v
$$

$\alpha, \xi \in \mathcal{R}$, we have $c \neq 0$. Further for both $i=1$ and $i=2$, the unique solution to

$$
\begin{equation*}
\min _{X \in \mathcal{S}_{m}} \varphi_{i}(\Delta) \quad \text { s.t. } \quad H_{+} Y_{R}=H_{+}^{R} Y_{R}, \quad \varphi_{1}(\Delta)=\operatorname{Tr} \Delta, \quad \varphi_{2}(\Delta)=\|\Delta\|_{F} \tag{3.18}
\end{equation*}
$$

satisfies $X A \in \mathcal{T}_{m}$ and is given by the choice $\xi=1 /|c|, \chi=-\xi p$ and $X_{P}=X_{R}+\xi p p^{T}$.
Proof. Since $\operatorname{det} A=\left(\alpha-u^{T} A_{R}^{-1} v\right) \operatorname{det} A_{R}=c \operatorname{det} A_{R}$, we get $c \neq 0$. Further,

$$
X A=\left[\begin{array}{cc}
\alpha \xi+\chi^{T} v & \xi u^{T}+\chi^{T} A_{R}  \tag{3.19}\\
\alpha \chi+X_{P} v & \chi u^{T}+X_{P} A_{R}
\end{array}\right]
$$

by (3.17). The terms $H_{+} Y_{R}, H_{+}^{R} Y_{R}$ in (3.18) can be equivalently rewritten as

$$
H_{+} Y_{R}=H_{+} Y\left[\begin{array}{c}
0^{T} \\
I
\end{array}\right]=S(X A)\left[\begin{array}{c}
0^{T} \\
I
\end{array}\right], \quad H_{+}^{R} Y_{R}=S_{R}\left(X_{R} A_{R}\right)=S\left[\begin{array}{c}
0^{T} \\
X_{R} A_{R}
\end{array}\right]
$$

By this, (3.19) and linear independence of columns of $S$ (which follows from $\operatorname{det} A \neq 0$ ), the equality $H_{+} Y_{R}=H_{+}^{R} Y_{R}$ is equivalent to

$$
\begin{equation*}
\xi u^{T}+\chi^{T} A_{R}=0^{T}, \quad \chi u^{T}+X_{P} A_{R}=X_{R} A_{R} \tag{3.20}
\end{equation*}
$$

For $X \in \mathcal{S}_{m}$, we obviously have $H_{+} \in \mathcal{S}_{N}$ and $\xi>0$ and (3.20) is equivalent to

$$
\chi=-\xi p, \quad X_{P}=X_{R}+\xi p p^{T}
$$

In view of this, we can rewrite the first column of $X A$ in (3.19) in the following way:

$$
\alpha \xi+\chi^{T} v=\xi c, \quad \alpha \chi+X_{P} v=-\alpha \xi p+X_{R} v+\xi\left(p^{T} v\right) p=X_{R} v-\xi c p \triangleq z
$$

By this and (3.19) - (3.20) we obtain

$$
X A=\left[\begin{array}{cc}
\xi c & 0^{T}  \tag{3.21}\\
z & X_{R} A_{R}
\end{array}\right], \quad(X A)^{-1}=\left[\begin{array}{cc}
1 /(\xi c) & 0^{T} \\
-A_{R}^{-1} X_{R}^{-1} z /(\xi c) & A_{R}^{-1} X_{R}^{-1}
\end{array}\right] .
$$

In view of Lemma 3.1, (3.21) and $X^{-1}=A(X A)^{-1}$ we can write $\Delta$ in the form

$$
\begin{aligned}
\Delta & =\left[\begin{array}{cc}
\alpha & v^{T} \\
u & A_{R}^{T}
\end{array}\right]\left[\begin{array}{cc}
\xi c & 0^{T} \\
z & X_{R} A_{R}
\end{array}\right]+\left[\begin{array}{cc}
\alpha & u^{T} \\
v & A_{R}
\end{array}\right]\left[\begin{array}{cc}
1 /(\xi c) & 0^{T} \\
-A_{R}^{-1} X_{R}^{-1} z /(\xi c) & A_{R}^{-1} X_{R}^{-1}
\end{array}\right]-A-A^{T} \\
& =\left[\begin{array}{cc}
\varrho-2 \alpha & v^{T} X_{R} A_{R}+u^{T} A_{R}^{-1} X_{R}^{-1}-u^{T}-v^{T} \\
A_{R}^{T} X_{R} v+X_{R}^{-1} A_{R}^{-T} u-u-v & A_{R}^{T} X_{R} A_{R}+X_{R}^{-1}-A_{R}-A_{R}^{T}
\end{array}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\varrho & =\left[\alpha, v^{T}\right] \cdot\left[\begin{array}{c}
\xi c \\
z
\end{array}\right]+\left[\alpha, u^{T}\right] \cdot\left[\begin{array}{c}
1 /(\xi c) \\
-A_{R}^{-1} X_{R}^{-1} z /(\xi c)
\end{array}\right] \\
& =\alpha(\xi c)+v^{T} z+\alpha /(\xi c)-u^{T} A_{R}^{-1} X_{R}^{-1} z /(\xi c) \\
& =\alpha(\xi c+1 /(\xi c))+\left(v^{T}-p^{T} X_{R}^{-1} /(\xi c)\right)\left(X_{R} v-\xi c p\right) \\
& =c(\xi c+1 /(\xi c))+v^{T} X_{R} v+p^{T} X_{R}^{-1} p
\end{aligned}
$$

by $p=A_{R}^{-T} u, z=X_{R} v-\xi c p$ and $c=\alpha-p^{T} v$. Obviously, we get minimum $\varrho$ for $\xi=1 /|c|$. Since all entries of $\Delta$ except for $\varrho-2 \alpha$ are independent of $\xi$ and $\varrho-2 \alpha \geq 0$ by $H_{+} \in \mathcal{S}_{N}$ and (3.3), solutions to both problems (3.18) are the same with $X \in \mathcal{S}_{m}$ by (3.17), $\xi>0$ and $X_{P}-\chi \chi^{T} / \xi=X_{R} \in \mathcal{S}_{m-1}$. Finally, $X A \in \mathcal{T}_{m}$ by (3.21) and $X_{R} A_{R} \in \mathcal{T}_{m-1}$.

To define our second new update (3.2) for $m \geq 2$, we start with $m=2$ and $X_{R}=A_{R}^{-1}$ symmetric positive definite of unit order, which corresponds to the BFGS update (2.1). Then we apply Theorem 3.3 iteratively, substituting $A, X$ from the previous iteration for $A_{R}, X_{R}$. We use this update only if all $c>0$ (otherwise the resultant VM method appears to be inefficient). Then the final matrix $X A$ is obviously lower triangular with unit diagonal entries, thus the QN equation $H_{+} y=s$ is satisfied by $H_{+} Y=S(X A)$; additional QN equations can be satisfied similarly as for the first new update, see Lemma 3.4.

The following lemma shows that such a matrix $X$, determined uniquely by the repeated application of Theorem 3.3, can also be constructed in a different way.
Lemma 3.3. Let $X \in \mathcal{S}_{m}$ and $A \in \mathcal{R}^{m \times m}$ be nonsingular with the factorization $A=U L$, where $U$ is upper triangular, $L$ lower triangular and $U, L$ have the same main diagonals. Then $X A$ is lower triangular with unit diagonal entries if and only if $X=U^{-T} U^{-1}$.
Proof. If $X=U^{-T} U^{-1}$, the matrix $X A=U^{-T} U^{-1} U L=U^{-T} L$ is obviously lower triangular and has unit diagonal entries in view of the assumption that $U, L$ have the same main diagonals.

Let $X \in \mathcal{S}_{m}$ and $X A \triangleq K$ is lower triangular with unit diagonal entries. Writing $X=R_{X}^{T} R_{X}$ with $R_{X}$ upper triangular, we have $K=R_{X}^{T} R_{X} U L$. This yields

$$
\begin{equation*}
R_{X} U=R_{X}^{-T} K L^{-1} \triangleq E \tag{3.22}
\end{equation*}
$$

i.e. the upper triangular matrix $R_{X} U$ is equal to the lower triangular matrix $R_{X}^{-T} K L^{-1}$. Thus $E$ is a diagonal matrix, $R_{X}=E U^{-1}$ and (3.22) implies $E=\left(E U^{-1}\right)^{-T} K L^{-1}=$ $E^{-T} U^{T} K L^{-1}$, or $E^{2}=U^{T} K L^{-1}$. Similarly as above we deduce that this diagonal matrix has unit diagonal entries, i.e. $E^{2}=I$ and $X=R_{X}^{T} R_{X}=\left(U^{-T} E\right)\left(E U^{-1}\right)=U^{-T} U^{-1}$.

The following lemma shows that QN equations $H_{+} Y_{(2)}=S_{(2)}$ can be satisfied under similar conditions as for the first new update.

Lemma 3.4. Let the assumptions of Lemmas 3.2 and 3.3 be satisfied and $X=U^{-T} U^{-1}$. Then $H_{+} Y_{(2)}=S_{(2)}$.

Proof. In view of Lemma 3.2 it suffices to prove (3.7). Suppose that

$$
U=\left[\begin{array}{cc}
U_{11} & U_{12}  \tag{3.23}\\
0 & U_{22}
\end{array}\right], \quad L=\left[\begin{array}{cc}
L_{11} & 0 \\
L_{21} & L_{22}
\end{array}\right]
$$

with $U_{22}, L_{22} \in \mathcal{R}^{\mu \times \mu}$. Since $A_{22}$ is symmetric nonsingular, from $A=U L$ and (3.23) we get first $A_{22}=U_{22} L_{22}=L_{22}^{T} U_{22}^{T}$. Thus the lower triangular matrix $L_{22} U_{22}^{-T}$ is equal to the upper triangular matrix $U_{22}^{-1} L_{22}^{T}$, where $U_{22}, L_{22}$ have the same main diagonals. This obviously yields $L_{22} U_{22}^{-T}=I$, i.e. $L_{22}=U_{22}^{T}$.

Further, $A=U L$ implies $A_{12}=U_{12} L_{22}$, or $U_{12}=A_{12} L_{22}^{-1}$, thus (3.23) gives

$$
U^{-1}=\left[\begin{array}{cc}
U_{11}^{-1} & -U_{11}^{-1} A_{12} A_{22}^{-1}  \tag{3.24}\\
0 & U_{22}^{-1}
\end{array}\right], \quad U^{-T}=\left[\begin{array}{cc}
U_{11}^{-T} & 0 \\
-A_{22}^{-1} A_{12}^{T} U_{11}^{-T} & U_{22}^{-T}
\end{array}\right]
$$

by $L_{22}^{-1} U_{22}^{-1}=A_{22}^{-1}$. From $X=U^{-T} U^{-1}$ we obtain $X_{11}=U_{11}^{-T} U_{11}^{-1}$ and subsequently

$$
X=\left[\begin{array}{cc}
X_{11} & -X_{11} A_{12} A_{22}^{-1}  \tag{3.25}\\
-A_{22}^{-1} A_{12}^{T} X_{11} & U_{22}^{-T} U_{22}^{-1}+A_{22}^{-1} A_{12}^{T} X_{11} A_{12} A_{22}^{-1}
\end{array}\right],
$$

thus (3.7) is satisfied by $U_{22}^{-T} U_{22}^{-1}=L_{22}^{-1} U_{22}^{-1}=A_{22}^{-1}$.
The second new update has the following property, relevant namely in connection with the corrections for conjugacy which use subsequent difference vectors, see Section 4.1.

Lemma 3.5. Let the assumptions of Lemma 3.3 be satisfied and $X=U^{-T} U^{-1}$. Then the matrix $H_{+}$given by (3.2) is invariant under the transformation $S \rightarrow S T_{S}, Y \rightarrow Y T_{Y}$ for arbitrary lower triangular matrices $T_{S}, T_{Y} \in \mathcal{R}^{m \times m}$ with the same main diagonals.
$\underset{\tilde{L}}{\text { Proof. Let }} \tilde{S}=S T_{S}, \tilde{Y}=Y T_{Y}, \tilde{A}=\tilde{S}^{T} \tilde{Y}$. Then $\tilde{A}=T_{S}^{T} A T_{Y}=\tilde{U} \tilde{L}$, where $\tilde{U}=T_{S}^{T} U$ and $\tilde{L}=L T_{Y}$ satisfy the same assumptions as $U, L$ in Lemma 3.3. Therefore corresponding $\tilde{X}$, determined uniquely by $\tilde{X}=\tilde{U}^{-T} \tilde{U}^{-1}=T_{S}^{-1} X T_{S}^{-T}$, see comments before Lemma 3.3, satisfies $\tilde{S} \tilde{X} \tilde{S}^{T}=S X S^{T}$. Further, $\tilde{Y} \tilde{A}^{-1} \tilde{S}^{T}=Y A^{-1} S^{T}$, which completes the proof.

## 4 Combination with vector corrections for conjugacy

It was shown in $[19,20,22]$ that the performance of limited-memory VM methods can often be improved by corrections for conjugacy which use previous difference vectors to correct the vectors $s, y$. These corrections can be written in the form $s \rightarrow \hat{s}=s+S_{C} \sigma$, $y \rightarrow \hat{y}=y+Y_{C} \eta$, where $S_{C}, Y_{C}$ are matrices with some selected columns of $S, Y$ without $s, y$. The vectors $\sigma, \eta$ are chosen so that $\hat{y}^{T} S_{C}=\hat{s}^{T} Y_{C}=0$, with potential modifications if $\hat{b}=\hat{s}^{T} \hat{y}$ is negative or too small. As we mentioned in Section 1, corresponding $\hat{s}, \hat{y}$ can be stored and used in next iterations instead of original $s, y$. However, this usually requires additional arithmetic operations.

Note that for quadratic functions, $\left[S_{C}, s\right]=S,\left[Y_{C}, y\right]=Y$ and the BFGS udate, these corrections represent the best improvement of convergence in some sense under
some conditions [20] and that for the theory in this section it is not significant, whether columns of $S_{C}, Y_{C}$ were corrected in previous iterations.

Although for one correction vector the improvement of numerical results can often be substantial [19], for our two new methods the benefit of additional correction vectors appears to be questionable. Therefore in this report we will consider only one correction vector. For this type of corrections and $S_{C}=\left[s_{-}\right], Y_{C}=\left[y_{-}\right]$, we can write

$$
\begin{equation*}
\hat{s}=s-\frac{s^{T} y_{-}}{b_{-}} s_{-}=\left(I-\frac{1}{b_{-}} s_{-} y_{-}^{T}\right) s, \quad \hat{y}=y-\frac{s_{-}^{T} y}{b_{-}} y_{-}=\left(I-\frac{1}{b_{-}} y_{-} s_{-}^{T}\right) y \tag{4.1}
\end{equation*}
$$

which satisfy (conjugacy of $\hat{s}, s_{-}$with respect to $B$ and $B_{+}$for $H_{+} \hat{y}=\hat{s}, H y_{-}=s_{-}$)

$$
\begin{equation*}
\hat{s}^{T} y_{-}=s_{-}^{T} \hat{y}=0 \tag{4.2}
\end{equation*}
$$

These corrections appear to improve efficiency only if $\hat{s}^{T} \hat{y} / b$ is positive and not too small and if $s^{T} y_{-}, s_{-}^{T} y$ are not too different.

If we replace $s, y$ by $\hat{s}, \hat{y}$ and construct some new update with the corrections (4.1), we get $H_{+}\left[y_{-}, \hat{y}\right]=\left[s_{-}, \hat{s}\right]$ by $\hat{s}^{T} y_{-}=\hat{y}^{T} s_{-}=0$ and Theorem 3.1 for the first update or by Lemma 3.4 for the second update, both with $S_{(2)}=\left[s_{-}, \hat{s}\right]$ and $Y_{(2)}=\left[y_{-}, \hat{y}\right]$. Thus

$$
\begin{equation*}
H_{+} y-s=H_{+} \hat{y}+\left(s_{-}^{T} y / b_{-}\right) s_{-}-s=\left(\left(s_{-}^{T} y-s^{T} y_{-}\right) / b_{-}\right) s_{-} . \tag{4.3}
\end{equation*}
$$

To satisfy the QN equations with both the corrected and uncorrected (original) latest difference vectors, in Section 4.2 we combine these corrections with the following type of corrections, which use subsequent difference vectors to correct previous columns of $S, Y$.

### 4.1 Corrections for conjugacy which use subsequent difference vectors

Corrections $S \rightarrow S T_{S}, Y \rightarrow Y T_{Y}$, where $T_{S}, T_{Y} \in \mathcal{R}^{m \times m}$ are lower triangular matrices with the same main diagonals, represent another type of corrections for conjugacy. They are implicit for our second new update, which is invariant under these transformations, see Lemma 3.5. If the factorization $A=U L$ exists with $U, L$ satisfying the assumptions of Lemma 3.3, we can e.g. simply get $\left(S T_{S}\right)^{T}\left(Y T_{Y}\right)=I$ setting $T_{S}=U^{-T}, T_{Y}=L^{-1}$.

To correct $s_{-}, y_{-}$in this way, we can use the following formulas

$$
\begin{equation*}
\tilde{s}_{-}=\left(I-(1 / b) s y^{T}\right) s_{-}, \quad \tilde{y}_{-}=\left(I-(1 / b) y s^{T}\right) y_{-}, \tag{4.4}
\end{equation*}
$$

analogous to (4.1), for $\tilde{s}_{-}^{T} \tilde{y}_{-}$positive and not too small (note that $\tilde{s}_{-}^{T} \tilde{y}_{-}=\hat{s}^{T} \hat{y}\left(b_{-} / b\right)$ ). These vectors satisfy

$$
\begin{equation*}
\tilde{s}_{-}^{T} y=s^{T} \tilde{y}_{-}=0 \tag{4.5}
\end{equation*}
$$

The second new update can also be understood as a precorrected BNS update:
Lemma 4.1. Let the assumptions of Lemma 3.3 be satisfied and $T_{Y}=L^{-1} D_{U}$, where $D_{U}$ is a diagonal matrix with the same main diagonal as $U$ (or $L$ ). Then the BFGS update (2.3) with $Y T_{Y}$ instead of $Y$ represents the update (3.2) with $X=U^{-T} U^{-1}$.

Proof. Let $A=U L$, where $U, L$ have the same main diagonals, and $\tilde{Y}=Y T_{Y}$. Then $S^{T} \tilde{Y}=A\left(L^{-1} D_{U}\right)=U D_{U}$, which is an upper triangular matrix with the main diagonal $D_{U}^{2}$. Substituting $U D_{U}, \tilde{Y}=Y L^{-1} D_{U}, D_{U}^{2}$ for $R, Y, D$ into (2.3), we get (3.2) with $X=U^{-T} U^{-1}$ by

$$
\left(U D_{U}\right)^{-T} D_{U}^{2}\left(U D_{U}\right)^{-1}=U^{-T} U^{-1}, \quad\left(U D_{U}\right)^{-T}\left(Y L^{-1} D_{U}\right)^{T}=A^{-T} Y^{T}
$$

### 4.2 Corrections which use previous and subsequent difference vectors

In this section we show, how the both of the QN equations $H_{+} \hat{y}=\hat{s}$ and $H_{+} y=s$ can be satisfied, using modified corrections (4.1) and (4.4). In this connection, from now on the vectors $\hat{s}, \hat{y}, \tilde{s}_{-}, \tilde{y}_{-}$will have a different meaning than up to now.

For the new type of corrections, which correct both $s, y \rightarrow \hat{s}, \hat{y}$ and $s_{-}, y_{-} \rightarrow \tilde{s}_{-}, \tilde{y}_{-}$, we replace the (conjugacy) properties (4.2), (4.5)) by $\hat{s}^{T} \tilde{y}_{-}=\tilde{s}_{-}^{T} \hat{y}=0$. Considering advantageous properties of the corrections (4.1), see [19, 20], we want to preserve them partially. By analogy with the BFGS update (see comments after Lemma 2.2 in [19]), we consider the satisfaction of $\hat{s}^{T} y_{-}=0$ for the new corrected vectors $\hat{s}, \hat{y}$ to be more important than $s_{-}^{T} \hat{y}=0$, which is confirmed by our numerical experiments.

Therefore we will not correct $y_{-}$, define $\hat{s}, \hat{y}$ by modified (4.1) with $s_{-}$replaced by a corrected vector $\tilde{s}_{-}$and simultaneously define $\tilde{s}_{-}$by (4.4) with $s, y$ replaced by $\hat{s}, \hat{y}$ :

$$
\begin{equation*}
\hat{s}=\left(I-\left(1 / \tilde{s}_{-}^{T} y_{-}\right) \tilde{s}_{-} y_{-}^{T}\right) s, \quad \hat{y}=\left(I-\left(1 / \tilde{s}_{-}^{T} y_{-}\right) y_{-} \tilde{s}_{-}^{T}\right) y, \quad \tilde{s}_{-}=\left(I-\left(1 / \hat{s}^{T} \hat{y}\right) \hat{s} \hat{y}^{T}\right) s_{-}, \tag{4.6}
\end{equation*}
$$

where $\tilde{s}_{-}^{T} y_{-}=b_{-}$by $\hat{s}^{T} y_{-}=0$. Obviously, these relations do not determine $\hat{s}, \hat{y}, \tilde{s}_{-}$ uniquely. Similarly as (4.3) we can obtain $H_{+} y-s=\left(\left(\tilde{s}_{-}^{T} y-s^{T} y_{-}\right) / b_{-}\right) \tilde{s}_{-}$. Thus to have $H_{+} y=s$, the transformation $s_{-} \rightarrow \tilde{s}_{-}$should satisfy the symmetry property $\tilde{s}_{-}^{T} y=s^{T} y_{-}$ (it can be proved that then $\hat{s}, \hat{y}, \tilde{s}_{-}$, if well defined, are determined by (4.6) uniquely).

The following theorem gives all corrected quantities, describes their properties and shows that all relations (4.6) are satisfied. Note that there is no need to compute $\tilde{s}_{-}$ explicitly; for the true handling of corrected quantities, see Section 5.

Theorem 4.1. Let $\alpha=s^{T} y_{-} / b_{-}, \bar{b}=b-\alpha s_{-}^{T} y \neq 0, \hat{s}=\left(s-\alpha s_{-}\right)\left(b-\alpha^{2} b_{-}\right) / \bar{b}, \hat{y}=$ $y-\alpha y_{-}, \hat{b}=\hat{s}^{T} \hat{y} \neq 0, \tilde{s}_{-}=P^{T} s_{-}, P=I-\left(1 / \hat{s}^{T} \hat{y}\right) \hat{y} \hat{s}^{T}, \gamma=s_{-}^{T} y-s^{T} y_{-}$. Then
(a) $\hat{s}^{T} y_{-}=\tilde{s}_{-}^{T} \hat{y}=0, b-\alpha^{2} b_{-}=\hat{b}, \tilde{s}_{-}^{T} y_{-}=b_{-}, \tilde{s}_{-}^{T} y=s^{T} y_{-}$,
(b) $\hat{y}=\left(I-\left(1 / b_{-}\right) y_{-} \tilde{S}_{-}^{T}\right) y, \hat{s}=s-\alpha \tilde{s}_{-}=\left(I-\left(1 / b_{-}\right) \tilde{s}_{-} y_{-}^{T}\right) s$.

Moreover, let $H_{+}$be the first or second new update of the form (3.2) with $A$ nonsingular and $s, y, s_{-}$replaced by $\hat{s}, \hat{y}, \tilde{s}_{-}$. If $X$ and $H_{+}$are symmetric positive definite, then
(c) $H_{+} \hat{y}=\hat{s}, H_{+} y_{-}=\tilde{s}_{-}, \hat{b}>0, H_{+} y=s$.

Proof. (a) We get $\hat{s}^{T} y_{-}=0$ in view of $\alpha=s^{T} y_{-} / b_{-}$, further $\tilde{s}_{-}^{T} \hat{y}=s_{-}^{T} P \hat{y}=0$ and

$$
\begin{equation*}
\hat{b}=\hat{s}^{T} \hat{y}=\hat{s}^{T}\left(y-\alpha y_{-}\right)=\hat{s}^{T} y=\left(b-\alpha s_{-}^{T} y\right)\left(b-\alpha^{2} b_{-}\right) / \bar{b}=b-\alpha^{2} b_{-} . \tag{4.7}
\end{equation*}
$$

Using again $\hat{s}^{T} y_{-}=0$, we obtain $\tilde{s}_{-}^{T} y_{-}=s_{-}^{T} P y_{-}=s_{-}^{T} y_{-}=b_{-}$, which yields

$$
\begin{equation*}
s_{-}-\tilde{s}_{-}=s_{-}-P^{T} s_{-}=s_{-}-\left(I-(1 / \hat{b}) \hat{s} \hat{y}^{T}\right) s_{-}=\left(s_{-}^{T} \hat{y} / \hat{b}\right) \hat{s}=(\gamma / \hat{b}) \hat{s} \tag{4.8}
\end{equation*}
$$

In view of $\hat{s}^{T} y=\hat{b}$, see (4.7), from (4.8) we get $\tilde{s}_{-}^{T} y=s_{-}^{T} y-\gamma=s^{T} y_{-}=\alpha b_{-}$.
(b) The last relation implies $\left(I-\left(1 / b_{-}\right) y_{-} \tilde{s}_{-}^{T}\right) y=y-\alpha y_{-}=\hat{y}$. In view of (4.8) the definition of $\hat{s}$ yields

$$
\begin{equation*}
s-\alpha s_{-}=\frac{\bar{b}}{\hat{b}} \hat{s}=\frac{b-\alpha^{2} b_{-}-\alpha\left(s_{-}^{T} y-\alpha b_{-}\right)}{\hat{b}} \hat{s}=\hat{s}-\alpha \frac{\gamma}{\hat{b}} \hat{s}=\hat{s}-\alpha\left(s_{-}-\tilde{s}_{-}\right), \tag{4.9}
\end{equation*}
$$

i.e. $s=\hat{s}+\alpha \tilde{s}_{-}$, therefore $\hat{s}=s-\alpha \tilde{s}_{-}=s-\left(s^{T} y_{-} / b_{-}\right) \tilde{s}_{-}=\left(I-\left(1 / b_{-}\right) \tilde{s}_{-} y_{-}^{T}\right) s$.
(c) From $\hat{s}^{T} y_{-}=\tilde{s}_{-}^{T} \hat{y}=0$ and $\tilde{s}_{-}^{T} y_{-}=b_{-}$, see (a), we get $H_{+}\left[y_{-}, \hat{y}\right]=\left[\tilde{s}_{-}, \hat{s}\right]$ by Theorem 3.1 for the first update or by Lemma 3.4 for the second update, both with $S_{(2)}=\left[\tilde{s}_{-}, \hat{s}\right], Y_{(2)}=\left[y_{-}, \hat{y}\right]$. This yields $\hat{b}=\hat{s}^{T} \hat{y}=\hat{y}^{T} H_{+} \hat{y}>0$ by assumption and further

$$
s-\alpha \tilde{s}_{-}=\hat{s}=H_{+} \hat{y}=H_{+}\left(y-\alpha y_{-}\right)=H_{+} y-\alpha \tilde{s}_{-}
$$

by (b), i.e. $H_{+} y=s$.

## 5 Implementation

In this section we assume that $H^{I}=\zeta I, \zeta=s^{T} y / y^{T} y>0$, and implement two new VM methods, applying the two new updates (Sections 3.1, 3.2) to difference vectors with possible corrections described in Theorem 4.1. To improve efficiency, the corrections $s \rightarrow \hat{s}, y \rightarrow \hat{y}$ are performed before updating and $\hat{s}, \hat{y}$ are stored and used instead of $s, y$. To indicate that all columns of $S, Y$ except for the latest (i.e. $s, y$ ) can be possibly corrected in previous iterations, we will write $\hat{s}_{i}, \hat{y}_{i}$ (and subsequently $\hat{b}_{i}$ ), $i<k$. To unify notation, after possible corrections (see Step 4 of Algorithm 5.1) we will often write $\hat{s}_{i}, \hat{y}_{i}, \hat{b}_{i}$ instead of $s_{i}, y_{i}, b_{i}$ also for $i=k$. Nevertheless, if we want to stress that we mean the original, uncorrected vectors, we will write $\stackrel{\circ}{i}_{i}, \grave{y}_{i}, \circ_{i}$ for any $i$.

The vector $\tilde{s}_{-}$is used only with the first new update and only together with corrections of $s, y$ and need not be computed. It suffices to consider the matrix $S T \triangleq \tilde{S}$ instead of $S$ (and subsequently $\tilde{A}=\tilde{S}^{T} Y=T^{T} A, \tilde{C}, \tilde{X}$ instead of $A, C, X$ ), where $T$ is the (loworder) identity matrix, except for $T_{m, m-1}=-\gamma / \hat{b}$ in view of $\tilde{s}_{-}=s_{-}(\gamma / \hat{b}) \hat{s}$, see (4.8); obviously $\operatorname{det} T=1$.

The matrix $H_{+}$modified in this way can be written in the following form

$$
\begin{align*}
H_{+} & =\tilde{S} \tilde{X} \tilde{S}^{T}+\zeta\left(I-\tilde{S} \tilde{A}^{-T} Y^{T}\right)\left(I-Y \tilde{A}^{-1} \tilde{S}^{T}\right) \\
& =S \bar{X} S^{T}+\zeta\left(I-S A^{-T} Y^{T}\right)\left(I-Y A^{-1} S^{T}\right), \quad \bar{X}=T \tilde{X} T^{T} \tag{5.1}
\end{align*}
$$

i.e. in the form (3.2) with $H^{I}=\zeta I$ and $X$ replaced by $\bar{X}$. Setting $T=I$ and $\bar{X}=X$ in case that we do not correct $\hat{s}_{-} \rightarrow \tilde{s}_{-}$, we will use the matrices $\tilde{S}, \tilde{A}, \tilde{C}, \tilde{X}$ instead of $S, A, C, X$ and the update (5.1) instead of (3.2) for the both new updates. Note that the matrix $\tilde{A}_{22}$ is diagonal by Theorem 4.1(a).

In view of (5.1), the direction vector $-H_{+} g_{+}$and an auxiliary vector $Y^{T} H_{+} g_{+}$(see comments to Procedure 5.2 below) can be calculated efficiently by

$$
\begin{align*}
-H_{+} g_{+} & =-\zeta g_{+}-S\left[\left(\bar{X}+\zeta A^{-T} Y^{T} Y A^{-1}\right) S^{T} g_{+}-\zeta A^{-T} Y^{T} g_{+}\right]+Y\left[\zeta A^{-1} S^{T} g_{+}\right]  \tag{5.2}\\
Y^{T} H_{+} g_{+} & =\zeta Y^{T} g_{+}+Y^{T} S\left[\left(\bar{X}+\zeta A^{-T} Y^{T} Y A^{-1}\right) S^{T} g_{+}-\zeta A^{-T} Y^{T} g_{+}\right]-Y^{T} Y\left[\zeta A^{-1} S^{T} g_{+}\right] \tag{5.3}
\end{align*}
$$

For the second new update with $\bar{X}=X=U^{-T} U^{-1}$, we use (5.2) and (5.3) in the form

$$
\begin{align*}
-H_{+} g_{+} & =-\zeta g_{+}-S\left[U^{-T}\left(\left(I+\zeta L^{-T} Y^{T} Y L^{-1}\right) q-\zeta L^{-T} Y^{T} g_{+}\right)\right]+Y\left[\zeta L^{-1} q\right]  \tag{5.4}\\
Y^{T} H_{+} g_{+} & =\zeta Y^{T} g_{+}+Y^{T} S\left[U^{-T}\left(\left(I+\zeta L^{-T} Y^{T} Y L^{-1}\right) q-\zeta L^{-T} Y^{T} g_{+}\right)\right]-Y^{T} Y\left[\zeta L^{-1} q\right] \tag{5.5}
\end{align*}
$$

where $q=U^{-1} S^{T} g_{+}$.

All corrections described in Section 4 and also both new updates appear to be more efficient than the L-BFGS (or BNS) method only if $A$ is sufficiently close to a symmetric matrix. Denoting $A=\left[a_{i j}\right]_{i, j=1}^{m}$, we use the values

$$
\begin{equation*}
\bar{\Delta}=\sum_{1 \leq i<j \leq m}\left(a_{i j}-a_{j i}\right)^{2} /\left(a_{i i} a_{j j}\right), \quad \bar{\delta}=\left(\hat{s}_{-}^{T} y-s^{T} \hat{y}_{-}\right)^{2} /\left(b \hat{b}_{-}\right) \tag{5.6}
\end{equation*}
$$

as measures of the deviation. Similarly as for the BFGS method with corrected vectors (to have VM matrices symmetric positive definite, see e.g. [17]), we require $\hat{b}>0$ and $\bar{b}>0$ (we can readily verify that $\bar{b}$ corresponds to $\hat{s}^{T} \hat{y}$ for the corrections (4.1)). Besides, as in [22] we can deduce that too small $\bar{b}, \hat{b}$ can deteriorate stability. Thus we do not correct $s, y$ if $\bar{\delta}>\delta_{1}$ or $\bar{b}<\delta_{2} b$ (all $\delta_{i} \in(0,1)$ ); this implies $\hat{b}>0$ by Lemma 6.3 . Due to our proof of global convergence, we also do not correct if $\max \left[\left|\hat{s}_{-}\right| /\left|\grave{s}_{-}\right|,\left|\hat{y}_{-}\right| /\left|\dot{y}_{-}\right|\right]>\theta, \theta>1$.

Due to $\bar{b}=\hat{b}-\alpha \gamma$, see (4.9), i.e. $1-\bar{b} / \hat{b}=\alpha \gamma / \hat{b}$, and $\left|\tilde{s}_{-}-\hat{s}_{-}\right| /|\hat{s}|=|\gamma| / \hat{b}$, see (4.8), in case of the first new update we also do not correct if $(\alpha \gamma / \hat{b})^{2}>\delta_{3}$ or $(\gamma / \hat{b})^{2}>\delta_{4}\left(\tilde{s}_{-}\right.$is too different from $\left.\hat{s}_{-}\right)$. Similarly for the second new update, if $(\alpha \gamma / \hat{b})^{2}>\delta_{5}$.

We use both new updates only for $\bar{\Delta} \leq \delta_{6}$. For the first new update, if all diagonal entries of $\tilde{A}_{22}$ are greater than $\varepsilon_{D} \operatorname{Tr} A, \varepsilon_{D} \in(0,1)$, we calculate $\tilde{C}$, using Theorem 3.1 with $\tilde{S}, \tilde{A}, \tilde{C}$ instead of $S, A, C$ (recall that all $b_{i}>0$ for $g_{i} \neq 0$ by (1.1)). Then, if all diagonal entries of $\tilde{C}$ are greater than $\delta_{7} \operatorname{Tr} A$ and all eigenvalues of $\tilde{C} \tilde{C}^{T}$ divided by $1+\left\|\tilde{A}_{22}^{-1} \tilde{A}_{21}\right\|_{F}^{2}$ (to guarantee global convergence, see Section 6) are greater than or equal to $\varepsilon_{E} \operatorname{Tr} A, \varepsilon_{E} \in(0,1)$, we calculate $\tilde{X}$, using Theorem 3.2 and (3.7) with $\tilde{S}, \tilde{A}, \tilde{C}, \tilde{X}$ instead of $S, A, C, X$. For the second new update, we factorize $A=U L$, where $U, L$ have the same main diagonals, and calculate $X=U^{-T} U^{-1}$, see Lemma 3.3. If some condition is not satisfied or if the factorization fails, we use the BNS method, see Section 2.

For calculation of eigenvalues and eigenvectors of low-order symmetric matrices, the well-known Jacobi iteration method (e.g. [6]) appears to be efficient. It is interesting that the increase in computational time for one iteration compared with the second new update or the standard BNS update is very small for $N$ large (it is independent of $N$ ).

For our proof of global convergence we need $\operatorname{det} \tilde{A} \neq 0$. This is guaranteed by Theorem 1.4.2 in [6] for the first new update, since $\tilde{A}_{22}$ and $\tilde{C}$ are nonsingular, or by Procedure 5.1 for the second update, since we require $L_{i i}^{2} \geq \varepsilon_{F} \max \left[\operatorname{Tr} A, \operatorname{Tr}\left(L^{T} L\right)\right], 1 \leq i \leq m$, $\varepsilon_{F} \in(0,1)$, see below; note that $L^{T} L=A$ for $A$ symmetric.

We first present two auxiliary procedures. Procedure 5.1, based on Lemma 5 in [21], is used for the factorization $A=U L$, Procedure 5.2 serves for updating of the basic matrices $S, Y, S^{T} Y=A, Y^{T} Y$; the submatrices of $S, Y$ with columns from previous iterations are denoted by $S_{P}, Y_{P}$ (i.e. $S=\left[S_{P}, \hat{s}\right], Y=\left[Y_{P}, \hat{y}\right]$ ). In comparison with the corresponding algorithm in [3], which uses only the main diagonal of $A$ and the part of $A$ above the diagonal, we need all entries of $A$ here. Thus we use an additional vector $Y_{P}^{T} s=-t Y_{P}^{T} H g$ (see Algorithm 5.1) to have the number of arithmetic operations approximately the same.

Procedure 5.1 ( $U L$ factorization of $A$ )
Given: A global convergence parameter $\varepsilon_{F} \in(0,1)$ and the $m \times m$ matrix $A$.
(i): Set $Q:=A, \nu:=m$ and $\kappa:=Q_{m m}$.
(ii): If $Q_{\nu \nu}<\varepsilon_{F} \operatorname{Tr} A$ then the factorization fails and return.
(iii): Set $Q_{i j}:=Q_{i j}-Q_{i \nu} Q_{\nu j} / Q_{\nu \nu}, i=1, \ldots, \nu-1, j=1, \ldots, \nu-1$. Set $\nu:=\nu-1$ and
then $\kappa:=\min \left[\kappa, Q_{\nu \nu}\right]$. If $\nu>1$ go to (ii).
(iv): Set $L_{i j}:=Q_{i j} / \sqrt{Q_{i i}}, U_{j i}:=Q_{j i} / \sqrt{Q_{i i}}$ for $1 \leq j \leq i \leq m$ and $L_{i j}:=U_{j i}:=0$ for $1 \leq i<j \leq m$. If $\kappa<\varepsilon_{F}\|L\|_{F}^{2}$ then the factorization fails. Return.

## Procedure 5.2 (Updating of basic matrices)

Given: $t, \hat{b} / \bar{b}, \alpha \in \mathcal{R}$, matrices $S_{P}, Y_{P}, S_{P}^{T} Y_{P}, Y_{P}^{T} Y_{P}$ and vectors $\hat{s}, \hat{y}, g_{+}, S_{P}^{T} g, Y_{P}^{T} g, Y_{P}^{T} H g$.
(i): Compute $S_{P}^{T} g_{+}, Y_{P}^{T} g_{+}, \hat{s}^{T} g_{+}, \hat{y}^{T} g_{+}, \hat{s}^{T} \hat{y}, \hat{y}^{T} \hat{y}$.
(ii): Compute $S_{P}^{T} y:=S_{P}^{T} g_{+}-S_{P}^{T} g, Y_{P}^{T} y:=Y_{P}^{T} g_{+}-Y_{P}^{T} g, Y_{P}^{T} s:=-t Y_{P}^{T} H g$.
(iii): Compute $S_{P}^{T} \hat{y}:=S_{P}^{T} y-\alpha S_{P}^{T} y_{-}, Y_{P}^{T} \hat{s}:=\left(Y_{P}^{T} s-\alpha Y_{P}^{T} s_{-}\right) \hat{b} / \bar{b}, Y_{P}^{T} \hat{y}:=Y_{P}^{T} y-\alpha Y_{P}^{T} y_{-}$.
(iv): Set $S:=\left[S_{P}, \hat{s}\right], Y:=\left[Y_{P}, \hat{y}\right], S^{T} g_{+}:=\left[S_{P}^{T} g_{+}, \hat{s}^{T} g_{+}\right], Y^{T} g_{+}:=\left[Y_{P}^{T} g_{+}, \hat{y}^{T} g_{+}\right]$.
(v): Set $A=S^{T} Y:=\left[\begin{array}{cc}S_{P}^{T} Y_{P} & S_{P}^{T} \hat{y} \\ \hat{s}^{T} Y_{P} & \hat{s}^{T} \hat{y}\end{array}\right], Y^{T} Y:=\left[\begin{array}{cc}Y_{P}^{T} Y_{P} & Y_{P}^{T} \hat{y} \\ \hat{y}^{T} Y_{P} & \hat{y}^{T} \hat{y}\end{array}\right]$ and return.

We now state the method in details. For simplicity, we do not describe stopping criteria and contingent restarts when some computed direction vector is not a sufficiently descent direction. Note that the order $\mu$ of $A_{22}$ is also used as a correction indicator and that the contingent restarts have occurred very rarely in our numerical experiments.

## Algorithm 5.1

Data: A maximum number $\hat{m}$ of columns $S, Y$, line search parameters $\varepsilon_{1}, \varepsilon_{2}, 0<\varepsilon_{1}<1 / 2$, $\varepsilon_{1}<\varepsilon_{2}<1$, tolerance parameters $\delta_{i} \in(0,1), i \in\{1, \ldots, 7\}$, a global convergence parameters $\theta>1, \varepsilon_{D}, \varepsilon_{E}, \varepsilon_{F} \in(0,1)$ and a chosen method number $n_{M} \in\{1,2\}$.
Step 1: Initiation. Choose starting point $x_{0} \in \mathcal{R}^{N}$, define the starting matrix $H_{0}:=I$ and the direction vector $d_{0}:=-g_{0}$ and initiate the iteration counter $k$ to zero.
Step 2: Line search. Set the update indicator $i_{U}$ to zero. Compute $x_{+}:=x+t d$, where $t$ satisfies (1.1), $g_{+}:=\nabla f\left(x_{+}\right), s:=t d, y:=g_{+}-g, b:=s^{T} y$ and $\zeta:=b / y^{T} y$. Set $\tilde{m}:=\min [k, \hat{m}-1], m:=\tilde{m}+1$ and define $H^{I}:=\zeta I$. If $k=0$ set $S:=[s], Y:=[y]$, $S^{T} Y:=\left[s^{T} y\right], Y^{T} Y:=\left[y^{T} y\right], X=[1 / b]$, compute $S^{T} g_{+}, Y^{T} g_{+}$and go to Step 9.
Step 3: Correction preparation. Set $\mu:=1$. If $m>1$ compute $\hat{b}, \bar{b}, \alpha, \gamma$ by Theorem 4.1 and $\bar{\delta}, \bar{\Delta}$ by (5.6). If $\bar{\delta}<\delta_{1}, \hat{b}>0, \bar{b}>\delta_{2} b$ and $\max \left[\left|\hat{s}_{-}\right| /\left|\stackrel{s}{-}_{-}\right|,\left|\hat{y}_{-}\right| /\left|\mathcal{y}_{-}\right|\right] \leq \theta$ set $\mu:=2$. If $n_{M}=1$ and $\left((\alpha \gamma / \hat{b})^{2}>\delta_{3}\right.$ or $\left.(\gamma / \hat{b})^{2}>\delta_{4}\right)$ or if $n_{M}=2$ and $(\alpha \gamma / \hat{b})^{2}>\delta_{5}$ set $\mu:=1$.
Step 4: Correction. If $\mu=1$ set $\hat{s}:=s, \hat{y}:=y$, otherwise compute $\hat{s}, \hat{y}$ by Theorem 4.1.
Step 5: Basic matrices updating. Using Procedure 5.2, form the matrices $S, Y, A, Y^{T} Y$.
Step 6: Update selection. If $\bar{\Delta}>\delta_{6}$ go to Step 9. Set $T:=I$. If $n_{M}=2$ go to Step 8. If $\mu=2$, form the matrix $T$ according to the second paragraph of Section 5 and define $\tilde{S}:=S T$ and $\tilde{A}:=T^{T} A$, otherwise define $\tilde{S}=S$ and $\tilde{A}=A$.
Step 7: VM update 1. Set $i_{U}:=1$. Use Theorems 3.2 and 3.1 with $\tilde{S}, \tilde{A}, \tilde{C}, \tilde{X}$ instead of $S, A, C, X$ to compute $\tilde{C}$, $\tilde{X}$, if all diagonal entries of $\tilde{A}_{22}$ are greater then $\varepsilon_{D} \operatorname{Tr} A$ and if all diagonal entries of $\tilde{C}$ are greater then $\delta_{7} \operatorname{Tr} A$ and if all eigenvalues of $\left(\tilde{C} \tilde{C}^{T}\right)^{1 / 2}$ divided by $1+\left\|\tilde{A}_{22}^{-1} \tilde{A}_{21}\right\|_{F}^{2}$ are greater than or equal to $\varepsilon_{E} \operatorname{Tr} A$, otherwise set $i_{U}:=0$. Go to Step 9 .
Step 8: VM update 2. Use Procedure 5.1 to factorize $A:=U L$. If the factorization fails, go to Step 9. Set $i_{U}:=2$ and compute $X:=U^{-T} U^{-1}$.

Step 9: Direction vector. Define $H_{+}$by (2.3) for $i_{U}=0$ or by (5.1) otherwise. Compute $d_{+}=-H_{+} g_{+}$and an auxiliary vector $Y^{T} H_{+} g_{+}$by (2.4)-(2.5) for $i_{U}=0$ or by (5.2)-(5.3) with $\bar{X}=T \tilde{X} T^{T}$ for $i_{U}=1$ or by (5.4)-(5.5) for $i_{U}=2$. Set $k:=k+1$. If $k \geq \hat{m}$ delete the first column of $S_{-}, Y_{-}$and the first row and column of $S_{-}^{T} Y_{-}$, $Y_{-}^{T} Y_{-}$to form $S_{P}, Y_{P}, S_{P}^{T} Y_{P}, Y_{P}^{T} Y_{P}$. Go to Step 2.

## 6 Global convergence

In this section we establish global convergence of Algorithm 5.1 in convex case and without restarts. Assumption 6.1 and Lemma 6.1 are presented in [19], Lemma 6.2 in [21].

Note that a suitable restarts technique can guarantee global convergence also for nonconvex $f$, see e.g. Algorithm 6.1 and comments in the beginning of Section 7 in [22]. Besides, there are some other possibilities how to establish global convergence of VM methods for non-convex $f$, see e.g. [9, 23].
Assumption 6.1. The objective function $f: \mathcal{R}^{N} \rightarrow \mathcal{R}$ is bounded from below and uniformly convex with bounded second-order derivatives (i.e. $0<\underline{G} \leq \underline{\lambda}(G(x)) \leq \bar{\lambda}(G(x)) \leq$ $\bar{G}<\infty, x \in \mathcal{R}^{N}$, where $\underline{\lambda}(G(x))$ and $\bar{\lambda}(G(x))$ are the lowest and the greatest eigenvalues of the Hessian matrix $G(x))$.
Lemma 6.1. Let the objective function $f$ satisfy Assumption 6.1. Then $\underline{G} \leq|ْ|^{2} / \grave{b} \leq \bar{G}$ and $\grave{b} /|\grave{s}|^{2} \geq \underline{G}(\stackrel{\circ}{\delta}, \dot{y}$ are original, uncorrected difference vectors, see Section 5).
Lemma 6.2. Let $K_{1}, K_{2} \in \mathcal{R}^{\nu \times \nu}, \nu>0$, be symmetric positive semidefinite matrices. Then $0 \leq \operatorname{Tr}\left(K_{1} K_{2}\right) \leq \operatorname{Tr} K_{1} \operatorname{Tr} K_{2}$. Moreover, if $K_{2}$ is symmetric positive definite, then $\operatorname{Tr}\left(K_{1} K_{2}^{-1}\right) \leq \operatorname{Tr} K_{1}\left(\operatorname{Tr} K_{2}\right)^{\nu-1} / \operatorname{det} K_{2}$.
Lemma 6.3. Let the assumptions of Theorem 3.1 be satisfied with $X_{11}=\left(C C^{T}\right)^{-1 / 2}$ and $A_{22} \in \mathcal{S}_{\mu}$ (symmetric positive definite of order $\mu$, see Section 3). Then $A^{T} X A \in \mathcal{S}_{m}$ and

$$
\begin{aligned}
\operatorname{Tr}\left(A^{T} X A\right)^{-1} & \leq\left(1+\left\|A_{22}^{-1} A_{21}\right\|_{F}^{2}\right) \operatorname{Tr} X_{11}+\operatorname{Tr} A_{22}^{-1}, \\
\operatorname{det} X^{-1} & =\operatorname{det} X_{11}^{-1} \operatorname{det} A_{22} .
\end{aligned}
$$

Proof. We have $X_{11}^{-1} \sim C^{T} X_{11} C \triangleq M$ by Theorem 3.2. Further, (3.11) and (3.12) imply

$$
A^{T} X A=\left[\begin{array}{cc}
M+A_{21}^{T} A_{22}^{-1} A_{21} & A_{21}^{T} \\
A_{21} & A_{22}
\end{array}\right] \in \mathcal{S}_{m}
$$

since the Schur complement of $A_{22}$ in $A^{T} X A$ is $M \in \mathcal{S}_{m-\mu}$. It can easily be verified that

$$
\left(A^{T} X A\right)^{-1}=\left[\begin{array}{cc}
M^{-1} & -M^{-1} A_{21}^{T} A_{22}^{-1} \\
-A_{22}^{-1} A_{21} M^{-1} & A_{22}^{-1}+A_{22}^{-1} A_{21} M^{-1} A_{21}^{T} A_{22}^{-1}
\end{array}\right] .
$$

The trace of a product of two matrices is independent of the order of multiplication (if the both products are defined), which gives $\operatorname{Tr}\left(A^{T} X A\right)^{-1}=\operatorname{Tr} M^{-1}+\operatorname{Tr}\left(A_{21}^{T} A_{22}^{-2} A_{21} M^{-1}\right)+$ $\operatorname{Tr} A_{22}^{-1}$. Using Lemma 6.2, we obtain

$$
\begin{aligned}
\operatorname{Tr}\left(A^{T} X A\right)^{-1} & \leq\left(1+\operatorname{Tr}\left(A_{21}^{T} A_{22}^{-2} A_{21}\right)\right) \operatorname{Tr} M^{-1}+\operatorname{Tr} A_{22}^{-1} \\
& =\left(1+\left\|A_{22}^{-1} A_{21}\right\|_{F}^{2}\right) \operatorname{Tr} X_{11}+\operatorname{Tr} A_{22}^{-1}
\end{aligned}
$$

and the rest follows from (3.9) by Theorem 1.4.2 in [6].

Lemma 6.4. Let $\hat{b}, \bar{b}, \alpha, \gamma$ are given by Theorem 4.1, $\delta, \tilde{\delta} \in(0,1)$ and suppose that $\bar{b}>\delta b$ and $|\alpha \gamma| \leq \tilde{\delta} \hat{b}$. Then $\hat{b} / \bar{b} \leq 1 /(1-\tilde{\delta})$ and $\hat{b}>(\delta / 2) b$.
Proof. In the same way as in (4.9) we get $\bar{b}=\hat{b}-\alpha \gamma$, therefore

$$
\hat{b}(1-\tilde{\delta}) \leq|\hat{b}|-\hat{b} \tilde{\delta} \leq|\hat{b}|-|\alpha \gamma| \leq|\hat{b}-\alpha \gamma|=\bar{b}, \quad \bar{b} \leq|\hat{b}|+|\alpha \gamma|<2 \hat{b},
$$

which yields $\hat{b} / \bar{b} \leq 1 /(1-\tilde{\delta})$ and $\hat{b}>\bar{b} / 2>(\delta / 2) b$.
Lemma 6.5. Let objective function $f$ satisfy Assumption 6.1. Then Algorithm 5.1 guarantees that the sequences $\left\{\left|\hat{s}_{k}\right|^{2} / \hat{b}_{k}\right\},\left\{\left|\hat{y}_{k}\right|^{2} / \hat{b}_{k}\right\}$ (corrected or uncorrected) are always bounded.
Proof. Since $\mid \stackrel{s}{\left.\right|^{2}} / \bar{b} \leq 1 / \underline{G}$ and $|\grave{y}|^{2} / \hbar \leq \bar{G}$ by Lemma 6.1, the assertion holds for $\hat{s}_{k}, \hat{y}_{k}, \hat{b}_{k}$ without corrections $\left(\mu_{k}=1\right)$. Let $\mu_{k}=2$. The safeguarding technique in Step 3 of Algorithm 5.1 guarantees

$$
\begin{equation*}
\hat{b}>0, \quad \bar{b}>\delta_{2} b, \quad\left|\hat{s}_{-}\right| \leq \theta\left|\grave{s}_{-}\right|, \quad\left|\hat{y}_{-}\right| \leq \theta\left|\grave{y}_{-}\right|, \quad|\alpha \gamma| \leq \tilde{\delta} \hat{b}, \tilde{\delta}=\sqrt{\max \left[\delta_{3}, \delta_{5}\right]} . \tag{6.1}
\end{equation*}
$$

Setting $\delta=\delta_{2}$ and using Lemma 6.4 , we have $\hat{b} / \bar{b} \leq 1 /(1-\tilde{\delta})$ and $\hat{b} / b>\delta / 2$. Similarly we obtain $\hat{b}_{-} / \hat{b}_{-}>\delta / 2$ for $\mu_{-}=2$, which is also true for $\mu_{-}=1$ by $\hat{b}_{-} / \hat{b}_{-}=1>\delta / 2$. Thus

$$
\begin{equation*}
\hat{b} / \bar{b} \leq 1 /(1-\tilde{\delta}), \quad \hat{b} / b>\delta / 2, \quad \hat{b}_{-} / \stackrel{\circ}{b}_{-}>\delta / 2 \tag{6.2}
\end{equation*}
$$

holds for any $\mu_{-}$. Further we get $\alpha^{2}<b / \hat{b}_{-}$by $\hat{b}=b-\alpha^{2} \hat{b}_{-}>0$, which yields

$$
\begin{aligned}
& \left|s-\alpha \hat{s}_{-}\right| \leq|s|+\sqrt{b / \hat{b}_{-}}\left|\hat{s}_{-}\right| \leq|s|+\sqrt{2 b / \delta} \theta\left|\stackrel{s}{s}_{-}\right| / \sqrt{\dot{b}_{-}} \leq \sqrt{b / \underline{G}}(1+\sqrt{2 / \delta} \theta), \\
& \left|y-\alpha \hat{y}_{-}\right| \leq|y|+\sqrt{b / \hat{b}_{-}}\left|\hat{y}_{-}\right| \leq|y|+\sqrt{2 b / \delta} \theta\left|\grave{y}_{-}\right| / \sqrt{\dot{b}_{-}} \leq \sqrt{b \bar{G}}(1+\sqrt{2 / \delta} \theta),
\end{aligned}
$$

by Lemma 6.1, $\hat{b}_{-}>(\delta / 2) \stackrel{\circ}{b}_{-},\left|\hat{s}_{-}\right| \leq \theta\left|\circ_{-}\right|$and $\left|\hat{y}_{-}\right| \leq \theta\left|\dot{y}_{-}\right|$, see (6.1)-(6.2). This implies

$$
\begin{aligned}
\frac{\left|\hat{s}_{k}\right|^{2}}{\hat{b}_{k}} & =\frac{\left|s_{k}-\alpha_{k} \hat{s}_{k-1}\right|^{2}\left(\hat{b}_{k} / \bar{b}_{k}\right)^{2}}{\hat{b}_{k}} \leq \frac{b_{k}(1+\sqrt{2 / \delta} \theta)^{2}}{\underline{G} \hat{b}_{k}(1-\tilde{\delta})^{2}} \leq \frac{2(1+\sqrt{2 / \delta} \theta)^{2}}{\delta \underline{G}(1-\tilde{\delta})^{2}} \\
\frac{\left|\hat{y}_{k}\right|^{2}}{\hat{b}_{k}} & =\frac{\left|y_{k}-\alpha_{k} \hat{y}_{k-1}\right|^{2}}{\hat{b}_{k}} \leq \frac{b_{k} \bar{G}}{\hat{b}_{k}}(1+\sqrt{2 / \delta} \theta)^{2} \leq \frac{2 \bar{G}}{\delta}(1+\sqrt{2 / \delta} \theta)^{2}
\end{aligned}
$$

by Theorem 4.1 and (6.2).
Theorem 6.1. Let objective function $f$ satisfy Assumption 6.1. Then Algorithm 5.1 generates a sequence $\left\{g_{k}\right\}$ that either satisfies $\lim _{k \rightarrow \infty}\left|g_{k}\right|=0$ or terminates with $g_{k}=0$ for some $k$.
Proof. Using Lemma 6.5, we can find $\theta_{1}, \theta_{2} \in \mathcal{R}$ satisfying

$$
\begin{align*}
& \left|\hat{s}_{k}\right|^{2} / \hat{b}_{k}<\theta_{1},  \tag{6.3}\\
& \left|\hat{y}_{k}\right|^{2} / \hat{b}_{k}<\theta_{2}
\end{align*}
$$

for all $k \geq 0$, where $\hat{s}_{k}, \hat{y}_{k}$ mean corrected or uncorrected vectors in the whole proof.
As we mentioned in Section 5, in all iterations we choose $H_{k}^{I}=\zeta_{k} I, \zeta_{k}=\stackrel{\circ}{b}_{k} /\left|\grave{y}_{k}\right|^{2}$, see Step 2. Denoting $B_{k}^{I}=\left(H_{k}^{I}\right)^{-1}$, Lemma 6.1 gives

$$
\begin{equation*}
\operatorname{Tr} B_{k}^{I}=\left(\left|\check{y}_{k}\right|^{2} / \circ_{k}\right) \operatorname{Tr} I \leq N \bar{G}, \quad \operatorname{det} B_{k}^{I}=\left(\left|\dot{y}_{k}\right|^{2} / \circ_{k}\right)^{N} \geq \underline{G}^{N}, \quad k \geq 0 . \tag{6.5}
\end{equation*}
$$

(i) Suppose that $i_{U}=0$, i.e. the BNS update (2.3) of $H_{k}^{I}$ is used and columns of $S, Y$ are $\hat{s}_{i}, \hat{y}_{i}$. This is equivalent to the recurrent application of the BFGS update (2.1) to $H_{k}^{I}$

$$
\begin{equation*}
H_{i+1}^{k+1}=\frac{1}{\hat{b}_{i}} \hat{s}_{i} \hat{S}_{i}^{T}+\left(I-\frac{1}{\hat{b}_{i}} \hat{s}_{i} \hat{y}_{i}^{T}\right) H_{i}^{k+1}\left(I-\frac{1}{\hat{b}_{i}} \hat{y}_{i} \hat{S}_{i}^{T}\right), \quad i \in\{k-\tilde{m}, \ldots, k\} \triangleq \mathcal{I}_{k}, \tag{6.6}
\end{equation*}
$$

where $H_{k-\tilde{m}}^{k+1}=H_{k}^{I}=\zeta_{k} I, H_{k+1}=H_{k+1}^{k+1}$. These updates satisfy (see [17])

$$
\begin{align*}
\operatorname{Tr} B_{i+1}^{k+1} & =\operatorname{Tr} B_{i}^{k+1}+\left|\hat{y}_{i}\right|^{2} / \hat{b}_{i}-\left|B_{i}^{k+1} \hat{s}_{i}\right|^{2} / \hat{s}_{i}^{T} B_{i}^{k+1} \hat{s}_{i},  \tag{6.7}\\
\operatorname{det} B_{i+1}^{k+1} & =\left(\hat{b}_{i} / \hat{s}_{i}^{T} B_{i}^{k+1} \hat{s}_{i}\right) \operatorname{det} B_{i}^{k+1}, \tag{6.8}
\end{align*}
$$

$i \in \mathcal{I}_{k}$, denoting $B_{i}^{k+1}=\left(H_{i}^{k+1}\right)^{-1}, i=k-\tilde{m}, \ldots, k+1$. This yields

$$
\begin{equation*}
\operatorname{Tr} B_{i}^{k+1} \leq N \bar{G}+m \theta_{2} \triangleq \theta_{3} \tag{6.9}
\end{equation*}
$$

by (6.4)-(6.5). Since $\hat{b}_{i} / \hat{s}_{i}^{T} B_{i}^{k+1} \hat{s}_{i}=\left(\hat{b}_{i} /\left|\hat{s}_{i}\right|^{2}\right)\left(\left|\hat{s}_{i}\right|^{2} / \hat{s}_{i}^{T} B_{i}^{k+1} \hat{s}_{i}\right) \geq 1 /\left(\theta_{1} \theta_{3}\right)$ by (6.3) and (6.9), for all $k>0$ in view of (6.9), (6.8) and (6.5) we get

$$
\begin{align*}
\operatorname{Tr} B_{k+1} & =\operatorname{Tr} B_{k+1}^{k+1} \leq \theta_{3},  \tag{6.10}\\
\operatorname{det} B_{k+1} & =\operatorname{det} B_{k+1}^{k+1} \geq \underline{G}^{N} /\left(\theta_{1} \theta_{3}\right)^{m} \triangleq \theta_{4} . \tag{6.11}
\end{align*}
$$

(ii) Let $i_{U}>0$, i.e. the update (5.1) is used (and columns of $S, Y$ are again $\hat{s}_{i}, \hat{y}_{i}$ ), where for $n_{M}=1$ we use matrices $\tilde{S}=S T, \tilde{A}, \tilde{C}, \tilde{X}$ instead of $S, A, C, X$ with $\operatorname{det} T=1$, see the beginning of Section 5. In view of $B_{k}^{I}=\left(1 / \zeta_{k}\right) I$, (6.5), $\tilde{S}^{T} \tilde{S}=T^{T}\left(S^{T} S\right) T$ and $\operatorname{det} \tilde{S}^{T} \tilde{S} \neq 0$ due to $\operatorname{det} \tilde{A} \neq 0$ (see Section 5), for $k>0$ Theorem 3 in [21] gives

$$
\begin{align*}
B_{k+1} & =\left(1 / \zeta_{k}\right) I-\left(1 / \zeta_{k}\right) \tilde{S}_{k}\left(\tilde{S}_{k}^{T} \tilde{S}_{k}\right)^{-1} \tilde{S}_{k}^{T}+Y_{k} \tilde{A}_{k}^{-1} \tilde{X}_{k}^{-1} \tilde{A}_{k}^{-T} Y_{k}^{T}  \tag{6.12}\\
\operatorname{Tr} B_{k+1} & \leq N \bar{G}+\operatorname{Tr}\left(Y_{k}^{T} Y_{k} \tilde{A}_{k}^{-1} \tilde{X}_{k}^{-1} \tilde{A}_{k}^{-T}\right)=N \bar{G}+\operatorname{Tr}\left(Y_{k}^{T} Y_{k}\left(\tilde{A}_{k}^{T} \tilde{X}_{k} \tilde{A}_{k}\right)^{-1}\right),  \tag{6.13}\\
\operatorname{det} B_{k+1} & =\zeta_{k}^{m-N} \operatorname{det} \tilde{X}_{k}^{-1} / \operatorname{det} \tilde{S}_{k}^{T} \tilde{S}_{k} \geq \underline{G}^{N-m} \operatorname{det} \tilde{X}_{k}^{-1} / \operatorname{det} S_{k}^{T} S_{k} \tag{6.14}
\end{align*}
$$

(ii-a) For $n_{M}=1$, Step 7 of Algorithm 5.1 guarantees $\operatorname{Tr} \tilde{A}_{22}^{-1} \leq \mu /\left(\varepsilon_{D} \operatorname{Tr} A\right.$ ) (recall that $\tilde{A}_{22}$ is diagonal, see Section 5) and $\omega \operatorname{Tr} \tilde{X}_{11} \leq(m-\mu) /\left(\varepsilon_{E} \operatorname{Tr} A\right)$, where $\omega=1+\left\|\tilde{A}_{22}^{-1} \tilde{A}_{21}\right\|_{F}^{2}$. Using Lemma 6.3 with $\tilde{A}, \tilde{X}$ instead of $A, X$, we have $\operatorname{Tr}\left(\tilde{A}^{T} \tilde{X} \tilde{A}\right)^{-1} \leq$ $m /\left(\varepsilon_{M} \operatorname{Tr} A\right)$, where $\varepsilon_{M}=\min \left[\varepsilon_{D}, \varepsilon_{E}\right]$. From (6.13) we get

$$
\begin{align*}
\operatorname{Tr} B_{k+1} & \leq N \bar{G}+\operatorname{Tr}\left(Y_{k}^{T} Y_{k}\right) \operatorname{Tr}\left(\tilde{A}_{k}^{T} \tilde{X}_{k} \tilde{A}_{k}\right)^{-1} \leq N \bar{G}+m \frac{\operatorname{Tr}\left(Y_{k}^{T} Y_{k}\right)}{\varepsilon_{M} \operatorname{Tr} A_{k}} \\
& =N \bar{G}+\frac{m \sum_{i \in \mathcal{I}_{k}}\left|\hat{y}_{i}\right|^{2}}{\varepsilon_{M} \sum_{i \in \mathcal{I}_{k}} \hat{b}_{i}} \leq N \bar{G}+\frac{m}{\varepsilon_{M}} \sum_{i \in \mathcal{I}_{k}} \frac{\left|\hat{y}_{i}\right|^{2}}{\hat{b}_{i}} \leq \theta_{5} \tag{6.15}
\end{align*}
$$

with $\theta_{5}=N \bar{G}+m^{2} \theta_{2} / \varepsilon_{M}>\theta_{3}$ by Lemma 6.2 , (6.4), $m>1$ and $\varepsilon_{M}<1$.
We proceed to estimate $\operatorname{det} B_{k+1}$. Similarly as above, Step 7 of Algorithm 5.1 guarantees $\operatorname{det} \tilde{A}_{22} \geq\left(\varepsilon_{D} \operatorname{Tr} A\right)^{\mu}$ and $\operatorname{det} \tilde{X}_{11}^{-1}>\left(\varepsilon_{E} \operatorname{Tr} A\right)^{m-\mu}$ by $\omega \geq 1$. Using Lemma 6.3 with $\tilde{A}, \tilde{X}$ instead of $A, X$, we have $\operatorname{det} \tilde{X}^{-1}>\left(\min \left[\varepsilon_{D}, \varepsilon_{E}\right] \operatorname{Tr} A\right)^{m}=\left(\varepsilon_{M} \operatorname{Tr} A\right)^{m}$. Using Lemma 6.2, (6.3) and the geometric mean - arithmetic mean inequality, from (6.14) we get
i.e.

$$
\left(\underline{G}^{m-N} \operatorname{det} B_{k+1}\right)^{1 / m} \geq \frac{\left(\operatorname{det} \tilde{X}_{k}^{-1}\right)^{1 / m}}{\operatorname{Tr}\left(S_{k}^{T} S_{k}\right) / m}>m \frac{\varepsilon_{M} \sum_{i \in \mathcal{I}_{k}} \hat{b}_{i}}{\sum_{i \in \mathcal{I}_{k}}\left|\hat{s}_{i}\right|^{2}} \geq \frac{m \varepsilon_{M}}{\sum_{i \in \mathcal{I}_{k}}\left|\hat{s}_{i}\right|^{2} / \hat{b}_{i}} \geq \frac{\varepsilon_{M}}{\theta_{1}},
$$

$$
\begin{equation*}
\operatorname{det} B_{k+1}>\left(\varepsilon_{M} / \theta_{1}\right)^{m} \underline{G}^{N-m} \triangleq \theta_{6} \tag{6.16}
\end{equation*}
$$

(ii-b) For $n_{M}=2$ we have $A_{k}=U_{k} L_{k}$ and $X_{k}=U_{k}^{-T} U_{k}^{-1}$, which yields $A_{k}^{T} X_{k} A_{k}=L_{k}^{T} L_{k}$. Using Lemma 6.2 , from (6.13) we get

$$
\begin{equation*}
\operatorname{Tr} B_{k+1} \leq N \bar{G}+\operatorname{Tr}\left(Y_{k}^{T} Y_{k}\left(L_{k}^{T} L_{k}\right)^{-1}\right) \leq N \bar{G}+\frac{\operatorname{Tr}\left(Y_{k}^{T} Y_{k}\right) \operatorname{Tr}\left(L_{k}^{T} L_{k}\right)^{m-1}}{\operatorname{det}\left(L_{k}^{T} L_{k}\right)} \tag{6.17}
\end{equation*}
$$

The conditions in Procedure 5.1 imply $\left(L_{k}\right)_{i i}^{2} \geq \varepsilon_{F} \max \left[\operatorname{Tr} A_{k}, \operatorname{Tr}\left(L_{k}^{T} L_{k}\right)\right], 1 \leq i \leq m$, thus $\operatorname{det}\left(L_{k}^{T} L_{k}\right) \geq \varepsilon_{F}^{m} \operatorname{Tr} A_{k}\left(\operatorname{Tr}\left(L_{k}^{T} L_{k}\right)\right)^{m-1}$. Using (6.17), Lemma 6.1 and (6.4) we get

$$
\begin{equation*}
\operatorname{Tr} B_{k+1} \leq N \bar{G}+\frac{\operatorname{Tr}\left(Y_{k}^{T} Y_{k}\right)}{\varepsilon_{F}^{m} \operatorname{Tr} A_{k}}=N \bar{G}+\frac{\sum_{i \in \mathcal{I}_{k}}\left|\hat{y}_{i}\right|^{2}}{\varepsilon_{F}^{m} \sum_{i \in \mathcal{I}_{k}} \hat{b}_{i}} \leq N \bar{G}+\varepsilon_{F}^{-m} \sum_{i \in \mathcal{I}_{k}} \frac{\left|\hat{y}_{i}\right|^{2}}{\hat{b}_{i}} \leq \theta_{7} \tag{6.18}
\end{equation*}
$$

$\theta_{7}=N \bar{G}+\varepsilon_{F}^{-m} m \theta_{2}>\theta_{3}$ by (6.9) and $\varepsilon_{F}<1$. Since $X^{-1}=U U^{T}$ and the main diagonals of $L, U$ are identical with $U_{i i}^{2}=L_{i i}^{2} \geq \varepsilon_{F} \operatorname{Tr} A=\varepsilon_{F} \sum_{i \in \mathcal{I}_{k}} \hat{b}_{i}$, we obtain similarly as above

$$
\left(\underline{G}^{m-N} \operatorname{det} B_{k+1}\right)^{1 / m} \geq \frac{\left(\operatorname{det}\left(U_{k} U_{k}^{T}\right)\right)^{1 / m}}{\operatorname{Tr}\left(S_{k}^{T} S_{k}\right) / m} \geq m \frac{\varepsilon_{F} \sum_{i \in \mathcal{I}_{k}} \hat{b}_{i}}{\sum_{i \in \mathcal{I}_{k}}\left|\hat{s}_{i}\right|^{2}} \geq \frac{m \varepsilon_{F}}{\sum_{i \in \mathcal{I}_{k}}\left|\hat{s}_{i}\right|^{2} / \hat{b}_{i}} \geq \frac{\varepsilon_{F}}{\theta_{1}}
$$

by (6.14), (6.3) and the geometric mean - arithmetic mean inequality. Thus we can write

$$
\begin{equation*}
\operatorname{det} B_{k+1} \geq \min \left[\left(\varepsilon_{F} / \theta_{1}\right)^{m} \underline{G}^{N-m}, \theta_{4}, \theta_{6}\right] \triangleq \theta_{8} \tag{6.19}
\end{equation*}
$$

(iii) Setting $\theta_{9}=\max \left[\theta_{5}, \theta_{7}\right]>\theta_{3}$, we always have $\operatorname{Tr} B_{i} \leq \theta_{9}$ and $\operatorname{det} B_{i} \geq \theta_{8}, i>1$, by (6.10)-(6.19). The lowest eigenvalue $\underline{\lambda}\left(B_{i}\right)$ of $B_{i}$ satisfies $\underline{\lambda}\left(B_{i}\right) \geq \operatorname{det} B_{i} /\left(\operatorname{Tr} B_{i}\right)^{N-1}$. Setting $q_{i}=H_{i}^{1 / 2} g_{i}$, we get

$$
\begin{equation*}
\frac{\left(s_{i}^{T} g_{i}\right)^{2}}{\left|s_{i}\right|^{2}\left|g_{i}\right|^{2}}=\frac{\stackrel{s}{i}_{T} B_{i} s_{i}}{s_{i}^{T T} s_{i}} \frac{g_{i}^{T} H_{i} g_{i}}{g_{i}^{T} g_{i}}=\frac{\stackrel{s}{i}_{T} B_{i} s_{i}}{s_{i}^{T} s_{i}} \frac{q_{i}^{T} q_{i}}{q_{i}^{T} B_{i} q_{i}} \geq \frac{\operatorname{det} B_{i}}{\left(\operatorname{Tr} B_{i}\right)^{N-1}} \frac{1}{\operatorname{Tr} B_{i}} \geq \frac{\theta_{8}}{\theta_{9}^{N}}, \quad i>1 \tag{6.20}
\end{equation*}
$$

which implies $\lim _{i \rightarrow \infty}\left|g_{i}\right|=0$, see Theorem 3.2 in [17] and relations (3.17)- (3.18) ibid.
One can show in the same way as in [10] that (6.20) with line search conditions (1.1) and Assumption 6.1 imply that the sequence $\left\{x_{i}\right\}$ is at least $R$-linearly convergent.

## 7 Numerical experiments

In this section, we compare our results with the results obtained by the L-BFGS method $[10,16]$ and by our latest limited-memory method [22].

All methods are implemented in the optimization software system UFO [15], which can be downloaded from www.cs.cas.cz/luksan/ufo.html. We use the following collections of test problems:

- Test 11 - 55 chosen problems from [13] (computed repeatedly five times for a better comparison), which are problems from the CUTE collection [2], some of them modified; used $N$ are given in Table 1, where the modified problems are marked with '*,
- Test 12-73 problems from [1], $N=10000$,
- Test $25-67$ chosen problems from [12], which are sparse test problems for unconstrained optimization, contained in the system UFO, $N=10000$.
The source texts and the reports corresponding to these test collections can be downloaded from the web page www.cs.cas.cz/luksan/test.html.

| Problem | $N$ | Problem | $N$ | Problem | $N$ | Problem | $N$ |
| :--- | :---: | :--- | :---: | :--- | :---: | :--- | :---: |
| ARWHEAD | 5000 | DIXMAANI | 3000 | EXTROSNB | 1000 | NONDIA | 5000 |
| BDQRTIC | 5000 | DIXMAANJ | 3000 | FLETCBV3* | 1000 | NONDQUAR | 5000 |
| BROYDN7D | 2000 | DIXMAANK | 3000 | FLETCBV2 | 1000 | PENALTY3 | 1000 |
| BRYBND | 5000 | DIXMAANL | 3000 | FLETCHCR | 1000 | POWELLSG | 5000 |
| CHAINWOO | 1000 | DIXMAANM | 3000 | FMINSRF2 | 5625 | SCHMVETT | 5000 |
| COSINE | 5000 | DIXMAANN | 3000 | FREUROTH | 5000 | SINQUAD | 5000 |
| CRAGGLVY | 5000 | DIXMAANO | 3000 | GENHUMPS | 1000 | SPARSINE | 1000 |
| CURLY10 | 1000 | DIXMAANP | 3000 | GENROSE | 1000 | SPARSQUR | 1000 |
| CURLY20 | 1000 | DQRTIC | 5000 | INDEF* | 1000 | SPMSRTLS | 4999 |
| CURLY30 | 1000 | EDENSCH | 5000 | LIARWHD | 5000 | SROSENBR | 5000 |
| DIXMAANE | 3000 | EG2 | 1000 | MOREBV* | 5000 | TOINTGSS | 5000 |
| DIXMAANF | 3000 | ENGVAL1 | 5000 | NCB20* | 1010 | TQUARTIC* | 5000 |
| DIXMAANG | 3000 | CHNROSNB* | 1000 | NCB20B* | 1000 | WOODS | 4000 |
| DIXMAANH | 3000 | ERRINROS* | 1000 | NONCVXU2 | 1000 |  |  |

Table 1: Dimensions for Test 11 - the modified CUTE collection.
We have chosen $\hat{m}=5$, which is an often used value in comparisons of limited-memory methods. In [17] the results for the L-BFGS method with $\hat{m}=3,5,17,29$ are compared and it is stated that the best CPU time is often obtained for small values of $\hat{m}$, but the algorithm tends to be less robust when $\hat{m}$ is small; this is also confirmed by our numerical experiments. Note that the required amount of storage is $2(\hat{m}+1) N$.

Furthermore, we have used $\delta_{1}=10^{-2}, \delta_{2}=\varepsilon_{E}=10^{-5}, \delta_{3}=\delta_{5}=0.025, \delta_{4}=0.05, \delta_{6}=0.5$, $\delta_{7}=\varepsilon_{D}=\varepsilon_{F}=10^{-7}, \theta=10^{3}, \varepsilon_{1}=10^{-4}, \varepsilon_{2}=0.9$ and the final precision $\left\|g\left(x^{\star}\right)\right\|_{\infty} \leq 10^{-6}$.

Table 2 contains the total number of function and also gradient evaluations (NFV) and the total computational time in seconds (Time).

| Method | Test 11 |  | Test 12 |  | Test 25 |  |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: |
|  | NFV | Time | NFV | Time | NFV | Time |
| L-BFGS | 79575 | 20.546 | 114083 | 132.00 | 501657 | 951.55 |
| Alg. 6.1 in [22] | 59735 | 14.566 | 64242 | 61.39 | 407421 | 714.81 |
| Alg. 5.1, met 1 | 60198 | 14.416 | 64815 | 62.56 | 372704 | 699.08 |
| Alg. 5.1, met 2 | 60083 | 14.422 | 63482 | 63.11 | 366824 | 917.49 |

Table 2: Comparison of the selected methods.
For a better demonstration of both the efficiency and the reliability, we compare selected optimization methods by using performance profiles introduced in [5]. The performance profile $\rho_{M}(\tau), \tau \geq 0$, is defined by the formula

$$
\rho_{M}(\tau)=\frac{\text { number of problems where } \log _{2}\left(\tau_{P, M}\right) \leq \tau}{\text { total number of problems }}
$$

where $\tau_{P, M}$ is the performance ratio of the number of function evaluations (or the time) required to solve problem $P$ by method $M$ to the lowest number of function evaluations (or the time) required to solve problem $P$. The ratio $\tau_{P, M}$ is set to infinity (or some large number) if method $M$ fails to solve problem $P$.

The value of $\rho_{M}(\tau)$ at $\tau=0$ gives the percentage of test problems for which the method $M$ is the best and the value for $\tau$ large enough is the percentage of test problems


Figure 7.1: Comparison of $\rho_{M}(\tau)$ for Test 11 and various methods for NFV and TIME.



Figure 7.2: Comparison of $\rho_{M}(\tau)$ for Test 12 and various methods for NFV and TIME.


Figure 7.3: Comparison of $\rho_{M}(\tau)$ for Test 25 and various methods for NFV and TIME.


Figure 7.4: Comparison of $\rho_{M}(\tau)$ for Test 25 and various methods for NFV and TIME.
that method $M$ can solve. The relative efficiency and reliability of each method can be directly seen from the performance profiles: the higher the particular curve, the better is the corresponding method. Figures 1-4, based on the results in Table 2, show graphical performance profiles for the tested methods (for Test 25, we compare our new methods separately to make the differences more visible). They demonstrate the efficiency of our methods in comparison with the L-BFGS method. We can also see that the numerical results for the new methods and for our method [22] are comparable.

## 8 Conclusions

In this contribution, we derive two new updates for general functions with minimum violation of the previous quasi-Newton equations in some sense, describe its properties and show how the corresponding methods can be advantageously combined with vector corrections for conjugacy.

Our experiments indicate that this approach can improve unconstrained large-scale minimization results significantly compared with the frequently used L-BFGS method.

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