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Abstract:

This is an unpublished two-page manuscript from 2000. We describe explicit formulae for componentwise bounds on solution of a system of linear equations $A_c x = b_c$ (A_c square) under perturbation of all data. To make the result numerically tractable, we avoid use of exact inverses, using instead some matrices R and M required only to satisfy certain inequalities. Hansen's optimality result is a special case of our theorem.

Keywords:
Linear equations, perturbation, bounds.

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A perturbation theorem for linear equations

We describe explicit formulae for componentwise bounds on solution of a system of linear equations $A_cx = b_c$ (A_c square) under perturbation of all data. To make the result numerically tractable, we avoid use of exact inverses, using instead some matrices R and M required only to satisfy certain inequalities. Hansen's optimality result is a special case of our theorem.

Notations used: I is the unit matrix, ϱ denotes the spectral radius, for $A = (a_{ij})$ we denote $|A| = (|a_{ij}|)$ and inequalities are understood componentwise. To save space, we write a/b instead of $\frac{a}{b}$.

Theorem. *Let $M \geq 0$ and R be arbitrary matrices satisfying*

$$MG + I \leq M, \tag{1}$$

where $G = |I - RA_c| + |R|\Delta$. Then for each A and b such that $|A - A_c| \leq \Delta$ and $|b - b_c| \leq \delta$, A is nonsingular and the solution of the system $Ax = b$ satisfies for each $i \in \{1, \ldots, n\}$

$$\min\{\underset{\sim}{x}_i/\alpha_i,\ \underset{\sim}{x}_i/\beta_i\} \leq x_i \leq \max\{\tilde{x}_i/\alpha_i,\ \tilde{x}_i/\beta_i\}, \tag{2}$$

where

$$\begin{aligned}
\underset{\sim}{x}_i &= -(M(|Rb_c| + |R|\delta))_i + m_i(Rb_c + |Rb_c|)_i \\
\tilde{x}_i &= (M(|Rb_c| + |R|\delta))_i + m_i(Rb_c - |Rb_c|)_i \\
\alpha_i &= 1 + (|r_i| - r_i)m_i + h_i \\
\beta_i &= 2m_i - 1 - (|r_i| + r_i)m_i - h_i \\
m_i &= M_{ii} \\
r_i &= (I - RA_c)_{ii} \\
h_i &= (M - MG - I)_{ii}
\end{aligned}$$

and $\beta_i \geq \alpha_i \geq 1$. Moreover, if $A_c = I$ and $\varrho(\Delta) < 1$ and if we take $R = I$ and $M = (I - \Delta)^{-1}$, then the bounds (2) are exact (i.e., attained).

Proof. 1) First we prove that each matrix A with $|A - A_c| \leq \Delta$ is nonsingular. Premultiplying the inequality (1) by the nonnegative matrix G yields $MG^2 + G + I \leq MG + I \leq M$ and by induction $\sum_{j=0}^{k} G^j \leq MG^{k+1} + \sum_{j=0}^{k} G^j \leq M$ for each $k \geq 0$, hence $\sum_0^\infty G^j$ is convergent which, as well known, implies that $\varrho(G) < 1$. Since $|I - RA| = |I - RA_c + R(A_c - A)| \leq |I - RA_c| + |R|\Delta = G$, we have $\varrho(I - RA) \leq \varrho(G) < 1$ which means that the matrix $RA = I - (I - RA)$ is nonsingular, hence A is nonsingular.

2) Next we prove that $\beta_i \geq \alpha_i \geq 1$ for each i. From the definition of h_i we have $m_i = (MG)_{ii} + 1 + h_i \geq m_i|r_i| + 1 + h_i$ which can be easily rearranged to $2m_i - 1 - (|r_i| + r_i)m_i - h_i \geq 1 + (|r_i| - r_i)m_i + h_i$, giving $\beta_i \geq \alpha_i$; the inequality $\alpha_i \geq 1$ follows from the nonnegativity of m_i and h_i.

3) Let x solve $Ax = b$ for some A, b with $|A - A_c| \leq \Delta$ and $|b - b_c| \leq \delta$. Then we have

$$x = (I - RA)x + Rb = (I - RA_c)x + R(A_c - A)x + Rb_c + R(b - b_c) \tag{3}$$

and taking absolute values gives

$$|x| \leq G|x| + |Rb_c| + |R|\delta. \tag{4}$$

Let $i \in \{1, \ldots, n\}$. Then from the ith equation in (3) we have

$$\begin{aligned}
x_i &\leq ((I - RA_c)x)_i + (|R|\Delta|x|)_i + (Rb_c)_i + (|R|\delta)_i \\
&= (G|x| + |Rb_c| + |R|\delta)_i + ((I - RA_c)x - |I - RA_c| \cdot |x| + Rb_c - |Rb_c|)_i.
\end{aligned} \tag{5}$$

Put $x' = (|x_1|, \ldots, |x_{i-1}|, x_i, |x_{i+1}|, \ldots, |x_n|)^T$. Then from (4) and (5) we have $x' \le G|x| + |Rb_c| + |R|\delta + ((I - RA_c)x - |I - RA_c| \cdot |x| + Rb_c - |Rb_c|)_i e_i$, where e_i is the ith column of I. Premultiplying this inequality by the nonnegative vector $e_i^T M$ and using the matrix $H := M - MG - I$, we obtain $(Mx')_i \le ((M - H - I)|x|)_i + ((I - RA_c)x - |I - RA_c| \cdot |x|)_i m_i + \tilde{x}_i$ and consequently

$$(M(x' - |x|))_i + (H|x|)_i + |x_i| + (|I - RA_c| \cdot |x| - (I - RA_c)x)_i m_i \le \tilde{x}_i. \tag{6}$$

Since $(M(x' - |x|))_i = m_i(x_i - |x_i|)$, $(H|x|)_i \ge h_i|x_i|$ and $(|I - RA_c| \cdot |x| - (I - RA_c)x)_i \ge |r_i| \cdot |x_i| - r_i x_i$, from (6) we finally obtain an inequality containing variable x_i only:

$$m_i(x_i - |x_i|) + h_i|x_i| + |x_i| + (|r_i| \cdot |x_i| - r_i x_i)m_i \le \tilde{x}_i. \tag{7}$$

If $x_i \ge 0$, then this inequality becomes $\alpha_i x_i \le \tilde{x}_i$, implying $x_i \le \tilde{x}_i/\alpha_i$, and if $x_i < 0$, then (7) turns into $\beta_i x_i \le \tilde{x}_i$, giving $x_i \le \tilde{x}_i/\beta_i$, which together yields

$$x_i \le \max\{\tilde{x}_i/\alpha_i,\ \tilde{x}_i/\beta_i\}. \tag{8}$$

In this way we have obtained the upper bound in (2). To prove the lower one, notice that $-x$ satisfies $A(-x) = -b$, where $|A - A_c| \le \Delta$ and $|(-b) - (-b_c)| \le \delta$. Hence we can use the previously obtained result if we set $b_c := -b_c$, which affects $\tilde{x}_i$ only. Then from (8) we get $-x_i \le \max\{-\underset{\sim}{x}_i/\alpha_i,\ -\underset{\sim}{x}_i/\beta_i\}$ which, after premultiplying by -1, gives the lower bound in (2).

4) Finally, to prove the optimality result for the case $A_c = I$ and $\varrho(\Delta) < 1$, take $R = I$ and $M = (I - \Delta)^{-1}$, then $M \ge 0$, $G = \Delta$ and (1) is satisfied as an equation; moreover, for each i we have $r_i = h_i = 0$, $\alpha_i = 1$, $\beta_i = 2m_i - 1$, hence (2) has the form

$$\min\{\underset{\sim}{x}_i,\ \underset{\sim}{x}_i/\beta_i\} \le x_i \le \max\{\tilde{x}_i,\ \tilde{x}_i/\beta_i\}. \tag{9}$$

To prove that the upper bound is really attained, let us fix an $i \in \{1, \ldots, n\}$ and define a diagonal matrix D by $D_{jj} = 1$ if $j \ne i$ and $(b_c)_j \ge 0$, $D_{jj} = -1$ if $j \ne i$ and $(b_c)_j < 0$, and $D_{jj} = 1$ if $j = i$, and let $\tilde{b} = Db_c + \delta$. Then it can be easily verified that $\tilde{x}_i = (M\tilde{b})_i$ holds. First, define $x' = DM\tilde{b}$. Since $M = (I - \Delta)^{-1}$ implies $\Delta M = M\Delta = M - I$, we have $(I - D\Delta D)x' = DM\tilde{b} - D(M - I)\tilde{b} = D\tilde{b} = b_c + D\delta$, which means that x' solves the system $(I - D\Delta D)x' = b_c + D\delta$ where $|(I - D\Delta D) - I| = \Delta$, $|(b_c + D\delta) - b_c| = \delta$ and $x'_i = e_i^T DM\tilde{b} = e_i^T M\tilde{b} = (M\tilde{b})_i = \tilde{x}_i$, which shows that $\tilde{x}_i$ is attained. Second, let $x'' = DM(\tilde{b} - 2(\tilde{x}_i/\beta_i)\Delta e_i)$ and define a diagonal matrix D' by $D'_{ii} = -1$ and $D'_{jj} = D_{jj}$ otherwise. Then $(I - D\Delta D')DM = DM - D\Delta(I - 2e_i e_i^T)M = DM - D(M - I) + 2D\Delta e_i e_i^T M = D + 2D\Delta e_i e_i^T M$, hence $(I - D\Delta D')x'' = (D + 2D\Delta e_i e_i^T M)(\tilde{b} - 2(\tilde{x}_i/\beta_i)\Delta e_i) = D\tilde{b} + 2\tilde{x}_i D\Delta e_i(-(1/\beta_i) + 1 - (2/\beta_i)(m_i - 1)) = D\tilde{b} = b_c + D\delta$, which shows that x'' is a solution to the system $(I - D\Delta D')x'' = b_c + D\delta$ where $|(I - D\Delta D') - I| = \Delta$, $|(b_c + D\delta) - b_c| = \delta$ and $x''_i = e_i^T DM(\tilde{b} - 2(\tilde{x}_i/\beta_i)\Delta e_i) = \tilde{x}_i - 2(\tilde{x}_i/\beta_i)(m_i - 1) = \tilde{x}_i/\beta_i$. This proves that $\tilde{x}_i/\beta_i$ is attained, hence also the upper bound $\max\{\tilde{x}_i,\ \tilde{x}_i/\beta_i\}$ in (9) is attained. The proof for the lower bound follows from the result just obtained by applying it to the case $b_c := -b_c$ as in the part 3). □

The quantities r_i and h_i correct the influence of the approximate inverses R and M; they vanish if $R = A_c^{-1}$ and $M = (I - G)^{-1} \ge 0$ are used. The last statement of the theorem is Hansen's optimality result [1] as reformulated in [2]. Matrices R and $M \ge 0$ satisfying (1) exist if and only if $\varrho(|A_c^{-1}|\Delta) < 1$ holds (Theorem 1 in [3]).

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