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Abstract:

The most often applied non-numerical uncertainty degrees are those taking their values in complete lattices, but also their weakened versions may be of interest. In what follows, we introduce and analyze possibilistic distributions and measures taking values in finite upper-valued possibilistic lattices, so that only for finite sets of such values their supremum is defined. For infinite sets of values of the finite lattice in question we apply the idea of the so called Monte-Carlo method: sample at random and under certain conditions a large enough finite subset of the infinite set in question, and take the supremum over this finite sample set as a "good enough" estimation of the undefined supremum of the infinite set. A number of more or less easy to prove assertions demonstrate the conditions when and in which sense the quality of the results obtained by replacing non-existing or non-accessible supremum values by their random estimations tend to the optimum results supposing that the probabilistic qualities of the statistical estimations increases as demanded by Monte-Carlo methods.

Keywords:
Complete lattice, upper-valued semilattice, lattice-valued possibilistic distribution, random samples from upper-valued semi-lattices, probabilistic algorithms, Monte-Carlo methods

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1 Introduction

As soon as two years after publication of the Zadeh's pioneering paper on real-valued fuzzy sets [16], J. A. Goguen applied the basic ideas of fuzziness to partially ordered sets with non-numerical values, so arriving at complete lattices as the most often used support set for non-numerically valued fuzziness degrees [8]. As excellent theoretical survey of complete lattice-valued fuzzy sets can be found in [2]. The reader is supposed to be familiar with the basic ideas and results of the papers [3] and [14] and with the most elementary stones of the formalized constructions from [11].

Shifting the fuzziness degrees from the real numbers in $[0,1]$ to elements of a complete lattice we make these degrees much more freely defined and much more vaguously relating these degrees to corresponding non-numerical structures. But, at the same time, the space with this weakened assumptions becomes much more open for applications than that with the real-valued fuzziness degrees.

In this paper, we will go on with this paradigma when still more weakening the conditions imposed on the structure from which fuzziness takes its degrees, namely, instead complete lattice we will define that structure by an upper semilattice. Let us recall that a partially ordered set $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ defines an upper semilattice, if for each *finite* set $A \subset T$ the supremum $\bigvee^{\mathcal{T}} A$ (w.r.to $\leq_{\mathcal{T}}$) is defined. If $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ were a complete lattice (e.g., the standard unit interval $\langle [0,1], \leq \rangle$), then each mapping $\pi : \Omega \to T$ defined on a nonempty set Ω for each $\omega \in \Omega$, defines a total set function $\Pi : \mathcal{P}(\Omega) \to T$. In other terms, the mapping π may be taken as a fuzzy or possibilistic distribution on the space Ω, implying uniquely the total fuzzy or possibilitic measure (set function) Π ascribing to each $A \subset \Omega$ the value $\bigvee^{\mathcal{T}}_{\omega \in A} \pi(\omega) \in T$. When replacing real-valued or complete lattice-valued structure $\mathcal{T}$ by upper semilattice $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$, the set function Π will be defined only for finite (or π-finite, if elements $\omega \in \Omega$ with identical values $\pi(\omega)$ are also taken as identical) subsets of Ω, hence, if Ω is infinite, $\Pi : \mathcal{P}(\Omega) \to T$ will be a partial mapping. So, when $A \subset \Omega$ is an infinite set and $\bigvee_{\omega \in A} \pi(\omega)$ is not defined, we would like to find a finite sequence $\omega_1, \omega_2, \ldots, \omega_N$ of elements of A such that the finite (hence, defined) supremum $\bigvee_{i=1}^{N} \pi(\omega_i)$ would replace or approximate, in a reasonable sense, the undefined value $\Pi(A)$. However, such approximations are offered and reasonably founded and processed by probability theory and mathematical statistics, taking finite random samples as elements from A.

The values $\pi(\omega)$ ascribed to elements of the space Ω may be seen from two points of view: as the degrees of fuzziness or possibility degrees defined on the support set Ω of a fuzzy set or possibilistic distribution π, but also as a $\mathcal{T}$-valued function $\pi : \Omega \to T$. Supposing that Ω is completed to a probability space $\langle \Omega, \mathcal{A}, P \rangle, \pi$ may be taken as $\mathcal{T}$-valued random variable. So, the value $\bigvee^{\mathcal{T}}_{\omega \in A}$ may be approximated (if it is defined) or extended (if not defined) by the value approximating or extending the expected value $\Pi(A)$ of the probability density $\{\pi(\eta(\omega)) : \omega \in \Omega\}$ defined on $\mathcal{A}$. Such a model enables to define the notions like statistical estimations related to the density $\pi(\eta(\cdot))$, namely, statistical estimations which may be, within the framework of $\mathcal{T}$-valued possibilistic distributions, taken as reasonable approximations and completions of the values related to $\mathcal{T}$-distributions.

Even if the extent of this contribution is rather limited, all the section 2 is devoted to a formalized definition of statistical estimation of values of upper semilattice-valued measures in order to prove that this notion can be completely defined and processed within the standard framework of the axiomatic probability theory. Section 3 shows that statistical estimation of possibilistic distributions meets the basic paradigmatic property of standard statistical estimations according to which reasonably defined qualities of such estimations improve with their size increasing. Finally, Section 4 offers the notion of π-quasi-supremum as a useful, even if not generally acceptable substitution of the notion of supremum at least in some particular cases of incomplete upper semilattices.

2 Statistical Estimations of Lattice-Valued Possibility Degrees – a Formalized Model

Let T be a nonempty set, let $\leq_{\mathcal{T}}$ be a partial ordering relation on T, so that $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ defines a p.o.set. Suppose, moreover, that $\mathcal{T}$ meets the conditions imposed on *upper semilattice*, so that, for each *finite* set $A_0 \subset T$ the supremum $\bigvee^{\mathcal{T}}_{t \in A_0} t$ ($\bigvee^{\mathcal{T}} A_0$ abbreviately) is defined. As the empty subset of T is also finite, $\bigvee^{\mathcal{T}}_{t \in \emptyset} = \bigvee^{\mathcal{T}} \emptyset$ is defined and denoted by $\mathbf{0}_{\mathcal{T}}$ as it obviously plays the role of the minimum or zero element of $\mathcal{T}$ (obviously, if T is infinite, the supremum element $\bigvee_{t\in T}^{\mathcal{T}} = \bigvee^{\mathcal{T}} T$ need not be defined).

Let Σ be a nonempty set, the elements of Σ will be denoted as η, η^*, η_i, and similarly. A mapping $\pi : \Sigma \to T$ is called a $\mathcal{T}$*-(valued possibilistic) distribution* on Σ, if $\bigvee^{\mathcal{T}} T$ (denoted also by $\mathbf{1}_{\mathcal{T}}$) is defined and if $\bigvee_{t\in\Sigma}^{\mathcal{T}} \pi(\eta) = \mathbf{1}_{\mathcal{T}}$ holds. This is the case if the space Σ is π-*finite*, i.e., if card $\{\pi(\eta) : \eta \in \Sigma\} < \infty$ holds and in this case all subsets of Σ are π-finite, hence, the possibilistic measure $\Pi(A) = \bigvee_{\eta\in A}^{\mathcal{T}} \pi(\eta)$ is defined for each $A \subset \Sigma$.

Our aim will be to replace or to extend the value $\bigvee_{\eta\in A}^{\mathcal{T}} \pi(\eta)$ by the value $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*)$, (an abbreviation for $(\bigvee_{n_i^*}^{\mathcal{T}})_{i=1}^{N} \pi(\eta_i^*)$), where $\eta_1^*, \eta_2^*, \dots \eta_N^*$ are "appropriately at random sampled" elements of the space Σ. The first formal notion needed in order to build the necessary mathematical construction is that of probability space.

Let Ω be a nonempty set, the elements of which are denoted by ω and are called *elementary random events.* Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ be a non-empty system of subsets of Ω which defines a σ-*field*, so that, for each $E_1, E_2, \dots \in \mathcal{A}$ also the sets $\Omega - E_i$ and $\bigcup_{i=1}^{\infty} E_i$ are in $\mathcal{A}$. Let $P : \mathcal{A} \to [0,1]$ be a mapping (set function on $\mathcal{A}$, as a matter of fact) such that $P(\Omega) = 1$ and $P(\bigcup_{i=1}^{\infty} E_i) = \Sigma_{i=1}^{\infty} P(E_i)$ for each sequence of mutually disjoint sets $E_1, E_2, \dots$ from $\mathcal{A}$, i.e., $E_i \cap E_j = \emptyset$ for each $i \neq j$. Such a set function P is called σ-*additive probability measure* defined on *measurable space* $\langle \Omega, \mathcal{A} \rangle$ and the ordered triple $\langle \Omega, \mathcal{A}, P \rangle$ is called *probability space.*

Let $\langle \Omega, \mathcal{A}, P \rangle$ be a probability space, let $\mathcal{X} = \langle X, \mathcal{S} \rangle$ be a *measurable space*, i.e., X is a nonempty set and $\mathcal{S}$ is a nonempty σ-field of subsets of X. A mapping $f : \Omega \to X$ is called *random variable,* defined on the probability space $\langle \Omega, \mathcal{A}, P \rangle$, if for each set $S \subset X$, $S \in \mathcal{S}$, the relation $\{\omega \in \Omega : f(\omega) \in S\} \in \mathcal{A}$ holds, consequently, the probability $P(\{\omega \in \Omega : f(\omega) \in S\})$ is defined. A sequence $\{f_1, f_2, \dots\}_{i=1}^{\infty}$ of random variables is called *independent and identically distributed sequence* of random variables (i.i.d. sequence, abbreviately), if for each $A \in \mathcal{S}$ and each $j = 1, 2, \dots$ the identity $P(\{\omega \in \Omega : f_j(\omega) \in A\}) = P(\{\omega \in \Omega : f_1(\omega) \in A\})$ holds and, moreover, if for each $1 \leq i, i \neq j$, and each $S_i, S_j \in \mathcal{S}$, the relation

$$P(\{\omega \in \Omega : f_i(\omega) \in S_i, f_j(\omega) \in S_j\}) \;=\; P(\{\omega \in \Omega : f_i(\omega) \in S_i\}) \cdot P(\{\omega \in \Omega : f_j(\omega) \in S_j\}) \tag{2.1}$$

is valid.

Take the space Σ of elementary possibilistic events, take a nonempty σ-field $\mathcal{E}$ of subsets of Σ so that the pair $\langle \Sigma, \mathcal{E} \rangle$ defines a measurable space, take a probability space $\langle \Omega, \mathcal{A}, P \rangle$. Let $\eta^* : \Omega \to \Sigma$ be a mapping such that, for each $E \in \mathcal{E}$, $\{\omega \in \Omega : \eta^*(\omega) \in E\} \subset \mathcal{A}$ holds, so that the probability $P(\{\omega \in \Omega : \eta^*(\omega) \in E\})$ is defined. Hence, $\eta^*(\omega) \in \Sigma$ is an at random sampled element of the elementary possibilistic space Σ. Combining the mapping η^* with the mapping $\pi : \Sigma \to T$ we obtain the mapping $\pi(\eta^*(\cdot)) : \Omega \to T$. Supposing that $\mathcal{F} \subset \mathcal{P}(T)$ is a σ-field of subsets of T and that $\{\omega \in \Omega : \pi(\eta^*(\omega)) \in F\} \in \mathcal{A}$ holds for each $F \in \mathcal{F}$, the mapping $\pi(\eta^*(\cdot)) : \Omega \to T$ defines a random variable on the probability space $\langle \Omega, \mathcal{A}, P \rangle$ which takes its values in the measurable space $\langle T, \mathcal{F} \rangle$. Informally defined, $\pi(\eta^*(\omega))$ is the possibility degree ascribed by the mapping (possibilistic distribution on Σ, if it is the case) π to the at random sampled element $\eta^*(\omega)$ of the space Σ of elementary possibilistic events.

Let $A \subset \Sigma$ be given, let $\eta_1^*, \eta_2^*, \dots$ be an infinite sequence of statistically independent and identically distributed random variables defined on the probability space $\langle \Omega, \mathcal{A}, P \rangle$, taking values in the measurable space $\langle \Sigma, \mathcal{E} \rangle$ and such that, for each $\omega \in \Omega$ and each $i = 1, 2, \dots, \eta_i^*(\omega) \in A \subset \Sigma$ holds. Hence, for each integer $N \geq 1$, $\langle \eta_1^*(\omega), \eta_2^*(\omega), \dots \eta_N^*(\omega) \rangle$ is a finite sequence of elements of A and $\langle \pi(\eta_1^*(\omega)), \pi(\eta_2^*(\omega)), \dots, \pi(\eta_N^*(\omega)) \rangle$ is the corresponding sequence of their possibility degrees defined by the mapping $\pi : \Sigma \to T, \mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$. Obviously, each $\pi(\eta_i^*(\omega)), i = 1, 2, \dots, \omega \in \Omega$, is an element of the upper semilattice $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$, consequently, the value $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega))$ is defined and belongs to T.

Supposing that $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\cdot)) : \Omega \to T$ defines a random variable which takes the probability space $\langle \Omega, \mathcal{A}, P \rangle$ into the measurable space $\langle T, \mathcal{F} \rangle$, i.e., if for each $F \in \mathcal{F}$ the relation $\{\omega \in \Omega : \bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega)) \in F\} \in \mathcal{A}$ holds, the mapping $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega)) : \Omega \to T$ is called the *statistical estimation* (if $\bigvee_{\eta\in A}^{\mathcal{T}} \pi(\eta)$ is defined) or the *statistical extension* (if $\bigvee_{\eta\in A}^{\mathcal{T}} \pi(\eta)$ is not defined) of the value of the partial $\mathcal{T}$-valued possibilistic measure Π, induced by π, to the set $A \subset \Sigma$, let us denote it by $\Pi(A)$.

Before going on with the mathematical considerations, some comments may be of use, let us begin with the terms *statistical estimations* and *statistical extension.* As a rule, the term estimation is used when some value is correctly and precisely defined, but for some reasons this value cannot be explicitly specified. E.g., the expected value of a random variable may be defined as a function of empirical values, but in practice only more or less good averages of a series of values taken from repeated random samples may be used as a statistical estimation of the expected value in question. When modifying the definition of expected value in such a way that expected value of an integer-valued random variable must be also defined by an integer, then the expected number of points occurring on dice when tossing is not defined and the value 3.5 may be taken as extension, but not as the expected value of the number of points on the tossed dice.

The difference between the two notions is obvious when considering the problem how to measure the quality of statistical estimations and extensions. For estimations, the closer is the estimation to the estimated value, or the closer to 1 is the probability, that these values are identical or sufficiently close to each other, the better is the estimation. For extensions the situation is more difficult. If $\Pi(A) = \bigvee_{\eta \in A}^{\mathcal{T}} \pi(\eta)$ is defined, then the statistical estimation $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega))$ is the best possible, if both the values are identical and in this case, with the probability one, the equality $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega)) = \bigvee_{i=1}^{\mathcal{T},N+1} \pi(\eta_i^*(\omega))$ holds. If $\bigvee_{\eta \in A}^{\mathcal{T}} \pi(\eta)$ is not defined, we may (and will) measure the quality of the extension $\Pi(A) = \bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega))$ by the criterion according to which the probability

$$P\left(\left\{\omega \in \Omega : \bigvee_{i=1}^{\mathcal{T},N+1} \pi(\eta_i^*(\omega)) > \bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega))\right\}\right), \tag{2.2}$$

i.e., the probability that the value of the “supremum” of the values $\pi(\eta), \eta \in A$, will increase when, taking into consideration one more sample from $A \subset \Sigma$ should be either 0 or as close to 0 as possible.

We have purposedly formalized the notions of statistical estimation and extension at a rather general and abstract level with the aim to demonstrate that this problem can be defined and solved at the same level of description and processing as it is common in standard works on probability theory. However, in order to arrive at some more explicit results, let us assume the following simplifying conditions to hold. The space Σ is supposed to be infinite and countable, and the σ-field $\mathcal{E}$ of measurable subsets of Σ is defined by the power-set $\mathcal{P}(\Sigma)$. So, random variables η^* are defined on $\langle \Omega, \mathcal{A}, P \rangle$ as mapping ascribing to each $\omega \in \Omega$ the value $\eta^*(\omega)$. For each $\eta \in \Sigma$ the value $P(\{\omega \in \Omega : \eta^*(\omega) = \eta\})$ is defined (and denoted, if no misunderstanding menaces) by $p(\omega)$. Consequently, for each $A \subset \Sigma$, the value $Pr(A)$ is defined by $Pr(A) = \Sigma_{\eta \in A} p(\eta)$.

3 Asymptotic Properties of Statistical Estimations of Upper-Semilattice-Valued Possibilistic Degrees

Let $\langle n_1^*, n_2^*, \dots \rangle$ be an infinite sequence of statistically independent random variables distributed identically with η^*, let $N = 1, 2, \dots$, let $\eta^*(\omega) \in A$ for each $\omega \in \Omega$ hold. Define

$$\Pi^N(\eta^*, \omega) = \bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega)). \tag{3.1}$$

The last supremum, hence, also the value $\Pi^N(\eta^*, \omega)$ is always defined. If $\Pi(A)$ is defined, then $\Pi^N(\eta^*, \omega)$ is called the *statistical estimation* of $\Pi(A)$, if $\Pi(A)$ is not defined, then $\Pi^N(\eta^*, \omega)$ is called the *statistical extension* of Π to A. In order to simplify our notation, we will use the term “statistical estimation of $\Pi(A)$” in both the cases, carefully keeping in mind the important differences staying behind both these approaches.

Lemma 3.1 *For each $A \subset \Sigma$, each $N = 1, 2, \dots$ and each $\omega \in \Omega$ the inequality $\Pi^N(\eta^*, \omega) \leq_{\mathcal{T}} \Pi^{N+1}(\eta^*, \omega)$ holds.*

If $\Pi(A) = \bigvee_{\eta \in A}^{\mathcal{T}} \pi(\eta)$ is defined, i.e., if A is a π-finite subset of Σ, then for each $N = 1, 2, \dots,$ and each $\omega \in \Omega$ the relation $\Pi^N(\eta^*, \omega) \leq_{\mathcal{T}} \Pi(A)$ holds.

Proof: Obvious. □

Definition 3.1 *Statistical estimation $\Pi^N(\eta^*,\omega)$ of the value $\Pi(A)$ for $A \subset \Sigma$ is statistically optimal, if*

$$P(\{\omega \in \Omega : \Pi^{N+1}(\eta^*,\omega) = \Pi^N(\eta^*,\omega)\}) = 1 \tag{3.2}$$

holds.

In a perhaps more intuitive setting, up to the cases with zero global probability P, the statistical estimation $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega))$ of $\Pi(A)$ cannot be improved, i.e., enlarged w.r.to the partial ordering $\leq_{\mathcal{T}}$ on T, no matter how large finite number of samples made by random variables η_i^*, $i > N$, may be taken.

Lemma 3.2 *Let $\Pi(A) = \bigvee_{\eta \in A}^{\mathcal{T}} \pi(\eta)$ be defined, let $\Pi^N(\eta^*,\omega) = \Pi(A)$ hold. Then $\Pi^N(\eta^*,\omega)$ is statistically optimal statistical estimation of the value $\Pi(A)$.*

Proof: Obvious. □

Under some more conditions also an assertion inverse to that of Lemma 3.2 may be stated and proved.

Theorem 3.1 *Let $A \subset \Sigma$ be π-finite, let Σ be an infinite countable set, let $\mathcal{E} = \mathcal{P}(\Sigma)$, let $\eta^* : \langle \Omega, \mathcal{A}, P\rangle \to \langle \Sigma, \mathcal{E}\rangle$ be such that $p(\eta) = P(\{\omega \in \Omega : \eta^*(\omega) = \eta\}) > 0$ holds iff $\eta \in A$, let $\langle \eta_1^*, \eta_2^*, \dots\rangle$ be an infinite sequence of statistically independent and identically distributed copies of the random variable η^*. Then the statistical estimation $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega))$ of the value $\Pi(A) = \bigvee_{\eta \in A}^{\mathcal{T}} \pi(\eta)$ is statistically optimal iff the identity $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega)) = \Pi(A)$ holds.*

Proof: Due to Lemma 3.2, the only we have to prove is that if $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega)) \neq \Pi(A)$, i.e., if $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega)) <_{\mathcal{T}} \Pi(A)$ is the case, then

$$P\left(\left\{\omega \in \Omega : \bigvee_{i=1}^{\mathcal{T},N+1} \pi(\eta_i^*(\omega)) >_{\mathcal{T}} \bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega))\right\}\right) > 0 \tag{3.3}$$

follows. Hence, we have to prove that if $\Pi^N(\eta^*,\omega) <_{\mathcal{T}} \Pi(A)$ holds, then with a positive probability the statistical estimation $\Pi^N(\eta^*,\omega)$ of $\Pi(A)$ can be improved when taken one more random sample $\eta_{N+1}^*(\omega) \in A$.

As for each $\omega \in \Omega$ and each $i = 1, 2, \dots$ holds that $\eta_i^*(\omega) \in A$ and $\{\pi(\eta_i^*(\omega)) : i = 1, 2, \dots\} \subset \{\pi(\eta) : \eta \in A\}$, both with the probability one, the inequality $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega)) <_{\mathcal{T}} \Pi(A)$ may happen only when there exists $\eta_0 \in A$ such that $\eta_0 \neq \eta_i^*(\omega), i = 1, 2, \dots, N$, and $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega)) \vee \pi(\eta_0) > \bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega))$ hold together. In other words, $\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega)) < \Pi(A)$ yields that there exists an element $\eta_0 \in A$ not sampled yet by the samples $\eta_1^*(\omega), \eta_2^*(\omega), \dots, \eta_N^*(\omega)$ but such that the value $\pi(\eta_0)$ augments the value $\Pi^N(\eta^*,\omega) = \bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega))$. However, with the positive probability $p(\eta_0)$ the case $\eta_{N+1}^*(\omega) = \eta_0$ occurs, so that $\Pi^{N+1}(\eta^*,\omega) > \Pi^N(\eta^*,\omega)$ holds with probability $p(\eta_0) > 0$. Hence, $\Pi^N(\eta^*,\omega)$ is not statistically optimal statistical estimation of $\Pi(A)$ and the assertion is proved. □

Theorem 3.2 *Let the notations and conditions of Theorem 3.1 hold with the only exception that the set A need not be π-finite. Let A_0 be finite subset of $A \subset \Sigma$ such that, for each $\eta_\star \in A$, the relation*

$$\Pi(A_0) = \bigvee_{\eta \in A_0}^{\mathcal{T}} \pi(\eta) = \bigvee_{\eta \in A_0 \cup \{\eta_\star\}}^{\mathcal{T}} \pi(\eta) = \Pi((A_0) \cup \{\eta_\star\}) \tag{3.4}$$

holds. Then $\Pi^N(\eta^*,\omega)$ tends to $\Pi(A_0)$ in probability P with N increasing, so that the relation

$$\lim_{N \to \infty} P(\{\omega \in \Omega : \Pi^N(\eta^*,\omega) = \Pi(A_0)\}) = 1 \tag{3.5}$$

is valid.

Proof: Let $A_0 = \{\eta^1, \eta^2, \dots, \eta^K\} \subset \Sigma$. An easy combinatoric consideration yields that

$$\begin{aligned}
& P(\{\omega \in \Omega : \{\eta^1, \eta^2, \dots, \eta^K\} \not\subset \{\eta_1^*(\omega), \eta_2^*(\omega), \dots, \eta_N^*(\omega)\}\}) \\
= \; & P\left(\bigcup_{j=1}^{K} (\{\omega \in \Omega : \eta^j \notin \{\eta_1^*(\omega), \dots, \eta_N^*(\omega)\}\})\right) \leq \\
\leq \; & \sum_{j=1}^{K} P(\{\omega \in \Omega : \eta^j \notin \{\eta_1^*(\omega), \eta_2^*(\omega), \dots, \eta_N^*(\omega)\}\}). \qquad (3.6)
\end{aligned}$$

For each $j = 1, 2, \dots, K$

$$P(\{\omega \in \Omega : \eta^j \notin \{\eta_1^*(\omega), \eta_2^*(\omega), \dots, \eta_N^*(\omega)\}\}) = (1 - p(\eta^j))^N \to 0 \qquad (3.7)$$

with $N \to \infty$ holds. Hence, given $\mathcal{E} > 0$, for each $j = 1, 2, \dots, K$ there exists $n_j \in \{1, 2, \dots\}$ such that for each $N \geq n_j$, $P(\{\omega \in \Omega : \eta^j \notin \{\eta_1^*(\omega), \dots, \eta_N^*(\omega)\}\}) < \mathcal{E}$ holds. Setting $N_0 \geq \max\{n_j : j \leq K\}$, we obtain that $P(\{\omega \in \Omega : \eta^j \notin \{\eta_1^*(\omega), \dots, \eta_N^*(\omega)\}\}) < \mathcal{E}$ holds for each $j \leq K$ supposing that $N \geq N_0$ is the case.

Consequently,

$$\begin{aligned}
& P(\{\omega \in \Omega : \{\eta^1, \dots, \eta^K\} \not\subseteq \{\eta_1^*(\omega), \dots, \eta_N^*(\omega)\} \leq \\
\leq \; & \sum_{i=1}^{K} P(\{\omega \in \Omega : \eta^j \notin \{\eta_1^*(\omega), \eta_2^*(\omega), \dots, \eta_N^*(\omega\}\}) \leq K\mathcal{E} \qquad (3.8)
\end{aligned}$$

follows, if $N \geq N_0$ is the case. As $\mathcal{E} > 0$ is arbitrary,

$$P(\{\omega \in \Omega : \{\eta^1, \dots, \eta^L\} \not\subset \{\eta_1^*(\omega), \dots, \eta_N^*(\omega)\}\}) \to 0 \qquad (3.9)$$

holds for each fixed L with $N \to \infty$, hence,

$$P(\{\omega \in \Omega : \{\eta^1, \dots, \eta^K\} \subset \{\eta_1^*(\omega), \dots, \eta_N^*(\omega)\}\}) \to 1 \qquad (3.10)$$

holds with $n \to \infty$. As $A_0 = \{\eta^1, \eta^2, \dots, \eta^K\}$, (3.10) yields that, for N increasing, with the probability increasing to 1 the relation

$$\begin{aligned}
& \Pi^N(\eta^*, \omega) = \bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i(\omega)) = \left(\bigvee_{j=1}^{\mathcal{T},K} \pi(\eta^j)\right) \vee^{\mathcal{T}} \left(\bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega))\right) = \\
= \; & \Pi(A_0) \vee^{\mathcal{T}} (\pi(\eta_1^*(\omega)) \vee^{\mathcal{T}} \pi(\eta_2^*(\omega)) \vee^{\mathcal{T}} \cdots \vee^{\mathcal{T}} \pi(\eta_N^*(\omega))) = \Pi(A_0), \qquad (3.11)
\end{aligned}$$

as $\Pi(A_0) \vee^{\mathcal{T}} \pi(\eta_\star) = \Pi(A_0)$ due to the assumptions imposed on A_0 and the principle of finite mathematical induction is applied. Hence, the relation

$$\lim_{N \to \infty} P(\{\omega \in \Omega : \Pi^N(\eta^*, \omega) = \Pi(A_0)\}) = 1 \qquad (3.12)$$

holds and the assertion is proved. □

4 Quasi-Supremum for Upper Semilattice-Valued Lattices

Let us reconsider, once more, the conditions of Theorem 3.2. If there exists a finite subset $A_0 \subset A$ meeting the condition (3.4), the value $\Pi(A_0)$ copies the properties of $\bigvee^{\mathcal{T}} A$ (if defined) at least in the sense that no element of A, joined with A_0, is able to make the value $\Pi(A_0)$ larger. The notion of π-quasi-supremum of A tries to define this property explicitly.

Definition 4.1 *Under the notation introduced above, the value* $\Pi(A_0) = \bigvee_{\eta \in A_0}^{\mathcal{T}} \pi(\eta)$ *is called the* π*-quasi-supremum of* A *and denoted by* $Q^\pi(A)$*, if* A_0 *is a finite subset of* $A \subset \Sigma$ *such that, for each* $\eta_\star \in A$, $\Pi(A_0 \cup \{\eta_\star\}) = \Pi(A_0)$ *holds. I.e.,* $\Pi(A_0)$ *is the* π*-quasi supremum of* A*, if*

$$\bigvee_{\eta \in A_0}^{\mathcal{T}} \pi(\eta) = \left(\bigvee_{\eta \in A_0}^{\mathcal{T}} \pi(\eta) \right) \vee^{\mathcal{T}} \pi(\eta_\star) \tag{4.1}$$

is valid for each $\eta_0 \in A$ *(in other notation, if* $\pi(\eta_\star) \leq_{\mathcal{T}} \Pi(A_0)$ *is the case).*

We have to prove that the value $Q^\pi(A)$, if defined, is defined uniquely (like it is the case for the standard supremum and infimum operations). Let $A \subset \Sigma$ be given, let $A_0 \subset A, B_0 \subset A$ be finite subsets of A such that both $\Pi(A_0)$ and $\Pi(B_0)$ define π-quasi-suprema $Q_1^\pi(A)$ and $Q_2^\pi(A)$. In this case, however, the identity $Q_1^\pi(A) = Q_2^\pi(A)$ follows. Indeed, let $A_0 = \{a_1, a_2, \dots a_K\} \subset A$, let $B_0 = \{b_1, b_2, \dots, b_L\} \subset A$. Then, applying (4.1) we obtain that

$$\begin{aligned} \Pi(A_0) &= \bigvee_{i=1}^{\mathcal{T},K} \pi(a_i) = \left(\bigvee_{i=1}^{\mathcal{T},K} \pi(a_i) \right) \vee^{\mathcal{T}} \pi(b_1) = \left(\bigvee_{i=1}^{\mathcal{T},K} \pi(a_i) \right) \vee^{\mathcal{T}} \pi(b_1) \vee^{\mathcal{T}} \pi(b_2) \\ &= \left(\bigvee_{i=1}^{\mathcal{T},K} \pi(a_i) \right) \vee^{\mathcal{T}} (\pi(b_1) \vee^{\mathcal{T}} \pi(b_2) \vee^{\mathcal{T}} \cdots \vee^{\mathcal{T}} \pi(b_2)) = \Pi(A_0) \vee^{\mathcal{T}} \Pi(B_0). \end{aligned} \tag{4.2}$$

Repeating this construction and consideration once more, but now starting from $\Pi(B_0)$ and adding, step by step, the values $\pi(a_1), \pi(a_2), \dots, \pi(a_K)$, we arrive at the equality $\Pi(B_0) = \Pi(B_0) \vee^{\mathcal{T}} \Pi(A_0)$, hence, the identity $\Pi(A_0) = \Pi(B_0)$ follows, so that the π-quasi-supremum $Q^\pi(A)$ is defined uniquely supposing that it is defined.

On the other side, the π-quasi-supremum $Q^\pi(A)$ need not be defined in general, i.e., it is possible that there is no finite $A_0 \subset A$ meeting the conditions imposed on $Q^\pi(A)$. E.g., consider $\Sigma = \{\eta^1, \eta^2, \dots\}$ and $T = \{t_1, t_2, \dots\}$ together with binary relation $\leq_{\mathcal{T}}$ such that $t_i <_{\mathcal{T}} t_j$ holds iff $i < j$ is the case (obviously, $T = \mathcal{N} = \{1, 2, \dots\}$ and the standard linear ordering $\leq$ on $\mathcal{N}$ will do). The pair $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ then defines the upper semi-lattice. Let $\pi : \Sigma \to T$ be defined by $\pi(\eta^i) = t_i$ for each $i \in \mathcal{N}$, let $A = \{\eta^{i_1}, \eta^{i_2}, \dots\}, i_1 < i_2 < \dots$ be any infinite subset of Σ. Then no finite subset $A_0 \subset A$ possesses the property that $\Pi(A_0) = \bigvee_{\eta \in A_0}^{\mathcal{T}} \pi(\eta)$ defines the π-quasi-supremum of A. Indeed, denote by $\alpha(A_0) \in \mathcal{N}$ the value $\alpha(A_0) = \max\{j \in \mathcal{N} : \eta^j \in A_0\}$. Then

$$\Pi(A_0) = \bigvee_{\eta \in A_0}^{\mathcal{T}} \pi(\eta) \ = \bigvee_{i=1, \eta^i \in A_0}^{\mathcal{T}, \alpha(A_0)} \pi(\eta^j) \ = \bigvee_{i=1}^{\mathcal{T}, \alpha(A_0)} t_i \ = \ t_{\alpha(A_0)}. \tag{4.3}$$

As the set A is infinite, there exists $j_0 \in \mathcal{N}$ such that $\eta^{j_0} \in A$ and $j_0 > \alpha(A_0)$ hold together. In this case, however, the relation $\pi(\eta^{j_0}) = t_{j_0} >_{\mathcal{T}} t_{\alpha(A_0)} = \Pi(A_0)$ follows (cf. (4.3)), so that the inequality $\Pi(A_0) \vee^{\mathcal{T}} \pi(\eta^{j_0}) = \pi(\eta^{j_0}) >_{\mathcal{T}} \Pi(A_0)$ holds. Hence, $\Pi(A_0)$ does not define the π-quasi-supremum of A.

It is perhaps worth being introduced explicitly, that the class of subsets $A \subset \Sigma$ for which π-quasi-supremum $Q^\pi(A)$ is defined is larger than the class of all π-finite subsets of Σ. Indeed, if A is π-finite, then there exists a finite set $A_0 \subset A$ such that $\pi(A_0) = \bigvee_{\eta \in A_0}^{\mathcal{T}} \pi(\eta) = \bigvee_{\eta \in A}^{\mathcal{T}} \pi(\eta)$, so that $\Pi(A_0)$ obviously defines the π-quasi-supremum of A. On the other side, when there exists a finite set $A_0 \subset A$ which defines the value $Q^\pi(A)$, it is possible that $A - A_0$ is an infinite set and the set of different values $\pi(\eta), \eta \in A - A_0$, is also infinite and for each $\eta \in A - A - 0$ the relation $\pi(\eta) \leq_{\mathcal{T}} \Pi(A_0)$ holds, hence, the set A is not π-finite.

Definition 4.2 *Let the notations and conditions of Definition 3.1 hold. Statistical estimation* $\Pi^N(\eta^*, \omega)$ *of the value* $\Pi(A)$ *for* $A \subset \Sigma$ *is* δ*-statistically optimal, where* δ *is a given real number, if the relation*

$$P(\{\omega \in \Omega : \Pi^{N+1}(\eta^*,\omega) = \Pi^N(\eta^*,\omega)\}) > 1-\delta \tag{4.4}$$

is valid.

Hence, $\Pi^N(\eta^*,\omega)$ is statistically optimal estimation of $\Pi(A)$ in the sense of Definition 3.1 iff $\Pi^N(\eta^*,\omega)$ is 0-statistically optimal in the sense of Definition 4.2.

Theorem 4.1 *Let $\langle \Omega, \mathcal{A}, P\rangle$ be a probability space, let $\langle \Sigma, \mathcal{P}(\Sigma)\rangle$ be the complete measurable space over a countable set Σ of elementary possibilistic states. Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}}\rangle$ be an upper semilattice, let $\pi : \Sigma \to T$ be a mapping such that $\bigvee^{\mathcal{T}}_{\eta\in\Sigma} \pi(\eta) = \mathbf{1}_{\mathcal{T}} = \bigvee^{\mathcal{T}}_{\eta\in\Sigma} \eta = \bigvee^{\mathcal{T}} T$ holds supposing that $\bigvee^{\mathcal{T}} T$ is defined. Let $A \subset T$ be given, let $\eta^* : \langle \Omega, \mathcal{A}, P\rangle \to \langle \Sigma, \mathcal{P}(\Sigma)\rangle$ be a random variable such that $P(\{\omega \in \Omega : \eta^*(\omega) = \eta\}) > 0$ is the case iff $\eta \in A$ holds.*

Let $\langle \eta_1^*, \eta_2^*, \dots\rangle$ be an infinite sequence of statistically independent random variables each of them being distributed identically with η^*, let $N = 1, 2, \dots,$ define

$$\Pi^N(\eta^*,\omega) = \bigvee_{i=1}^{\mathcal{T},N} \pi(\eta_i^*(\omega)). \tag{4.5}$$

Then, for each $\delta > 0$, the assertion

$$\lim_{N\to\infty} P\left(\left\{ \omega \in \Omega \ : \ \Pi^N(\eta^*,\omega) \ \begin{array}{l}\textit{defines a } \delta\textit{-statistically optimal sta-}\\ \textit{tistical extimation of the value } \Pi(A)\end{array} \right\}\right) = 1 \tag{4.6}$$

holds.

Proof: Let $\delta > 0$ be given. According to the conditions imposed on η^* and consequently, on each $\eta_1^*, \eta_2^*, \dots,$ there exits a finite set $A_0 \subset \Sigma$ such that

$$\begin{aligned} P(A_0) &= P(\{\omega \in \Omega : \eta^*(\omega) \in A_0\}) = P(\{\omega \in \Omega : \eta_i^*(\omega) \in A_0\}) = \\ &= \sum_{\eta \in A_0} P(\{\omega \in \Omega : \eta_1^*(\omega) = \eta\}) > 1-\delta \end{aligned} \tag{4.7}$$

holds. Hence, if $\Pi^N(\eta^*,\omega) \supset A_0$ is the case, then the inequality $\Pi^{N+1}(\eta^*,\omega) >_{\mathcal{T}} \Pi^N(\eta^*,\omega)$ may happen to be valid only when $\eta_{N+1}^*(\omega) \in A - A_0$ holds. However, the probability of this random event does not exceed δ, as proved in (4.7). As shown in Theorem 3.1, for each finite $A_0 \subset A$ the inclusion $\Pi^N(\eta^*,\omega) \supset A_0$ holds with the probability increasing to 1 with $N \to \infty$, the same limit assertion is valid for the probability that $\Pi^N(\eta^*,\omega)$ defines a δ-statistically optimal statistical estimation of the value $\Pi(A)$. The theorem is proved. □

5 Conclusions

In this contribution we analyzed an alternative mathematical model of uncertainty quantification and processing which combines two qualitatively different approaches to the idea of uncertainty. The first one takes the uncertainty in the sense of fuzziness and vagueness formalized above by the notion of possibilistic space $\langle E, \mathcal{E}\rangle$, the other approach is that of randomness, formalized by the standard notion of probability space and probability algorithm. What may be perhaps of interest is the mutual relation of both the uncertainty processing tools which copies the structure of probability algorithms, well-known from numerous theoretical and practical procedures.

At least the two directions of further developing of the basic ideas of mixed uncertainty quantification and processing models might be considered. First, more sophisticated details of the probability algorithm sketched above and perhaps some of its interesting applications should be analyzed and discussed. Second, different combinations of various models of uncertainty quantification and processing should be considered. E.g., in the first step probability algorithm for classical real-valued quantification are applied, but the quality of the achieved results, e.g., the distance of these results from the ideally perfect masterpiece, are quantified in the terms of a possibilistic lattice-valued measure.

The authors hope to have a possibility, sometimes in the future, to return to these and related problems more closely.

Important note concerning the references: for the reader's convenience, the list of references contains not only the items namely referred in the text, but also some works thematically tightly close to the subject of this paper, so making its understanding more easy.

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