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Abstract:

Considering complete lattice as the structures in which non-numerical fuzzy sets or, in our case, possibilistic measures take their values can be proved as too strong and restrictive for some practical cases when processing uncertainty degrees. Hence, some weaker structures over uncertainty degrees may be worth being considered and, in what follows, we investigate, for these purposes, possibilistic measures taking their values in upper semilattices. Two alternative models how to modify the original upper semilattice-valued mapping in order to obtain structures embeddable in a complete lattice are proposed and analyzed. In order to compare the qualities of the original mapping taken as a partial upper semilattice-valued possibilistic measure and the resulting complete lattice-valued possibilistic measure a complete lattice-valued entropy function applicable to both the kinds of possibilistic measure is introduced and its values for the original (partial upper semilattice-valued) and the resulting complete lattice-valued possibilistic measures are proved to be identical.

Keywords:
Complete lattice, upper valued semilattice, lattice-valued possibilistic distribution, lattice-valued entropy function embedding of incomplete lattice into a complete one

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1 Introduction, Preliminaries, Problem Formulation

The origins of possibilistic measures go back to the notion of fuzzy sets as introduced, in its real-valued version, by L. A. Zadeh in [16]. Given a nonempty space X, fuzzy set in X is a mapping $A : X \to [0,1]$ ascribing to each $x \in X$ the value $A(x) \in [0,1]$ taken as the degree of uncertainty, in the sense of fuzziness or vagueness, to which x belongs to A. In order to quantify the total amount of this uncertainty contained in the standard (crisp) subset $Y \subset X$ the value $\Pi_A(Y) = \sup\{A(x) : x \in Y\}$ was proposed and analyzed in more detail, again by L. A. Zadeh in [18]. He also proposed the term *possibilistic* (or *possibility*) *distribution* for the mapping $A : X \to [0,1]$, and the term *possibilistic* (or *possibility*) *measure* for the mapping $\Pi_A : \mathcal{P}(X) \to [0,1]$ induced by A on the power-set of all subsets of X.

When processing real-valued fuzzy sets, the most important operations over their values are those of supremum and infimum. In spite of the operations of addition (summary or series taking), substraction, multiplication, etc., which are the key ones for the standard measure and probability theory and which are closely related with the structures over the real line in general and the unit interval in particular, operations of supremum and infimum can be defined in a great variety of structures including the non-numerical ones. It was why very soon, within two-years distance after the Zadeh's pioneering work, the first mathematical model of fuzzy sets with non-numerical fuzziness degrees was presented in [8] by J. A. Goguen. This idea was re-written in the terms of lattice-valued possibilistic measures and carefully and in detail analyzed by G. de Cooman in [2].

To proceed in our explanation, some more formal apparatus will be necessary and the reader is supposed to be familiar with its most elementary foundations. Let $\mathcal{T} = \langle T, \leq_T \rangle$ be a partially ordered set, hence, T is a nonempty set (support of $\mathcal{T}$) and $\leq_T$ is a reflective, antisymmetric and transitive binary relation on T. Given $\emptyset \neq A \subset T$, the supremum $\bigvee A$ and infimum $\bigwedge T$ are defined in the standard way, if necessary, we write $\bigvee^{\mathcal{T}} A$ and $\bigwedge^{\mathcal{T}} A$ to avoid possible misunderstanding. In general, neither $\bigvee A$ nor $\bigwedge A$ need be defined for every $A \subset T$, the more specific examples are classified as follows:

$\mathcal{T} = \langle T, \leq_T \rangle$ is an upper (a lower, resp.) semilattice, if for each *finite* $A \subset T$ the supremum $\bigvee A$ (the infimum $\bigwedge A$, resp.) is defined. $\mathcal{T} = \langle T, \leq_T \rangle$ is a complete upper (a complete lower, resp.) semilattice, if for each $A \subset T$ the supremum $\bigvee A$ (the infimum $\bigwedge A$), resp.) is defined.

$\mathcal{T} = \langle T, \leq_T \rangle$ is a lattice, if it is an upper semi-lattice and a lower semi-lattice. $\mathcal{T} = \langle T, \leq_T \rangle$ is a complete lattice, if it is a complete upper semi-lattice and a complete lower semilattice.

The most simple examples may read as follows. The system $\mathcal{P}(\Omega)$ of a nonempty space Ω defines a complete lattice and the system of all finite subsets of Ω defines a complete lower semilattice and an (incomplete, if Ω is finite) upper semilattce, both with standard set inclusion as partial ordering. Other examples may be $(a,b) \subset R = (-\infty, \infty)$ for incomplete lattice and $\langle a,b \rangle \subset R$ for complete lattice, both with respect to the standard partial (linear, in this case) ordering $\leq$ on R.

In what follows, we focus our attention to the p.o.set $\langle T, \leq_{\mathcal{T}} \rangle$ such that, for each *nonempty* subset $A \subset T$ the infimum $\bigwedge^{\mathcal{T}} A = \bigwedge^{\mathcal{T}}_{t \in A} t$ is defined, and for each *finite* subset $A \subset T$ (including the empty set) the supremum $\bigvee^{\mathcal{T}} A = \bigvee^{\mathcal{T}}_{t \in A}$ is defined. Such a p.o.set $\langle T, \leq_{\mathcal{T}} \rangle$ will be called $*$-*lattice* and denoted by $\mathcal{T}^*$.

Let Ω be a nonempty set, let $\mathcal{T} = \langle T, \leq \rangle$ be a p.o.set, let $\pi : \Omega \to T$ be a total, i.e., for each $\omega \in \Omega$ defined mapping. If $\mathcal{T}$ defines a complete lattice and the condition of normalization $\bigvee_{\omega \in \Omega} \pi(\omega) = \mathbf{1}_{\mathcal{T}} = \bigvee T$ holds, then π defines a $\mathcal{T}$-*(valued possibilistic) distribution* on Ω, for each $A \subset \Omega$ the value $\Pi(A) = \bigvee_{\omega \in A} \pi(\omega)$ is defined ($\pi(\emptyset) = \oslash_{\mathcal{T}} = \bigwedge T$) and the (total) mapping $\Pi : \mathcal{P}(\Omega) \to T$ is called the $\mathcal{T}$-*(valued possibilistic) measure* induced by π on $\mathcal{P}(\Omega)$, obviously, $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$. This case of complete lattices and possibilistic distributions over then is analyzed and investigated in detail in [2, 3, 10], and elsewhere, and it is the most simple in the sense that the existence of all supremum and infimum values in question is assumed a priori.

Let Ω, T and $\pi : \Omega \to T$ be as above, but this time let $\mathcal{T}^* = \langle T, \leq_T \rangle$ be an $*$-lattice. So, for each $\emptyset \neq A \subset T$ the infimum value $\bigwedge A$ is defined in $\mathcal{T}^*$, but the supremum value $\bigvee^{T}$ is defined only if A is finite. The intuition behind is very simple – such a lattice describes the behaviour of non-negative characteristics whose positive values may take no matter which finite sizes: but there is no universal upper bound over the possible values.

Taking $\mathcal{T}^*$ as the structure over T, the definition of $\Pi(A)$ may fail for infinite sets $A \subset \Omega$ and the definition of π^*-possibilistic distribution π and $\mathcal{T}^*$-possibilistic measure Π may fail as well. If $\mathcal{T}^*$ is an $*$-lattice, the supremum value $\bigvee_{\omega \in A} \pi(\omega)$ is obviously defined for nonempty finite A's$\subset \Omega$, but also for such infinite A's for which the number of different values $\pi(\omega)$, $\omega \in A$ is finite, there subsets of Ω will be called *π-finite*. Each nonempty finite subset of Ω is obviously also π-finite for each mapping $\pi : \Omega \to T$, but the inverse implication need not be the case. Indeed, take $\Omega = \{\omega_1, \omega_2, \dots\}, \pi(\omega_j) = t_1 \in T$, if j is odd, $\pi(\omega_j) = t_2$ for j even. Then $\Pi(A)$ is defined for each $\emptyset \neq A \subset \Omega$, as each $\emptyset \neq A \subset \Omega$ is π-finite. Hence, the mapping $\pi : \Omega \to T$ does not define a $\mathcal{T}$-possiblistic distribution on Ω and does not induce a $\mathcal{T}$-possiblistic measure Π on $\mathcal{P}(\Omega)$ iff there exists an (obviously infinite) subset $A_0 \subset \Omega$ such that the set $\{\pi(\omega) : \omega \in A_0\}$ is infinite. So, the most simple (from the abstract mathematical point of view) approximation of the mapping $\pi : \Omega \to T$, with respect to $*$-lattice $\mathcal{T}^* = \langle T, \leq_{\mathcal{T}} \rangle$ by a possibilistic distribution on Ω enabling to define possibilistic measure Π on $\mathcal{P}(\Omega)$ reads as follows.

Let $\mathcal{T}^* = \langle T, \leq_{\mathcal{T}} \rangle$ be an $*$-lattice, let Ω be a nonempty space, let $\pi : \Omega \to T$ be a mapping, let $\emptyset \neq \Omega_0 \subset \Omega$ be a π-finite subset of Ω, so that the value $t_0 = \bigvee_{\omega \in \Omega_0} \pi(\omega) = \Pi(\Omega_0)$ is defined, let $t_1 \in T$ be a fixed value such that $t_1 \leq t_0$ holds. Define the mapping $\pi^0 : \Omega \to T$ in this way: $\pi^0(\omega) = \pi(\omega)$, if $\omega \in \Omega_0, \pi^0(\omega) = t_1$, if $\omega \in \Omega - \Omega_0$. Hence, the set $\{\pi^0(\omega) : \omega \in \Omega\}$ is finite, as it is a subset of the set $\{\pi^0(\omega) : \omega \in \Omega_0\} \cup \{t_1\}$. Consequently, the space Ω is π^0-finite and the identity

$$\Pi^0(\Omega) = \bigvee_{\omega \in \Omega} \pi^0(\omega) = \bigvee_{\omega \in \Omega_0} \pi(\omega) = t_0 \tag{1.1}$$

holds, as $\pi^0(\omega) = t_1 \leq t_0$ holds for each $\omega \in \Omega - \Omega_0$. So, $\pi^0 : \Omega \to T$ defines a $\mathcal{T}_0$-valued possibilistic distribution on Ω, where $\mathcal{T}_0 = \langle T_0, \leq_{\mathcal{T}} \restriction T_0 \rangle, T_0 = \{t \wedge t_0 : t \in T\}$, and this possibilistic distribution induces the $\mathcal{T}_0$-possibilistic measure Π^0 on $\mathcal{P}(\Omega)$.

2 Lattice-Valued Possibilistic Entropy

When choosing the π-finite set $\Omega_0 \subset \Omega$ from which the construction of the possibilistic distribution π^0 on Ω begins, at the first sight it seems to be intuitive to choose the greatest π-finite subset $\Omega_0 \subset \Omega$. However, up to the most simple case when the whole set Ω is π-finite, the greatest π-finite proper subset of Ω does not exist. Indeed, if Ω is not π-finite, then no $\Omega_0 \subset \Omega$ such that $\Omega - \Omega_0$ is π-finite may be π-finite. If this were the case, then Ω would be also π-finite, as $\Omega = \Omega_0 \cup (\Omega - \Omega_0)$ holds. So, if Ω is not π-finite, then for each π-finite $\Omega_0 \subset \Omega$ the set $\Omega - \Omega_0$ is infinite and Ω_0 may be extended just to an Ω_1 such that $\Omega_0 \subset \Omega_1 \subset \Omega$ holds, but $\Omega_1 - \Omega_0$ is finite and $\Omega - \Omega_1$ is infinite.

Now, the problem is twofold. First, to find whether, and in which sense, the possibilistic distribution π^0 could play the role in a reasonable sense similar to that one played by π on $\mathcal{T}$ supposing that $\mathcal{T}$ were a complete lattice. Second, two free parameters, the set Ω_0 and the value $t_1 (\leq t_0 = \Pi(\Omega_0))$ enter our construction of π^0 as free parameters and we may and will ask, how the changes of these values influence some reasonable quality criterion applied to the resulting possibilistic distribution π^0 on Ω. For these purposes let us recall the notion of lattice-valued possibilistic entropy function.

Definition 2.1 *Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be a complete lattice, let $\pi : \Omega \to T$ be a $\mathcal{T}$-possibilistic distribution on a nonempty space, so that $\bigvee_{\omega \in \Omega} \pi(\omega) = \mathbf{1}_{\mathcal{T}}$ $(= \bigvee_{t \in T} t)$. The $\mathcal{T}$-(valued possibilistic) entropy function I will be defined by the Sugeno integral $I(\pi)$ of the nonincreasing (in $\pi(\omega)$) lattice-valued function $\Pi(\Omega - \{\omega\})$. Hence,*

$$\begin{aligned} I(\pi) &= \int \Pi(\Omega - \{\omega\}) d\pi(\omega) = \bigvee_{\omega \in \Omega} [\pi(\omega) \wedge \Pi(\Omega - \{\omega\})] = \\ &= \bigvee_{\omega \in \Omega} [\pi(\omega) \wedge \bigvee_{\omega_1 \in \Omega, \omega_1 \neq \omega} \pi(\omega_1)]. \end{aligned} \tag{2.1}$$

The weak point of this entropy function consists in the fact that if there are at least two different $\omega_1, \omega_2 \in \Omega$ such that $\pi(\omega_1) = \pi(\omega_2) = \mathbf{1}_{\mathcal{T}}$, then

$$I(\pi) = \Pi(\Omega - \{\omega_1\}) \wedge \pi(\omega_1) = \mathbf{1}_{\mathcal{T}} \wedge \mathbf{1}_{\mathcal{T}} = \mathbf{1}_{\mathcal{T}}, \tag{2.2}$$

as $\omega_2 \in \Omega - \{\omega_1\}$ holds. Nevertheless, in what follow we will apply this entropy function to the distributions π^0. In this case, and with the values $t_1 = t_0 = \bigvee_{\omega \in \Omega_0} \pi(\omega)$, we obtain that $I(\pi^0) = t_0 = \mathbf{1}_{\mathcal{T}_0}$, as t_0 is the unit (the maximum) element in $\mathcal{T}_0 = \langle T_0, \leq_{\mathcal{T}} \restriction T_0 \rangle$, $T_0 = \{t \in T, t \leq t_0\}$, supposing that the space Ω is not π-finite. The same will be the case when $\pi(\omega_1) = \pi(\omega_2) = t_0$ for at least two $\omega_1, \omega_2 \in \Omega_0$. Consequently, the application of I to π_0 will lead to non-trivial results when there exists just one $\omega_0 \in \Omega$ such that $\pi(\omega_0) = t_0$.

Theorem 2.1 *Let $\mathcal{T}^* = \langle T, \leq_{\mathcal{T}} \rangle$ be an $*$-lattice, let Ω be a nonempty space, let $\pi : \Omega \to T$ be a (total) mapping. Let Ω_0 be a fixed π-finite subset of Ω so that the value $t_0 = \Pi(\Omega_0) = \bigvee_{\omega \in \Omega_0} \pi(\omega) \in T$ is defined. Let $\pi^0(\omega) = \pi(\omega)$, if $\omega \in \Omega_0$, let $\pi^0(\omega) = t_1$, if $\omega \in \Omega - \Omega_0$, where $t_1 < t_0$ is a fixed element of T, let there exist $\omega_0 \in \Omega_0$ such that $\pi(\omega_0) = t_0$ is the case, let I be the lattice-valued entropy function defined by (2.2). Then the relation*

$$I(\pi^0) = t_1 \vee \left(\bigvee_{\omega \in \Omega_0, \omega \neq \omega_0} \pi(\omega) \right) = t_1 \vee \Pi(\Omega - \{\omega_0\}) \tag{2.3}$$

holds.

Proof: As shown in Section 1, the mapping π^0 defines a $\mathcal{T}_0$-valued possibility distribution on the complete lattice $\mathcal{T}_0$ over T_0. The entropy value $I(\pi^0)$ may be separated into three items as follows.

$$\begin{aligned} I(\pi^0) &= \bigvee_{\omega \in \Omega} (\pi^0(\omega) \wedge \Pi^0(\Omega - \{\omega\})) = \\ &= \left[\bigvee_{\omega \in \Omega_0 - \{\omega_0\}} (\pi^0(\omega) \wedge \Pi^0(\Omega - \{\omega\})) \right] \vee \\ & \quad \left[\bigvee_{\omega = \omega_0} (\pi^0(\omega) \wedge \Pi^0(\Omega - \{\omega\})) \right] \vee \\ & \quad \left[\bigvee_{\omega \in \Omega - \Omega_0} (\pi^0(\omega) \wedge \Pi^0(\Omega - \{\omega\})) \right]. \end{aligned} \tag{2.4}$$

For each $A \subset \Omega, \pi^0(\omega)$ takes only finitely many values, if ω ranges over A, so that each $A \subset \Omega$ is π^0-finite and the values $\Pi^0(A)$, in particular, the values $\Pi^0(\Omega - \{\omega\})$, are defined for each $\omega \in \Omega$.

Let us distinguish two cases: (i) there is only *one* $\omega_0 \in \Omega_0$ such that $\pi(\omega_0) = t_0 = \bigvee_{\omega \in \Omega_0} \pi(\omega)$, and (ii) there are at least *two* such ω_0's, let us denote them ω_0^1, ω_0^2. If (i) is the case, then $\Pi^0(\Omega - \{\omega\}) = t_0$, if $\omega \neq \omega_0$, and $\Pi^0(\Omega - \{\omega_0\}) = t_1 \vee \bigvee_{\omega \in \Omega, \omega \neq \omega_0} \pi(\omega)$ for ω_0, if (ii) is valid, then $\Pi^0(\Omega - \{\omega\}) = \bigvee_{\omega \in \Omega_0} \pi(\omega) = t_0$ holds for each $\omega \in \Omega$. Hence, the values $\pi^0(\omega) \wedge \Pi^0(\Omega - \{\omega\})$ are in T and range over a finite subset of T, so that the supremum values occuring in (2.4) are defined.

So, if (i) holds, then for each $\omega \in \Omega_0 - \{\omega_0\}$ the relations $\omega_0 \in \Omega - \{\omega\}$ and $\Pi^0(\Omega - \{\omega\}) = \pi(\omega_0) = t_0$ hold. Consequently, for each $\omega \in \Omega_0 - \{\omega_0\}$, $\pi^0(\omega) \wedge \Pi^0(\Omega - \{\omega\}) = \pi^0(\omega) \wedge t_0 = \pi^0(\omega)$, hence, we obtain that the relation

$$\bigvee_{\omega \in \Omega_0, \omega \neq \omega_0} [\pi^0(\omega) \wedge \Pi^0(\Omega - \{\omega\})] = \bigvee_{\omega \in \Omega_0, \omega \neq \omega_0} \pi^0(\omega) = \bigvee_{\omega \in \Omega_0, \omega \neq \omega_0} \pi(\omega) \tag{2.5}$$

is valid, as for each $\omega \in \Omega_0$ the values $\pi^0(\omega)$ and $\pi(\omega)$ are identical. The following line in (2.4) yields that

$$\pi^0(\omega_0) \wedge \pi^0(\Omega - \{\omega_0\}) = t_0 \wedge \bigvee_{\omega \in \Omega_0, \omega \neq \omega_0} \pi^0(\omega). \tag{2.6}$$

If $\omega \in \Omega - \Omega_0$ is the case, then $\omega \neq \omega_0$ follows (as $\omega_0 \in \Omega_0$), so that $\omega_0 \in \Omega - \{\omega\}$ and $\Pi^0(\Omega - \{\omega\}) = t_0$ hold as well, hence,

$$\bigvee_{\omega \in \Omega - \Omega_0} [\pi^0(\omega) \wedge \Pi^0(\Omega - \{\omega\})] = \bigvee_{\omega \in \Omega_0} [t_1 \wedge t_0] = t_1, \tag{2.7}$$

according to the definition of $\pi^0(\omega)$ for $\omega \in \Omega - \Omega_0$. Combining together (2.5), (2.6), and (2.7), we obtain that in the case (i), $I(\pi^0) = t_1 \vee [\bigvee_{\omega \in \Omega_0, \omega \neq \omega_0} \pi(\omega)]$ holds.

If (ii) is the case (i.e., $\pi(\omega_0^1) = \pi(\omega_0^2) = \bigvee_{\omega \in \Omega_0} \pi(\omega)$ holds for different $\omega_0^1, \omega_0^2 \in \Omega_0$,) then $\Pi^0(\Omega - \{\omega\}) = \pi^0(\omega_0)$ for each $\omega \in \Omega$, so that

$$I(\pi^0) = \bigvee_{\omega \in \Omega} (\pi^0(\omega) \wedge \pi^0(\omega)) = \bigvee_{\omega \in \Omega} \pi^0(\omega) = \pi^0(\omega_0) \tag{2.8}$$

holds for both $\omega_0 = \omega_0^1, \omega_0 = \omega_0^2$. Consequently, for both ω_0^1, ω_0^2, either ω_0^2 or ω_0^1 is in $\Omega - \{\omega_0^1\}$ (in $\Omega - \{\omega_0^2\}$, resp.), so that

$$\bigvee_{\omega \in \Omega} \pi^0(\omega) = \bigvee_{\omega \in \Omega, \omega \neq \omega_0} \pi^0(\omega) = \pi^0(\omega_0) = I(\pi^0) \tag{2.9}$$

follows and the assertion is proved also in the case (ii). □

Consequently, the value $I(\pi^0)$ is nontrivial (in the sense that it is smaller than the maximum value $\bigvee_{\omega \in \Omega_0} \pi(\omega)$) only when there exists only one $\omega_0 \in \Omega_0$ such that $\pi(\omega_0) = \bigvee_{\omega \in \Omega_0} \pi(\omega)$ and this value $\pi(\omega_0)$ is strictly greater than each $\pi(\omega), \omega \neq \omega_0$, in the sense that $\bigvee_{\omega \in \Omega, \omega \neq \omega_0} \pi(\omega) < \pi(\omega_0)$ holds.

3 A Simple Illustrative Example

As an illustration, the following simple example may be introduced. Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be an $*$-lattice, e.g., the set of all non-negative integers under their standard ordering, let Ω be a nonempty space, let $\pi : \Omega \to T$ be given, let $t \in T$ be fixed. Take $\omega_1 \in \Omega$ such that $t < \pi(\omega_1)$ holds, take $\Omega_1 = \{\omega_1\}$. The set Ω_1 is obviously finite, hence, π-finite, so that we may define the mapping $\pi^1 : \Omega \to T$ analogously to π^0 above and we obtain: $\pi^1(\omega_1) = \pi(\omega_1), \pi^1(\omega) = t < \pi(\omega_1)$ for each $\omega \in \Omega - \{\omega_1\}$. According to (2.1) we define the entropy function I and we obtain for π^1, that

$$\begin{aligned} I(\pi^1) &= \bigvee_{\omega \in \Omega} [\pi^1(\omega) \wedge \Pi^1(\Omega - \{\omega\})] = \\ &= [\pi^1(\omega_1) \wedge \Pi^1(\Omega - \{\omega_1\})] \vee \bigvee_{\omega \in \Omega, \omega \neq \omega_1} [\pi^1(\omega) \wedge \Pi^1(\Omega - \{\omega\})]. \end{aligned} \tag{3.1}$$

As $\pi(\omega) = t$ for each $\omega \neq \omega_1, \Pi^1(\Omega - \{\omega_1\}) = t$ follows. In the same case, $\omega_1 \in \Omega - \{\omega\}$, hence, $\Pi^1(\Omega - \{\omega\}) = \pi(\omega_1)$ holds, so that (3.1) yields that

$$I(\pi^1) = [\pi^1(\omega_1) \wedge t] \vee \bigvee_{\omega \in \Omega, \omega \neq \omega_1} (t \wedge \pi^1(\omega_1)) = t \wedge \pi^1(\omega_1) = t \tag{3.2}$$

holds due to the assumption that $t < \pi(\omega_1)$ is valid.

Obviously, when taking $\omega_2 \in \Omega, \omega_2 \neq \omega_1$, setting $\Omega_2 = \{\omega_2\}$, introducing the mapping $\pi^2 : \Omega \to T$ analogously to π^1 but now with Ω_2 (π is the same as above), and supposing that $t < \pi^2(\omega_2)(= \pi(\omega_2))$ holds for the same $t \in T$ as before, we obtain that $I(\pi^2) = t \wedge \pi^2(\omega_2) = t$ holds as well, so that $I(\pi^*) = t$ for each π^* induced from π by $\Omega_* = \{\omega^*\}$ such that $t < \pi(\omega^*)$ holds.

Let us combine Ω_1 and Ω_2 in this way. Keep $\mathcal{T}, \pi, \Omega, \omega_1, \omega_2$ and t as above and set $\Omega_{12} = \{\omega_1, \omega_2\}$. Consequently, define $\pi^{12} : \Omega \to T$ in this way: $\pi^{12}(\omega_1) = \pi(\omega_1)$, $\pi^{12}(\omega_2) = \pi(\omega_2), \pi^{12}(\omega) = t$ ($<_{\mathcal{T}} \pi(\omega_1), \pi(\omega_2)$, according to the assumptions) for each $\omega \in \Omega - \{\omega_1, \omega_2\}$. For the entropy value $I(\pi^{12})$ we obtain that

$$
\begin{aligned}
I(\pi^{12}) &= \bigvee_{\omega\in\Omega} (\pi^{12}(\omega) \wedge \Pi^{12}(\Omega - \{\omega\})) = \\
&= [(\pi^{12}(\omega_1)) \wedge \Pi^{12}(\Omega - \{\omega_1\})] \vee [(\pi^{12}(\omega_2)) \wedge \Pi^{12}(\Omega - \{\omega_2\})] \vee \\
&\vee \bigvee_{\omega\in\Omega-\{\omega_1,\omega_2\}} [\pi^{12}(\omega) \wedge \Pi^{12}(\Omega - \{\omega\})] = \\
&= [\pi(\omega_1) \wedge \pi(\omega_2)] \vee [\pi(\omega_2) \wedge \pi(\omega_1)] \vee [t \wedge \Pi(\{\omega_1,\omega_2\})] = \pi(\omega_1) \wedge \pi(\omega_2),
\end{aligned}
\tag{3.3}
$$

applying the inequalities $t <_{\mathcal{T}} \pi(\omega_1), t <_{\mathcal{T}} \pi(\omega_2)$, and the elementary relations valid in p.o.sets.

Let us analyze the same situation applying Theorem 2.1 to the particular case when $\Omega_{12} = \{\omega_1, \omega_2\}$ and $\pi(\omega_1) > t, \pi(\omega_2) > t$ holds. So, $\Omega_{12} = \{\omega_1\} \cup \{\omega_2\}$, hence $\bigvee_{\omega\in\Omega_0} \pi(\omega) = \pi(\omega_1) \vee \pi(\omega_2)$ (here and below Ω_0 denotes Ω_{12}) is defined and we suppose, according to the conditions imposed in Theorem 2.1, that there exists $\omega_0 \in \Omega_0$ such that $\pi(\omega_0) = \pi(\omega_1) \vee \pi(\omega_2)$. This may be the case only when $\pi(\omega_1) \leq \pi(\omega_2)$ (hence, $\omega_0 = \omega_2$ and $\pi(\omega_0) = \pi(\omega_2)$) holds, or when $\pi(\omega_2) \leq \pi(\omega_1)$ (hence, $\omega_0 = \omega_1$ and $\pi(\omega_0) = \pi(\omega_1)$) is the case, without any loss of generality we may suppose that $\pi(\omega_1) \leq \pi(\omega_2)$ holds. Relation (2.3) then yields that

$$
I(\pi^{12}) = t \vee \Pi^0(\Omega_0 - \{\omega_0\}) = t \vee \pi(\omega_1) = \pi(\omega_1) = \pi(\omega_1) \wedge \pi(\omega_2), \tag{3.4}
$$

as in (3.3).

Lemma 3.1 *Let the notations and conditions of Theorem 2.1 hold, let* $t_1 = \oslash_{\mathcal{T}}$ $(= \bigwedge_{t\in T} t)$, *let* $\omega_1 \in \Omega - \Omega_0$ *be such that* $\pi(\omega_1) \geq \pi(\omega_0) = \bigvee_{\omega\in\Omega_0} \pi(\omega)$ *hold, let* $\Omega_1 = \Omega_0 \cup \{\omega_1\}$, *let* $\pi^1(\omega) = \pi(\omega)$ *for each* $\omega \in \Omega_1$, *let* $\pi^1(\omega) = \oslash_{\mathcal{T}} (= t_1$ *in our particular case) for each* $\omega \in \Omega - \Omega_1$. *Then*

$$
I(\pi^1) = I(\pi^0) \vee \pi(\omega_0) = \Pi^0(\Omega_0) = \Pi^1(\Omega_1 - \{\omega_1\}). \tag{3.5}
$$

Proof: As the set Ω_0 is supposed to be π-finite, the same is valid for Ω_1 and there exists $\omega \in \Omega$, namely ω_1, such that $\pi^1(\omega_1) = \bigvee_{\omega\in\Omega_1} \pi^1(\omega)$. Hence, applying Theorem 2.1, we obtain that

$$
\begin{aligned}
I(\pi^1) &= \bigvee_{\omega\in\Omega_1-\{\omega_1\}} \pi^1(\omega) = \bigvee_{\omega\in\Omega_0} \pi(\omega) = \Pi^0(\Omega_0) = \Pi^0(\Omega_0 - \{\omega_0\}) \vee \Pi^0(\{\omega_0\}) = \\
&= I(\pi^0) \vee \pi(\omega_0)
\end{aligned}
\tag{3.6}
$$

and the assertion is proved. □

The following corollary of Lemma 3.1 is easy to verify.

Corollary 3.1 *If* $\pi(\omega_1) < \pi(\omega_0) = \bigvee_{\omega\in\Omega_0} \pi(\omega)$ *is the case, then the identity*

$$
I(\pi^1) = \bigvee_{\omega\in\Omega_1-\{\omega^1\}} \Pi(\omega) = \bigvee_{\omega\in\Omega_0} \pi(\omega) = \bigvee_{\omega\in\Omega_0-\{\omega_0\}} \pi(\omega) = I(\pi^0) \tag{3.7}
$$

easily follows.

4 An Alternative Approximation of Possibilistic Distributions for ∗-Lattice-Valued Mappings

As above, let $\mathcal{T}^* = \langle T, \leq_{\mathcal{T}} \rangle$ be an asterisk lattice, let Ω be a nonempty set, let $\pi : \Omega \to T$ be a mapping, let $q_0 \in T$ be given, let $\pi_0 : \Omega \to T$ be defined by $\pi_0(\omega) = \pi(\omega) \wedge q_0$. Then π_0 is a mapping which takes its values in the structure $\mathcal{T}_0 = \langle T_0, \leq_{\mathcal{T}} \restriction T_0 \rangle$, where $T_0 = \{t \in T : t \leq q_0\} = \{t \wedge q_0 : t \in T\}$, and $\leq_{\mathcal{T}} \restriction T_0$ is the restriction of $\leq_{\mathcal{T}}$ on T_0, so that $t_1 \leq_{\mathcal{T}} \restriction T_0\ t_2$ holds iff $t_1, t_2 \in T_0$ and $t_1 \leq_{\mathcal{T}} t_2$ is the case. Let there exist $\omega_0 \in \Omega$ such that $\pi(\omega_0) \geq q_0$ holds, and the mapping $\pi_0 : \Omega \to T_0$ defines a lattice-valued possibilistic distribution on Ω taking values in the complete lattice $\mathcal{T}_0$.

We will suppose, that given $q_0 \in T$, the structure $\mathcal{T}_0 = \langle T_0, \leq_{\mathcal{T}} \restriction T_0 \rangle$ defines a complete lattice. In this case, the restriction $\leq_{\mathcal{T}} \restriction T_0$ of the partial ordering $\leq_{\mathcal{T}}$ on T_0 also defines a partial ordering so that what remains to be proved is the existence of the supremum $\bigvee^{\mathcal{T}_0} A$ and the infimum $\bigwedge^{\mathcal{T}_0} A$ for each $\emptyset \neq A \subset T_0$. For $\bigwedge^{\mathcal{T}_0} A$ we obtain easily that this infimum exists due to the existence of the infimum $\bigwedge^{\mathcal{T}_0} A$ (recall that $\mathcal{T}^*$ is an asterisk lattice) and both the infimum values are identical. If $\bigvee^{\mathcal{T}} A$ is defined for $A \subset T_0$, then $\bigvee^{\mathcal{T}} A \leq q_0$ holds, so that $\bigvee^{\mathcal{T}_0} A$ also exists and is identical with $\bigvee^{\mathcal{T}} A$. If $\bigvee^{\mathcal{T}} A$ does not exists, then the value q_0 is the only upper bound for all elements of A in the sense of the relation $\leq_{\mathcal{T}} \restriction T_0$, so that this value defines the supremum value $\bigvee^{\mathcal{T}_0} A$ of the set $A \subset T_0$. Consequently, the p.o.set $\mathcal{T}_0 = \langle T_0, \leq_{\mathcal{T}} \restriction T_0 \rangle$ defines a complete lattice. As $\pi(\omega_0) \geq q_0$ is supposed to hold, we obtain that $\pi_0(\omega_0) = \pi(\omega_0) \wedge q_0 = q_0 = \mathbf{1}_{\mathcal{T}_0}$ is valid, so that $\bigvee_{\omega \in \Omega} \pi_0(\omega) = q_0 = \mathbf{1}_{\mathcal{T}_0}$, hence, π_0 defines a $\mathcal{T}_0$-valued possibilsitic distribution on Ω.

Theorem 4.1 *Let the $*$-lattice $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$, nonempty set Ω, mapping $\pi : \Omega \to T$ and the entropy function $I(\pi)$ be defined as above, let $q_0 \in T$ be given such that there exists $\omega_0 \in \Omega$ with the property $\pi(\omega_0) \geq q_0$, let $\pi_0 : \Omega \to T$ be defined by $\pi_0(\omega) = \pi(\omega) \wedge q_0$ for each $\omega \in \Omega$, here $\wedge$ denotes the infimum operation in $\mathcal{T}$. Then the value $\Pi_0(A) = \bigvee_{\omega \in A}^{\mathcal{T}_0} \pi_0(\omega)$ is defined for each $A \subset \Omega$ and the relation*

$$I(\pi_0) = \Pi_0(\Omega - \{\omega_0\}) \tag{4.1}$$

holds.

Proof: Let $\omega_0 \in \Omega$ be such that $\pi(\omega_0) \geq q_0$, hence, $\pi_0(\omega_0) = \pi(\omega_0) \wedge q_0 = q_0$ holds. For $I(\pi_0) = \bigvee_{\omega \in \Omega} (\pi_0(\omega) \wedge \Pi_0(\Omega - \{\omega_0\}))$ we obtain that if $\omega \neq \omega_0$, then $\omega_0 \in \Omega - \{\omega\}$ is the case, so that $\Pi_0(\Omega - \{\omega\}) = \bigvee_{\omega_1 \in \Omega, \omega_1 \neq \omega} \pi(\omega_1) = \pi(\omega_0) = q_0$ follows. If $\omega = \omega_0$, we obtain that

$$\pi_0(\omega) \wedge \Pi_0(\Omega - \{\omega\}) = \pi(\omega_0) \wedge \Pi_0(\Omega - \{\omega_0\}) = q_0 \wedge \Pi_0(\Omega - \{\omega_0\}) = \Pi_0(\Omega - \{\omega_0\}). \tag{4.2}$$

Combining both the cases together, the result

$$\begin{aligned} I(\pi_0) &= (\pi(\omega_0) \wedge \Pi_0(\Omega - \{\omega_0\})) \vee \bigvee_{\omega \in \Omega, \omega \neq \omega_0}^{\mathcal{T}_0} (\pi_0(\omega) \wedge \Pi_0(\Omega - \{\omega\})) = \\ &= (q_0 \wedge \Pi_0(\Omega - \{\omega_0\})) \vee \bigvee_{\omega \in \Omega - \{\omega_0\}}^{\mathcal{T}_0} (\pi_0(\omega) \wedge \pi_0(\omega_0)) = \\ &= \Pi_0(\Omega - \{\omega_0\}) \vee \bigvee_{\omega \in \Omega - \{\omega_0\}}^{\mathcal{T}_0} \pi_0(\omega) = \\ &= \Pi_0(\Omega - \{\omega_0\}) \vee \Pi_0(\Omega - \{\omega_0\}) = \Pi_0(\Omega - \{\omega_0\}) \end{aligned} \tag{4.3}$$

follows, so that the assertion is proved. □

Corollary 4.1 *The value $I(\pi_0)$ in (4.1) is defined uniquely no matter how large the set $\{\omega \in \Omega : \pi_0(\omega) = q_0\}$ may be.*

Proof: Supposing that there exists only one ω_0 such that $\pi_0(\omega_0) = q_0$ (i.e., such that $\pi(\omega_0) \geq q_0$ holds), the assertion is obvious. If there exist at least two elements $\omega_0, \omega_1 \in \Omega$ such that $\pi_0(\omega_0) = \pi_0(\omega_1) = q_0$ holds, then $\omega_1 \in \Omega - \{\omega_0\}$, hence, $\Pi_0(\Omega - \{\omega_1\}) = \Pi_0(\Omega - \{\omega_0\}) = q_0 = I(\pi_0)$. So, $I(\pi_0)$ is defined uniquely and takes the maximum value q_0 (let us recall that q_0 defines the maximum or the unit element of the complete lattice $\mathcal{T}_0 = \langle T_0, \leq_{\mathcal{T}_0} \rangle$). □

5 Sequential Applications of Approximation Operations to Lattice-Valued Mappings

In order to be able to define repeated applications of the operation $(\cdot)^0$ leading from a mapping $\pi : \Omega \to T$ to the mapping $\pi^0 : \Omega \to T$ given $\Omega_0 \subset \Omega$ and $t_1 \in T$, let us rewrite this operation in the following way. Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be an $*$-lattice, let π be a mapping which takes Ω into T, let Ω_0 be a π-finite subset of Ω, let $t \in T$ be given. Then $\varphi(\Omega_0, t)$ is the operation ascribing to π the mapping $\varphi(\Omega_0, t)(\pi) : \Omega \to T$ such that

$$(\varphi(\Omega_0, t)(\pi))(\omega) = \pi(\omega), \text{ if } \omega \in \Omega_0, (\varphi(\Omega_0, t)(\pi))(\omega) = t, \text{ if } \omega \in \Omega - \Omega_0.$$

Hence, the particular mapping $\pi^0 : \Omega \to T$, defined and analyzed in more detail above, would be denoted by $(\varphi(\Omega_0, t_1))(\pi)$.

Let $t \in T$, let $Q = \langle \Omega_1, \Omega_2, \dots \rangle$ be an infinite sequence of nonempty π-finite subsets of Ω, let $\varphi^{(n)}(Q, t)$ be the operation ascribing to each $\pi : \Omega \to T$ the mapping $(\varphi^{(n)}(Q, t))(\pi) : \Omega \to T$ defined by induction on n in this way: $(\varphi^{(1)}(Q, t))(\pi) = (\varphi(\Omega_1, t))(\pi)(= \pi^0$ in the former notation with $\Omega_0 = \Omega_1, t_1 = t)$,

$$(\varphi^{(n)}(Q, t))(\pi) = (\varphi(\Omega_n, t))((\varphi^{(n-1)}(Q, t))(\pi)). \tag{5.1}$$

Theorem 5.1 *Under the notations and conditions introduced in (5.1) the relation*

$$(\varphi^{(n)}(Q, t))(\pi) = \left(\varphi \left(\left(\bigcap_{i=1}^{n} \Omega_i \right), t \right) \right)(\pi) \tag{5.2}$$

holds. Hence, $((\varphi^{(n)}(Q, t))(\pi)(\omega) = \pi(\omega)$ *holds, if* $\omega \in \Omega_i$ *holds for each* $1 \leq i \leq n$, *and* $((\varphi^{(n)}(Q, t))$ $(\pi))(\omega) = t$ *otherwise.*

Proof: Like the definition of $\varphi^{(n)}$, also the proof is by induction on n. For $n = 1$ the relation (5.2) reduces to

$$(\varphi^{(1)}(Q, t))(\pi) = \left(\varphi \left(\bigcap_{i=1}^{1} \Omega_i, t \right) \right)(\pi) = (\varphi(\Omega_1, t))(\pi) \tag{5.3}$$

which agrees with the definition of $\varphi^{(1)}(Q, t)$ in (5.1). Let (5.2) hold for $n = 1$, denote the set $\bigcap_{i=1}^{n} \Omega_i$ by $\Omega^{(n)}$. Hence, we have to prove that

$$(\varphi(\Omega_n, t))(\varphi(\Omega^{n-1}, t)(\pi)) = (\varphi(\Omega^{(n)}, t))(\pi) \tag{5.4}$$

holds.

Let $\omega \in \bigcap_{i=1}^{n} \Omega_i$, hence, $\omega \in \Omega_i$ for each $i \leq n$, so that $((\varphi(\Omega_i, t))(\pi))(\omega) = \pi(\omega)$ is valid for each $i \leq n - 1$. As $\omega \in \Omega_n$ holds as well,

$$((\varphi(\Omega_n, t))(\pi))(\omega) = ((\varphi^{(n-1)}(Q, t))(\pi))(\omega) = \pi(\omega), \tag{5.5}$$

hence, $((\varphi^{(n)}(Q, t))(\pi))(\omega) = \pi(\omega)$.

Let $\omega \in \bigcap_{i=1}^{n-1} \Omega_i$, but $\omega \in \Omega - \Omega_n$ be the case. Then $((\varphi(\Omega_n, t))(\pi))(\omega) = t$ no matter which the value $((\varphi^{(n-1)}(Q, t))(\pi))(\omega)$ may be, so that $((\varphi^{(n)}(Q, t))(\pi))(\omega) = t$ holds. If $((\varphi^{(n-1)}(Q, t))(\pi))(\omega) = t$, then the value $((\varphi^{(n)}(Q, t))(\pi))(\omega) = t$ follows either as the copy of the value $((\varphi^{(n-1)}(Q, t))(\pi))(\omega)$, if $\omega \in \Omega_n$ holds, or the same value $((\varphi^{(n)}(Q, t))(\pi)(\omega) = t$ follows according to the rule how to process the value $((\varphi^{(n-1)}(Q, t))(\pi))(\omega)$ for $\omega \in \Omega - \Omega_n$. To conclude, $((\varphi^{(n)}(Q, t))(\pi))(\omega) = \pi(\omega)$, if $\omega \in \bigcap_{i=1}^{n} \Omega_i$ holds, this value being t otherwise. The assertion is proved. □

As the most simple example let us introduce the case with $n = 2$ and $\Omega_1 = \Omega_2$ $(= \Omega_0)$, hence, let us analyze the repeated application of the operation $(\cdot)^0$ to the mapping $\pi : \Omega \to T$. So, we have to compute the mapping $\pi^0(\pi^0) : \Omega \to T$. We obtain that $(\pi^0(\pi^0))(\omega) = \pi^0(\omega) = \pi(\omega)$ for $\omega \in \Omega_0$. If $\omega \in \Omega - \Omega_0$ is the case, then $\pi^0(\pi^0(\omega)) = t$ no matter which the actual value $\Pi^0(\omega)$ may be according to the rule defining the value $\pi^0(\omega)$ for $\omega \in \Omega - \Omega_0$. So, $\pi^0(\pi^0(\omega)) = \pi(\omega)$, if $\omega \in \Omega_0, \pi^0(\pi^0(\omega)) = t$, if $\omega \in \Omega - \Omega_0$, so that $\pi^0(\pi^0(\omega)) = \pi^0(\omega)$ for every $\omega \in \Omega$, in other terms, the operation $(\cdot)^0$ is idempotent.

For the alternative approximation π_0 of the mapping π the situation is analogous, the operation $(\cdot)_0$ is also idempotent. Let us recall that, given $\pi : \Omega \to T$, the mapping $\pi_0 : \Omega \to T$ is defined by $\pi_0(\omega) = \pi(\omega) \wedge q_0$ for certain $q_0 \in T$. We obtain that the relation

$$\pi_0(\pi_0(\omega)) = (\pi_0(\omega)) \wedge q_0 = (\pi(\omega) \wedge q_0 = \pi(\omega) \wedge q_0 = \pi_0(\omega) \tag{5.6}$$

is valid, so that $(\cdot)_0$ is also idempotent.

The following assertion is almost self-evident.

Lemma 5.1 *Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be an $*$-lattice, let $\pi : \Omega \to T$ be as above, let for each $i = 1, 2, \ldots, q_i$ be an element of T such that there exists $\omega_i \in \Omega$ with the property $q_i \leq_{\mathcal{T}} \pi(\omega_i)$, let Ψ_i be the operator transforming the mapping π into the mapping $\Psi_i(\pi) : \Omega \to T$ defined, for each $\omega \in \Omega$, by $(\Psi_i(\pi))(\omega) = \pi(\omega) \wedge q_i$. let $\Psi^{(n)}$ be the operator defined by induction as $(\Psi^{(1)}(\pi))(\omega) = \pi_0(\omega) = \pi(\omega) \wedge q_1$,*

$$(\Psi^{(n)}(\pi))(\omega) = \varphi_n((\Psi^{(n-1)}(\pi))(\omega)) = (\Psi^{(n-1)}(\pi))(\omega) \wedge q_n. \tag{5.7}$$

Then $\Psi^{(n)}(\omega) = \pi(\omega) \wedge (\bigwedge_{i=1}^n q_i)$.

6 Conclusions

When taking into consideration uncertainty quantifications with non-numerical degrees, complete lattices were quite naturally the first structures coming into one's mind, into which such uncertainty values should be embedded. Indeed, as finite and infinite suprema and infima of non-numerical uncertainty degrees are defined for each complete lattice-valued systems, the considerations over uncertainty or possibility degrees taking their values in an appropriate complete lattice are simplified by the general assumption that suprema and infima of all sets of possibility degrees are always defined, hence, their existence need not be, case by case, either proved or assumed to be defined in some particular cases, and replaced by weaker properties in the remaining cases.

On the other side, we should like also to investigate possibility degrees taking their values in weaker structures than complete lattices, e.g., in upper semilattices. Such semilattices describe the qualitative properties of non-negative quantities which are finite but not covered by a common upper bounds. Such quantities may be processed just by finite and qualitative relations like "the value of an object A (its price, e.g.) is greater than (not lesser that, resp.) than the value (price) of an object B" and, in general, not every two objects are comparable with each other in this sense.

In this paper we have proposed and analyzed the solution perhaps the most simple and conservative one from the methodological (even if perhaps not too easy from the computational) point of view. Namely, we have subjected the original upper semilattice to certain modifications (deformations) transforming this upper semilattice into a complete lattice preserving at least some important properties of the original upper semilattice, expressed and proved in assertions on the values taken by lattice-valued entropy function for the original as well as for the resulting structures. Two models of modifications (deformations) are proposed and analyzed, both of them depending on free parameters the choice of which involves the properties and qualities of the resulting complete lattice related to the outcoming upper semilattice.

At present, the author tries to open another way how to overcome the problem that the supremum value $\Pi(A) = \bigvee_{\omega \in A} \pi(\omega)$ is not defined for infinite sets $A \subset \Omega$ with infinite sets of values of upper-semilattice-valued mappings $\pi : \Omega \to T$. This approach is based on the principal paradigma of mathematical statistics according to which "appropriately", in the sense which can be precized and formalized within the framework of standard probability theory and mathematical statistics, chosen or "sampled" (when using standard tern of mathematical statistics), finite subset $A_0 \subset A$ may be used instead of the whole $A \subset \Omega$ so that the value $\Pi(A_0) = \bigvee_{\omega \in A_0} \pi(\omega)$ is defined and either approximates (estimates) sufficiently closely the value $\Pi(A)$, if this value is defined, or the value $\Pi(A_0)$ in a reasonable sense extends the mapping Π from finite and π-finite subsets of Ω to A. A more detailed manuscript is just under preparation and the author hopes to submit it for publication as soon as possible.

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