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Abstract:

A theorem of the alternatives for the equation $|Ax| - |B||x| = b$ ($A, B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$) is proved and several consequences are drawn. In particular, a class of matrices A, B is identified for which the equation has exactly 2^n solutions for each positive right-hand side b.

Keywords:
Absolute value equation, triple absolute value equation, alternatives, solution set, interval matrix, regularity.

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1 Introduction

We consider here the equation

$$|Ax| - |B||x| = b, \tag{1.1}$$

where $A, B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, which we call a *triple absolute value equation*. This equation could also be written in the form

$$|Ax| - C|x| = b,$$
$$C \geq 0,$$

but we prefer the one-line expression (1.1). As far as known to us, nobody has studied this equation as yet.

In the main result of this paper we show that for each $A, B \in \mathbb{R}^{n\times n}$ exactly one of the following two alternatives holds: (i) for each $b > 0$ the equation (1.1) has exactly 2^n solutions and the set $\{\, Ax \mid |Ax| - |B||x| = b \,\}$ intersects interiors of all orthants of $\mathbb{R}^n$, (ii) the equation (1.1) has a nontrivial solution for some $b \leq 0$. In Corollary 2 we show that, even more, if the property mentioned in (i) holds for *some* $b_0 > 0$, then it is shared by *any* $b > 0$, and in Corollary 3 we prove that if A is nonsingular and the condition

$$\varrho(|A^{-1}||B|) < 1 \tag{1.2}$$

is satisfied, then (i) holds, so that for each $b > 0$ the equation (1.1) has exactly 2^n solutions. As it will be shown later, these results follow from necessary and/or sufficient conditions for regularity/singularity of interval matrices when applied to the interval matrix $[A - |B|, A + |B|]$. In turn, our results enable us to add three more such necessary and sufficient conditions to the list of forty of them surveyed in [11] (Proposition 7 below).

Nearest in form to the equation (1.1) is the *absolute value equation*

$$Ax + B|x| = b \tag{1.3}$$

which has been resently studied by Mangasarian [2], [3], [4], Mangasarian and Meyer [5], Prokopyev [7], and Rohn [10], [12]. There is, however, a big difference between these two equations: while the equation (1.3) has under the condition (1.2) exactly one solution for each b, the equation (1.1) under the same condition has exactly 2^n solutions for each $b > 0$. This sharp difference between both the equations is to be ascribed to the absence/presence of the absolute value of the term Ax.

The particular circumstances of discovery of the main theorem are briefly mentioned in the personal note in Section 8.

2 Notations

We use the following notations. Matrix inequalities, as $A \leq B$ or $A < B$, are understood componentwise. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. The same notations also apply to vectors that are considered one-column matrices. For each $y \in \{-1, 1\}^n$ we denote

$$T_y = \operatorname{diag}(y_1, \ldots, y_n) = \begin{pmatrix} y_1 & 0 & \ldots & 0 \\ 0 & y_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & y_n \end{pmatrix},$$

and $\mathbb{R}^n_y = \{x\ ;\ T_y x \geq 0\}$ is the orthant prescribed by the ± 1-vector y. Notice that $T_y^{-1} = T_y$ for such a y. $\varrho(A)$ stands for the spectral radius of A. Given $A, B \in \mathbb{R}^{n\times n}$, the set

$$[A - |B|, A + |B|] = \{\, S \mid |S - A| \leq |B| \,\}$$

is an interval matrix; it is called regular if each $S \in [A - |B|, A + |B|]$ is nonsingular, and it is said to be singular otherwise (i.e., if it contains a singular matrix).

3 Theorem of the alternatives

To simplify formulations, let us say that the equation (1.1) is *exponentially solvable* for a particular right-hand side b if it has exactly 2^n solutions and the set

$$\{\, Ax \mid |Ax| - |B||x| = b \,\} \tag{3.1}$$

intersects interiors of all orthants of $\mathbb{R}^n$. The following theorem is the main result of this paper.

Theorem 1. *For each $A, B \in \mathbb{R}^{n\times n}$ exactly one of the following two alternatives holds:*

(i) the equation (1.1) is exponentially solvable for each $b > 0$,

(ii) the equation (1.1) has a nontrivial solution for some $b \leq 0$.

Proof. Consider the following two options for the interval matrix $[A - |B|, A + |B|]$:

(i') $[A - |B|, A + |B|]$ is regular,

(ii') $[A - |B|, A + |B|]$ is singular.

We shall prove that the assertions (i), (ii) are equivalent to (i'), (ii'), respectively. Since exactly one of (i'), (ii') always holds, the same will be true for (i), (ii).

(i)$\Rightarrow$(i'). Let (i) hold. Take any $b_0 > 0$, then for each ± 1-vector $y \in \mathbb{R}^n$ there exists a solution x_y of the equation $|Ax| - |B||x| = b_0$ such that $Ax_y \in \mathbb{R}^n_y$. Since x_y satisfies $|Ax_y| = |B||x_y| + b_0 > |B||x_y|$, the condition (v) of Theorem 3.1 in [9] is met and consequently the interval matrix $[A - |B|, A + |B|]$ is regular.

(i')$\Rightarrow$(i). If (i') holds, then for each ± 1-vector y the interval matrix

$$[A - | - T_y|B||, A + | - T_y|B||] = [A - |B|, A + |B|]$$

is regular, hence by Proposition 4.2 in [10] the equation

$$Ax - T_y|B||x| = T_y b \tag{3.2}$$

has a unique solution x_y. This x_y then satisfies

$$T_y A x_y - |B||x_y| = b, \tag{3.3}$$

which implies

$$T_y A x_y = |B||x_y| + b \geq b > 0, \tag{3.4}$$

hence Ax_y belongs to the interior of $\mathbb{R}^n_y$ and $T_yAx_y = |Ax_y|$, which in view of (3.3) means that x_y is a solution of (1.1). Conversely, let x solve (1.1). Put $y_i = 1$ if $(Ax)_i \geq 0$ and $y_i = -1$ otherwise ($i = 1, \ldots, n$), then $T_yAx = |Ax|$, so that x is a solution of

$$T_yAx - |B||x| = b$$

and thus also of (3.2). Because of the above-stated uniqueness of solution of (3.2), this implies that $x = x_y$. In this way we have proved that the solution set of (1.1) consists precisely of the points x_y for all possible ± 1-vectors $y \in \mathbb{R}^n$. Thus to prove that (1.1) has exactly 2^n solutions, it will suffice to show that all the x_y's are mutually different. To this end, take two ± 1-vectors y and y', $y \neq y'$. Then $y_iy'_i = -1$ for some i. From (3.4) it follows that $y_i(Ax_y)_i > 0$ and $y'_i(Ax_{y'})_i > 0$ and by multiplication $y_i(Ax_y)_iy'_i(Ax_{y'})_i > 0$, hence $(Ax_y)_i(Ax_{y'})_i < 0$, which clearly shows that $x_y \neq x_{y'}$.

(ii)⇔(ii'). Existence of a nontrivial solution of (1.1) for some $b \leq 0$ is equivalent to existence of a nontrivial solution of the inequality

$$|Ax| \leq |B||x|, \tag{3.5}$$

which, by Proposition 2.2 in [10], is in turn equivalent to singularity of the interval matrix $[A - |B|, A + |B|]$.

This proves the theorem. □

4 Consequences

We can draw some consequences from Theorem 1 and its proof.

Corollary 2. *If the equation (1.1) is exponentially solvable for some $b_0 > 0$, then it is exponentially solvable for each $b > 0$.*

Proof. Indeed, in the proof of Theorem 1, implication "(i)⇒(i')", we showed that exponential solvability of the equation (1.1) for some $b_0 > 0$ implies regularity of $[A - |B|, A + |B|]$ and thus, by "(i')⇒(i)", also exponential solvability for each $b > 0$. □

Corollary 3. *If A is nonsingular and*

$$\varrho(|A^{-1}||B|) < 1 \tag{4.1}$$

holds, then the equation (1.1) is exponentially solvable for each $b > 0$.

Proof. By the well-known Beeck's result in [1], the condition (4.1) implies regularity of the interval matrix $[A - |B|, A + |B|]$ and thus, by the equivalence "(i)⇔(ii)" established in the proof of Theorem 1, it also implies exponential solvability of (1.1) for each $b > 0$. □

Corollary 4. *If A is nonsingular and*

$$\max_j (|A^{-1}||B|)_{jj} \geq 1 \tag{4.2}$$

holds, then the equation (1.1) is not exponentially solvable for any $b > 0$.

Proof. It follows from [8], Corollary 5.1, (iii) that the condition (4.2) implies singularity of the interval matrix $[A - |B|, A + |B|]$, which, by the proof of Theorem 1 and by Corollary 2, precludes exponential solvability of (1.1) for any $b > 0$. $\square$

For $A, B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, denote

$$X(A, B, b) = \{\, x \mid |Ax| - |B||x| = b \,\},$$

i.e., the solution set of (1.1) (attention: not to be mixed with (3.1)). Observe that if $x \in X(A, B, b)$, then $-x \in X(A, B, b)$, hence the solutions appear in $X(A, B, b)$ in pairs $(x, -x)$. Thus, unless $b = 0$, the cardinality of $X(A, B, b)$, if finite, is even.

Corollary 5. *If the equation $|Ax| - |B||x| = b_0$ is exponentially solvable for some $b_0 > 0$, then for each $b > 0$ we have*

$$X(A, B, b) = \{\, x_y \mid y \in \{-1, 1\}^n \,\},$$

where for each $y \in \{-1, 1\}^n$, x_y is the unique solution of the absolute value equation

$$T_y Ax - |B||x| = b. \tag{4.3}$$

Proof. This has been proved in the "(i')⇒(i)" part of the proof of Theorem 1. $\square$

Corollary 6. *Under the assumptions of Corollary 5, we have $x_{-y} = -x_y$ for each $y \in \{-1, 1\}^n$.*

Proof. Since x_y is a solution of (4.3), it follows that $-x_y$ solves the equation

$$T_{-y} Ax - |B||x| = b,$$

and in view of the uniqueness of solution of this equation we have that $x_{-y} = -x_y$. $\square$

The equation (4.3) can be solved in a finite number of steps by a very efficient algorithm **absvaleqn** described in [12]. Corollary (6) reduces the number of x_y's to be computed from 2^n to 2^{n-1} (e.g., it suffices to consider only the y's with $y_n = 1$).

5 Example

The following computation was performed in MATLAB. Using the randomly generated data

```
>> n=3; rand('state',1); A=2*rand(n,n)-1, B=2*rand(n,n)-1, b=rand(n,1),
A =
    0.9056    0.1963    0.6736
    0.4081    0.6815    0.0374
    0.9078   -0.1144   -0.9556
```

```
B =
   -0.2482   -0.6009    0.8205
    0.7972   -0.3938    0.0506
   -0.1420    0.0766   -0.3863
b =
    0.0345
    0.7153
    0.7687
```

we obtain the solution set consisting of two solutions

```
X =
   18.4720  -18.4720
   27.7448  -27.7448
    6.0199   -6.0199
```

After scaling the matrix B by

```
>>B=0.1*B
B =
   -0.0248   -0.0601    0.0820
    0.0797   -0.0394    0.0051
   -0.0142    0.0077   -0.0386
```

we find that the solution set now consists of $2^3 = 8$ solutions

```
X =
    0.3921   -0.3921    0.1794   -0.1794    0.3679   -0.3679
    0.9521   -0.9521    1.0756   -1.0756   -1.3774    1.3774
   -0.5830    0.5830   -0.8066    0.8066   -0.3196    0.3196

    0.5740   -0.5740
   -1.5452    1.5452
   -0.0991    0.0991
```

and the set (3.1)

```
>> AX=A*X
AX =
    0.1492   -0.1492   -0.1697    0.1697   -0.1526    0.1526
    0.7870   -0.7870    0.7761   -0.7761   -0.8005    0.8005
    0.8041   -0.8041    0.8107   -0.8107    0.7968   -0.7968

    0.1497   -0.1497
   -0.8224    0.8224
    0.7925   -0.7925
```

intersects interiors of all the orthants. Hence, the equation $|Ax| - |0.1 \cdot B||x| = b$ is exponentially solvable for this right-hand side b (and thus it is exponentially solvable for each $b' > 0$).

6 Regularity conditions

Checking regularity of interval matrices is a co-NP-complete problem [6]. Forty necessary and sufficient regularity conditions were surveyed in [11]; the results of this paper enable us to add three more items to the list.

Proposition 7. *For a square interval matrix* $[A-\Delta, A+\Delta]$*, the following assertions are equivalent:*

(a) $[A-\Delta, A+\Delta]$ *is regular,*

(b) the equation

$$|Ax| - \Delta|x| = b \tag{6.1}$$

is exponentially solvable for each $b > 0$*,*

(c) the equation (6.1) is exponentially solvable for some right-hand side $b_0 > 0$*,*

(d) the equation

$$|Ax| - \Delta|x| = e$$

is exponentially solvable.

Proof. In the light of Theorem 1 and Proposition 7 we see that (a)⇒(b)⇒(d)⇒(c)⇒(b)⇒(a) holds, which proves the mutual equivalence of all the assertions. □

7 Conclusion

We have investigated the case of $b > 0$. For a general right-hand side b there seems not to be an easy clue to the cardinality of the solution set of (1.1). This should be a subject of further research.

8 Personal note

I am a little ashamed to admit that I discovered Theorem 1 during the Christmas Eve mass on December 24, 2006 in St Francis Church in Prague.

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