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Chvalovský, Karel
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Non-Equivalence of Some Implicational Deduction Theorems

Post-Graduate Student:
MGR. KAREL CHVALOVSKÝ
Institute of Computer Science of the ASCR, v. v. i.
Pod Vodárenskou věží 2
182 07 Prague 8, CZ
Department of Logic, Charles University
Celetná 20
116 42, Prague 1, CZ
chvalovsky@cs.cas.cz

Supervisor:
MGR. MARTA BÍLKOVÁ, PH.D.
Institute of Computer Science of the ASCR, v. v. i.
Pod Vodárenskou věží 2
182 07 Prague 8, CZ
bilkova@cs.cas.cz

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The paper is substantially based on the forthcoming paper by Petr Cintula (principal author) and the author.

Abstract

We show that some classes of logics in the hierarchy of Implicational Deduction Theorems, defined in the forthcoming paper [1], are not equal. This completes the picture of hierarchy of these logics.

1. Introduction

One of the most important theorems of classical propositional logic is the Deduction Theorem, independently discovered by Herbrand [2] and Tarski [3], which connects provability and implication. In its most popular form it says

$$\Gamma, \varphi \vdash \psi \text{ iff } \Gamma \vdash \varphi \to \psi.$$

It enables us to find some proofs much easier. However, this theorem does not hold in all logics. For example in logics without contraction, we usually have so called Local Deduction Theorem, which says that there exists some natural k such that

$$\Gamma, \varphi \vdash \psi \text{ iff } \Gamma \vdash \underbrace{\varphi \to (\dots(\varphi}_{k\text{-times}} \to \psi)\dots).$$

The problem is that generally we do not have any (reasonable) upper bound on k.

We can try to estimate k somehow. The immediate idea is to count how many times the assumption φ is used in the proof of ψ. This idea is captured in the forthcoming paper [1] where the situation is shown not to be so easy. Authors define some hierarchy of logics with Implicational Deduction Theorems and investigate relations between its members. It is shown that this hierarchy collapses on some level. In this paper we show in full details that some of its members are not the same.

This is shown by presenting, for the remaining case, a counter-example. It is worth to note that this counter-example was found with the help of computer. For further details, proofs and references we refer the reader to the forthcoming paper [1] mentioned already.

2. Preliminaries

We use some standard terminology from the theory of logical calculi (see e.g. [4])—a *propositional language* $\mathcal{L}$ (a set of logical connectives with some finite arity, in this paper we have just one binary connective called implication $\to$ and we use the following convention: $\varphi \to^0 \psi = \psi$ and $\varphi \to^{i+1} \psi = \varphi \to (\varphi \to^i \psi)$), the set of $\mathcal{L}$-*formulae* $Fle_{\mathcal{L}}$ over some fixed countably infinite set of propositional variables and $\mathcal{L}$-*substitutions*. An $\mathcal{L}$-*theory* Γ is a set of $\mathcal{L}$-formulae. An $\mathcal{L}$-*consecution* $\Gamma \rhd \varphi$ is a pair consisting of a theory Γ and a formula φ.

A *logic* $\mathbf{L}$ in the language $\mathcal{L}$ is a structural consequence relation (in the sense of Tarski) on $Fle_{\mathcal{L}}$. That is, $\mathbf{L}$ is a set of relations between theories and formulae (writing $\Gamma \vdash_{\mathbf{L}} \varphi$, and $\Gamma \vdash_{\mathbf{L}} \Gamma'$ as an abbreviation for $\Gamma \vdash_{\mathbf{L}} \varphi$ for each $\varphi \in \Gamma'$) satisfying the following conditions:

(i) If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathbf{L}} \varphi$.
(ii) If $\Gamma \vdash_{\mathbf{L}} \Gamma'$ and $\Gamma' \vdash_{\mathbf{L}} \varphi$, then $\Gamma \vdash_{\mathbf{L}} \varphi$.
(iii) If $\Gamma \vdash_{\mathbf{L}} \varphi$, then there is a finite set $\Gamma' \subseteq \Gamma$ s.t. $\Gamma' \vdash_{\mathbf{L}} \varphi$.
(iv) If $\Gamma \vdash_{\mathbf{L}} \varphi$, then $\sigma(\Gamma) \vdash_{\mathbf{L}} \sigma(\varphi)$ for any $\mathcal{L}$-substitution σ.

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The previous conditions are called reflexivity, cut, finitarity and structurality.

Definition 1 *An* axiomatic system $\mathcal{AX}$ *is a set of finitary consecutions closed under substitutions. The members of* $\mathcal{AX}$ *with non-empty theories are called* deductive rules*, these with empty theories are called* axioms. *We say that* $\mathcal{AX}$ *is* MP-based *if modus ponens is its only deduction rule.*

Note that we have only finitary rules, and axioms as well as rules are presented by schemata.

Definition 2 *Let* $\mathcal{AX}$ *be an axiomatic system. An* $\mathcal{AX}$-proof *of the formula* φ *in theory* Γ *is a finite tree labelled by formulae satisfying*

(i) the root is labelled by φ*,*
(ii) leaves by either axioms or elements of Γ*,*
(iii) if a node is labelled by ψ *and its preceding nodes are labelled by* $\psi_1, \ldots, \psi_n$ *then* $\{\psi_1, \ldots, \psi_n\} \rhd \varphi \in \mathcal{AX}$.

If such a proof exists we write $\Gamma \vdash^p_{\mathcal{AX}} \varphi$.

We say that $\mathcal{AX}$ is an *axiomatic system* for (a presentation of) a logic $\mathbf{L}$ iff $\mathbf{L} = \vdash^p_{\mathcal{AX}}$. A logic $\mathbf{L}$ is *MP-based* if it has some MP-based presentation.

2.1. Matrix models and semantics

A *matrix* $\mathbf{M}$ for $\mathcal{L}$ is a pair $\langle A, D\rangle$, where A is an $\mathcal{L}$-*algebra* and $D \subseteq A$ is the *set of designated elements* of $\mathbf{M}$. An $\mathbf{M}$-*evaluation* for matrix $\mathbf{M} = \langle A, D\rangle$ is a mapping $e : Fle_{\mathcal{L}} \to A$ which commutes with all connectives in $\mathcal{L}$.

Logics can be defined semantically through logical matrices. Any class of $\mathcal{L}$-matrices $\mathcal{C}$ is called *matrix semantics* for $\mathcal{L}$. We say $\Gamma \models_{\mathcal{C}} \varphi$ iff for each $\mathbf{M} \in \mathcal{C}$, $\mathbf{M} = \langle A, D\rangle$, and each evaluation e in $\mathbf{M}$, $e(\varphi) \in D$ whenever $e[\Gamma] \subseteq D$.

Notice that $\models_{\mathcal{C}}$ is a logic for $\mathcal{C}$ being a finite set of finite $\mathcal{L}$-matrices. By relaxing either of the finiteness conditions we obtain a consequence relation, but not necessarily a finitary one.

3. Implicational Deduction Theorems

In this section we define some basic notions concerning the study of Implicational Deduction Theorems. First, we define an analog to a Local Deduction Theorem.

Definition 3 *A logic* $\mathbf{L}$ *has* Simple Implicational Deduction Theorem *(IDT$_0$) if for each theory* Γ *and formulae* φ, ψ*:*

$$\Gamma, \varphi \vdash_{\mathbf{L}} \psi \text{ iff there is } n \text{ such that } \Gamma \vdash_{\mathbf{L}} \varphi \to^n \psi.$$

We immediately obtain the following important property of logics with IDT$_0$, which is a consequence of our assumptions concerning finitarity.

Lemma 1 *A logic* $\mathbf{L}$ *with IDT$_0$ is MP-based.*

Now we present a finer analysis of Local Deduction Theorems arising from the idea of counting number of occurrences of φ in the leaves of some proof of ψ in Γ and φ.

Definition 4 *Let* $n > 0$. *A logic* $\mathbf{L}$ *has* n-Implicational Deduction Theorem *(IDT$_n$) if*

(i) $\mathbf{L}$ *has an MP-based presentation* $\mathcal{AX}$*,*
(ii) for each theory Γ*, formula* ψ*, mutually different formulae* φ_i*,* $1 \le i \le n$*, and for each* $\mathcal{AX}$*-proof* $\mathcal{P}$ *of* ψ *in* $\Gamma \cup \{\varphi_i \mid 1 \le i \le n\}$*:*

$$\Gamma \vdash \varphi_1 \to^{j_1} (\varphi_2 \to^{j_2} \ldots (\varphi_n \to^{j_n} \psi) \ldots),$$

where j_i *is the number of occurrences of* φ_i *in the leaves of* $\mathcal{P}$.

It may seem that eg. IDT$_2$ can be obtained just by double application of IDT$_1$, but it is not true.

Example 1 *Let we assume that*

$$\varphi, \psi, \varphi \to (\psi \to \chi) \vdash \chi. \tag{1}$$

IDT$_2$ gives

$$\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi) \tag{2}$$

and IDT$_1$ gives

$$\psi, \varphi \to (\psi \to \chi) \vdash \varphi \to \chi, \tag{3}$$

but now we cannot use IDT$_1$ once again to obtain (2). *We only know that* $\varphi \to \chi$ *is provable from* ψ *and* $\varphi \to (\psi \to \chi)$*, but we do not know how many times* ψ *has to be used.*

From now on, we shall use IDT$_n$ also for the class of all logics satisfying IDT$_n$. The meaning will be obvious from context.

In the paper [1] we prove

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Theorem 1

(i) *If a logic* $\mathbf{L}$ *has* IDT_n *then* $\mathbf{L}$ *has* IDT_m *for any* $m \leq n$.
(ii) *If a logic* $\mathbf{L}$ *has* IDT_3 *then* $\mathbf{L}$ *has* IDT_m *for any* $m \geq 3$.
(iii) $IDT_0 \neq IDT_1$ *and* $IDT_1 \neq IDT_2$.

The previous theorem shows that the hierarchy of logics with Implicational Deduction Theorems has the following properties

$$\mathrm{IDT}_0 \supsetneq \mathrm{IDT}_1 \supsetneq \mathrm{IDT}_2 \supseteq \mathrm{IDT}_3 = \mathrm{IDT}_4 = \dots$$

The remaining problem is whether $\mathrm{IDT}_2 = \mathrm{IDT}_3$ or not, which is the result of this paper. We solve this problem negatively in the next section. Before we proceed we recall an important characterisation lemma:

Lemma 2 *Let* $\mathbf{L}$ *be a logic and* $n > 0$ *then* $\mathbf{L}$ *has* IDT_n *iff*

(i) $\mathbf{L}$ *is MP-based,*
(ii) $\vdash_{\mathbf{L}} \varphi \to \varphi$,
(iii) *for each natural* a_i, b_i, *for* $1 \leq i \leq n$, *holds*

$$\varphi_1 \to^{a_1} (\dots(\varphi_n \to^{a_n} (\chi \to \psi))\dots), \quad \varphi_1 \to^{b_1} (\dots(\varphi_n \to^{b_n} \chi)\dots) \vdash_{\mathbf{L}} \varphi \to^{a_1+b_1} \dots(\varphi_n \to^{a_n+b_n} \psi)\dots).$$

4. $\mathbf{IDT}_2 \neq \mathbf{IDT}_3$

In this section we prove that there is a logic with IDT_2 but without IDT_3. The proof is based on the matrix $\mathbf{M}$ in Table 1. The only denoted element of $\mathbf{M}$ is $\mathbf{1}$. Let us note that the matrix was found with the help of computer.

$\to$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	1	1	a	a	a	c	d	f
b	1	1	1	a	a	b	d	e
c	1	1	1	1	a	a	c	e
d	1	1	1	1	1	a	a	d
e	1	1	1	1	1	1	a	b
f	1	1	1	1	1	1	1	a
g	1	1	1	1	1	1	1	1

Table 1: Model $\mathbf{M}$.

From now on, we use the following notation. We abbreviate $e(\varphi) =_{\mathbf{M}} \mathbf{1}$ by $\varphi = \mathbf{1}$, because model $\mathbf{M}$ is fixed and the evaluation is obvious from the context. We shall also abbreviate it simply saying φ is $\mathbf{1}$.

Lemma 3 *The logic* $\mathbf{L}$ *given by model* $\mathbf{M}$ *does not have* IDT_3.

Proof: There is a proof of

$$\varphi, \psi, \varphi \to (\psi \to \chi) \vdash \chi,$$

where φ, ψ and $\varphi \to (\psi \to \chi)$ are used only once. IDT_3 would give

$$\vdash (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)),$$

which is not true. Consider an evaluation $e(\varphi) = \mathbf{a}$, $e(\psi) = \mathbf{b}$ and $e(\chi) = \mathbf{g}$. ∎

Now we shall show that $\mathbf{L}$ has IDT_2. First, we establish some useful properties of $\mathbf{M}$.

Observation 1 *The following statements are true in* $\mathbf{M}$*:*

(i) $\varphi \to \varphi = \mathbf{1}$,
(ii) $\mathbf{1} \to \varphi = \varphi$,
(iii) *if* $\varphi = \mathbf{1}$ *and* $\varphi \to \psi = \mathbf{1}$ *then* $\psi = \mathbf{1}$ *(MP holds in* $\mathbf{M}$*)*.

Definition 5 *We define the* ordering $<$ *on the elements of* $\mathbf{M}$ *by*

$$\mathbf{g} < \mathbf{f} < \mathbf{e} < \mathbf{d} < \mathbf{c} < \mathbf{b} < \mathbf{a} < \mathbf{1}.$$

We use $x \leq y$ with the standard meaning $x = y$ or $x < y$. In the very same way as for $=$ we use $\varphi \leq \psi$ which means that for given evaluation e it holds that $e(\varphi) \leq e(\psi)$.

Lemma 4 *The model* $\mathbf{M}$ *has the following properties:*

(i) $\psi \leq \varphi \to \psi$,
(ii) $\varphi_0 \leq \varphi$ *implies* $\varphi \to \psi \leq \varphi_0 \to \psi$,
(iii) $\psi_0 \leq \psi$ *implies* $\varphi \to \psi_0 \leq \varphi \to \psi$.

These properties of implication in $\mathbf{M}$ play a very important role in the rest of the section.

Lemma 5 *For any evaluation such that* $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ *are different from* $\mathbf{1}$ *holds*

$$\varphi_1 \to (\varphi_2 \to (\varphi_3 \to (\varphi_4 \to \psi))) = \mathbf{1}.$$

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Proof: By Lemma 4 the worst case is $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \mathbf{a}$ and $\psi = \mathbf{g}$, but even for such evaluation lemma holds. ∎

Observation 1 and Lemma 5 are very important. From Lemma 2 we immediately obtain that we only need to check finitely many cases to show that logic $\mathbf{L}$ given by model $\mathbf{M}$ has IDT$_2$.

Corollary 1 *The logic* $\mathbf{L}$ *given by model* $\mathbf{M}$ *has IDT*$_2$ *iff*

$$\varphi_1 \to^{a_1} (\varphi_2 \to^{a_2} \to (\chi \to \psi)) = \mathbf{1} \quad (4)$$

and

$$\varphi_1 \to^{b_1} (\varphi_2 \to^{b_2} \chi) = \mathbf{1} \quad (5)$$

imply

$$\varphi_1 \to^{a_1+b_1} (\varphi_2 \to^{a_2+b_2} \psi) = \mathbf{1} \quad (6)$$

for any $a_1 + a_2 + b_1 + b_2 < 4$.

Now we can proceed by exhaustive checking of all possible variants, or we can simplify our work significantly as shown by following three lemmata.

Lemma 6 *Given* (4) *and* (5)*, and if*

(i) $\psi = \mathbf{1}$,
(ii) $\chi = \mathbf{1}$,
(iii) $\chi = \mathbf{g}$,
(iv) $\varphi_1 = \mathbf{g}$ *(for* $a_1 + b_1 > 0$*)*,
(v) $\varphi_2 = \mathbf{g}$ *(for* $a_1 + b_1 > 0$*)*,

then the condition (6) *holds.*

Proof: Cases (i), (iv) and (v) are evident. If $\chi = \mathbf{1}$ then

$$\begin{aligned} \mathbf{1} &= \varphi_1 \to^{a_1} (\varphi_2 \to^{a_2} (\mathbf{1} \to \psi)) \\ &= \varphi_1 \to^{a_1} (\varphi_2 \to^{a_2} \psi) \\ &\leq \varphi_1 \to^{a_1+b_1} (\varphi_2 \to^{a_2+b_2} \psi). \end{aligned}$$

If $\chi = \mathbf{g}$ then $\chi \leq \psi$ and hence

$$\begin{aligned} \mathbf{1} &= \varphi_1 \to^{b_1} (\varphi_2 \to^{b_2} \chi) \\ &\leq \varphi_1 \to^{b_1} (\varphi_2 \to^{b_2} \psi) \\ &\leq \varphi_1 \to^{a_1+b_1} (\varphi_2 \to^{a_2+b_2} \psi). \end{aligned}$$

∎

Lemma 7 *Given* (4) *and* (5)*, and if*

(i) $a_1 + a_2 = 0$,
(ii) $b_1 + b_2 = 0$,

then the condition (6) *holds.*

Proof: Case (i) gives $\chi \to \psi = \mathbf{1}$ hence $\chi \leq \psi$ and then (5) implies (6). Case (ii) is even easier, because $\chi = \mathbf{1}$ and then (4) implies (6). ∎

Lemma 8 *Given* (4) *and* (5)*, and if*

(i) $a_1 = 1$, $a_2 = 0$, $b_1 = 1$, $b_2 = 0$,
(ii) $a_1 = 1$, $a_2 = 0$, $b_1 = 0$, $b_2 = 1$,
(iii) $a_1 = 0$, $a_2 = 1$, $b_1 = 0$, $b_2 = 1$,
(iv) $a_1 = 2$, $a_2 = 0$, $b_1 = 1$, $b_2 = 0$,
(v) $a_1 = 2$, $a_2 = 0$, $b_1 = 0$, $b_2 = 1$,
(vi) $a_1 = 0$, $a_2 = 2$, $b_1 = 0$, $b_2 = 1$,
(vii) $a_1 = 1$, $a_2 = 1$, $b_1 = 0$, $b_2 = 1$,

then the condition (6) *holds.*

Proof: Let us show case (i). Other cases are more or less similar. We have

$$\varphi_1 \to (\chi \to \psi) = \mathbf{1}, \quad (7)$$
$$\varphi_1 \to \chi = \mathbf{1}. \quad (8)$$

Now we have $\varphi_1 \leq \chi$ from (8) and hence $\chi \to \psi \leq \varphi_1 \to \psi$. From (7) we have $\varphi_1 \leq \chi \to \psi$. So we have $\varphi_1 \leq \varphi_1 \to \psi$. ∎

We can now on assume that $\varphi_1 \neq \mathbf{1}$ and $\varphi_2 \neq \mathbf{1}$, because $\mathbf{1} \to \varphi = \varphi$ and consequently $\varphi_1 = \mathbf{1}$ and $\varphi_2 = \mathbf{1}$ is the same as $a_1 = b_1 = 0$ and $a_2 = b_2 = 0$, respectively. In such case all the following instances lead to the cases solved already.

The remaining cases have to be checked separately and in more details. We analyse all possible evaluations and show that (4) and (5) imply (6).

Lemma 9 *Given* (4) *and* (5)*, and if* $a_1 = 0$, $a_2 = 1$, $b_1 = 1$, $b_2 = 0$*, then the condition* (6) *holds.*

Proof: We need to show that

$$\varphi_2 \to (\chi \to \psi) = \mathbf{1}, \quad (9)$$
$$\varphi_1 \to \chi = \mathbf{1} \quad (10)$$

imply

$$\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}. \quad (11)$$

We prove it by cases. First, if $\psi = \mathbf{a}, \ldots, \mathbf{d}$ then it is easy to show that $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$ (we assume $\varphi_1 \neq \mathbf{1}$ and $\varphi_2 \neq \mathbf{1}$). The only interesting cases are the following:

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- $\cdot\ \psi = \mathbf{e}$
  - $\cdot\ \varphi_2 = \mathbf{a}$
    - $\cdot\ \varphi_1 = \mathbf{a}, \mathbf{b}$ then $\varphi_1 \leq \chi$ from (10), hence $\chi \geq \mathbf{b}$ and therefore $\varphi_2 \to (\chi \to \psi) < \mathbf{1}$.
    - $\cdot\ \varphi_1 = \mathbf{c}, \ldots$ then $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$.
  - $\cdot\ \varphi_2 = \mathbf{b}$
    - $\cdot\ \varphi_1 = \mathbf{a}$ then $\chi \geq \mathbf{a}$ and therefore $\varphi_2 \to (\chi \to \psi) < \mathbf{1}$.
    - $\cdot\ \varphi_1 = \mathbf{b}, \ldots$ then $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$.
  - $\cdot\ \varphi_2 = \mathbf{c}, \ldots$ then $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$.
- $\cdot\ \psi = \mathbf{f}$
  - $\cdot\ \varphi_2 = \mathbf{a}, \mathbf{b}$
    - $\cdot\ \varphi_1 = \mathbf{a}, \mathbf{b}, \mathbf{c}$ then $\chi \geq \mathbf{c}$ and therefore $\varphi_2 \to (\chi \to \psi) < \mathbf{1}$.
    - $\cdot\ \varphi_1 = \mathbf{d}, \ldots$ then $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$.
  - $\cdot\ \varphi_2 = \mathbf{c}$
    - $\cdot\ \varphi_1 = \mathbf{a}, \mathbf{b}$ $\chi \geq \mathbf{b}$ then $\varphi_2 \to (\chi \to \psi) < \mathbf{1}$.
    - $\cdot\ \varphi_1 = \mathbf{c}, \ldots$ then $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$.
  - $\cdot\ \varphi_2 = \mathbf{d}, \ldots$ then $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$.
- $\cdot\ \psi = \mathbf{g}$
  - $\cdot\ \varphi_2 = \mathbf{a}$
    - $\cdot\ \varphi_1 = \mathbf{a}, \ldots, \mathbf{e}$ then $\chi \geq \mathbf{e}$ and therefore $\varphi_2 \to (\chi \to \psi) < \mathbf{1}$.
    - $\cdot\ \varphi_1 = \mathbf{f}, \ldots$ then $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$.
  - $\cdot\ \varphi_2 = \mathbf{b}, \mathbf{c}$
    - $\cdot\ \varphi_1 = \mathbf{a}, \ldots, \mathbf{d}$ then $\chi \geq \mathbf{d}$ and therefore $\varphi_2 \to (\chi \to \psi) < \mathbf{1}$.
    - $\cdot\ \varphi_1 = \mathbf{e}, \ldots$ then $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$.
  - $\cdot\ \varphi_2 = \mathbf{d}$
    - $\cdot\ \varphi_1 = \mathbf{a}, \mathbf{b}, \mathbf{c}$ then $\chi \geq \mathbf{c}$ and therefore $\varphi_2 \to (\chi \to \psi) < \mathbf{1}$.
    - $\cdot\ \varphi_1 = \mathbf{d}, \ldots$ then $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$.
  - $\cdot\ \varphi_2 = \mathbf{e}$
    - $\cdot\ \varphi_1 = \mathbf{a}$ then $\chi \geq \mathbf{a}$ and therefore $\varphi_2 \to (\chi \to \psi) < \mathbf{1}$.
    - $\cdot\ \varphi_1 = \mathbf{b}, \ldots$ then $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$.
  - $\cdot\ \varphi_2 = \mathbf{f}, \ldots$ then $\varphi_1 \to (\varphi_2 \to \psi) = \mathbf{1}$.

■

Let us point out that if $\mathbf{f} \leq \psi$ and φ_0, φ_1 and φ_2 are different from $\mathbf{1}$, then

$$\varphi_0 \to (\varphi_1 \to (\varphi_2 \to \psi)) = \mathbf{1}.$$

So if $a_1 + a_2 + b_1 + b_2 = 3$ then we need to check only $\psi = \mathbf{g}$ case.

Lemma 10 *Given* (4) *and* (5)*, and if* $a_1 = 0$*,* $a_2 = 2$*,* $b_1 = 1$*,* $b_2 = 0$*, then the condition* (6) *holds.*

Proof: We need to show that

$$\varphi_2 \to (\varphi_2 \to (\chi \to \psi)) = \mathbf{1}, \tag{12}$$

$$\varphi_1 \to \chi = \mathbf{1} \tag{13}$$

imply

$$\varphi_1 \to (\varphi_2 \to (\varphi_2 \to \psi)) = \mathbf{1}. \tag{14}$$

We prove it by cases:

- $\cdot\ \psi = \mathbf{g}$
  - $\cdot\ \varphi_2 = \mathbf{a}$
    - $\cdot\ \varphi_1 = \mathbf{a}, \mathbf{b}, \mathbf{c}$ then $\varphi_1 \leq \chi$ from (13), hence $\chi \geq \mathbf{c}$ and therefore (12) does not hold.
  - $\cdot\ \varphi_2 = \mathbf{b}$
    - $\cdot\ \varphi_1 = \mathbf{a}$ then $\chi \geq \mathbf{a}$ and therefore (12) does not hold.

In all other cases (14) holds. ■

Lemma 11 *Given* (4) *and* (5)*, and if* $a_1 = 1$*,* $a_2 = 0$*,* $b_1 = 2$*,* $b_2 = 0$*, then the condition* (6) *holds.*

Proof: We need to show that

$$\varphi_1 \to (\chi \to \psi) = \mathbf{1}, \tag{15}$$

$$\varphi_1 \to (\varphi_1 \to \chi) = \mathbf{1} \tag{16}$$

imply

$$\varphi_1 \to (\varphi_1 \to (\varphi_1 \to \psi)) = \mathbf{1}. \tag{17}$$

The only case to show is:

- $\cdot\ \psi = \mathbf{g}$
  - $\cdot\ \varphi_1 = \mathbf{a}$ then $\chi \geq \mathbf{d}$ from (16) and therefore (15) does not hold.

In all other cases (17) holds. ■

Corollary 2 *Given* (4) *and* (5)*, and if* $a_1 = 0$*,* $a_2 = 1$*,* $b_1 = 0$*,* $b_2 = 2$*, then the condition* (6) *holds.*

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Lemma 12 *Given* (4) *and* (5)*, and if* $a_1 = 0$, $a_2 = 1$, $b_1 = 2$, $b_2 = 0$*, then the condition* (6) *holds.*

Proof: We need to show that

$$\varphi_2 \to (\chi \to \psi) = \mathbf{1}, \tag{18}$$
$$\varphi_1 \to (\varphi_1 \to \chi) = \mathbf{1} \tag{19}$$

imply

$$\varphi_1 \to (\varphi_1 \to (\varphi_2 \to \psi)) = \mathbf{1}. \tag{20}$$

We prove it by cases:

- $\psi = \mathbf{g}$
  - $\varphi_2 = \mathbf{a}$
    - $\varphi_1 = \mathbf{a}, \mathbf{b}$ by (19) $\chi \geq \mathbf{e}$ and so (18) fails.
  - $\varphi_2 = \mathbf{b}$
    - $\varphi_1 = \mathbf{a}$ by (19) $\chi \geq \mathbf{d}$ and therefore (18) fails.
  - $\varphi_2 = \mathbf{c}$
    - $\varphi_1 = \mathbf{a}$ by (19) $\chi \geq \mathbf{d}$ and therefore (18) fails.

In all other cases (20) holds. ∎

Lemma 13 *Given* (4) *and* (5)*, and if* $a_1 = 1$, $a_2 = 0$, $b_1 = 0$, $b_2 = 2$*, then the condition* (6) *holds.*

Proof: We need to show that

$$\varphi_1 \to (\chi \to \psi) = \mathbf{1}, \tag{21}$$
$$\varphi_2 \to (\varphi_2 \to \chi) = \mathbf{1} \tag{22}$$

imply

$$\varphi_1 \to (\varphi_2 \to (\varphi_2 \to \psi)) = \mathbf{1}. \tag{23}$$

We prove it by cases:

- $\psi = \mathbf{g}$
  - $\varphi_2 = \mathbf{a}$
    - $\varphi_1 = \mathbf{a}, \mathbf{b}, \mathbf{c}$ by (22) $\chi \geq \mathbf{d}$ and so (21) fails.
  - $\varphi_2 = \mathbf{b}$
    - $\varphi_1 = \mathbf{a}$ by (22) $\chi \geq \mathbf{e}$ and therefore (21) fails.

In all other cases (23) holds. ∎

Lemma 14 *Given* (4) *and* (5)*, and if* $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, $b_2 = 0$*, then the condition* (6) *holds.*

Proof: We need to show that

$$\varphi_1 \to (\varphi_2 \to (\chi \to \psi)) = \mathbf{1}, \tag{24}$$
$$\varphi_1 \to \chi = \mathbf{1} \tag{25}$$

imply

$$\varphi_1 \to (\varphi_1 \to (\varphi_2 \to \psi)) = \mathbf{1}. \tag{26}$$

We prove it by cases:

- $\psi = \mathbf{g}$
  - $\varphi_2 = \mathbf{a}$
    - $\varphi_1 = \mathbf{a}, \mathbf{b}$ by (25) $\chi \geq \mathbf{b}$ and so (24) fails.
  - $\varphi_2 = \mathbf{b}$
    - $\varphi_1 = \mathbf{a}$ by (25) $\chi \geq \mathbf{a}$ and therefore (24) fails.
  - $\varphi_2 = \mathbf{c}$
    - $\varphi_1 = \mathbf{a}$ by (25) $\chi \geq \mathbf{a}$ and therefore (24) fails.

In all other cases (26) holds. ∎

Lemma 15 *Given* (4) *and* (5)*, and if* $a_1 = 1$, $a_2 = 0$, $b_1 = 1$, $b_2 = 1$*, then the condition* (6) *holds.*

Proof: We need to show that

$$\varphi_1 \to (\chi \to \psi) = \mathbf{1}, \tag{27}$$
$$\varphi_1 \to (\varphi_2 \to \chi) = \mathbf{1} \tag{28}$$

imply

$$\varphi_1 \to (\varphi_1 \to (\varphi_2 \to \psi)) = \mathbf{1}. \tag{29}$$

We prove it by cases:

- $\psi = \mathbf{g}$
  - $\varphi_2 = \mathbf{a}$
    - $\varphi_1 = \mathbf{a}$ by Lemma 11.
    - $\varphi_1 = \mathbf{b}$ by (28) $\chi \geq \mathbf{d}$ and therefore (27) fails.
  - $\varphi_2 = \mathbf{b}$
    - $\varphi_1 = \mathbf{a}$ by (28) $\chi \geq \mathbf{d}$ and therefore (27) fails.
    - $\varphi_1 = \mathbf{b}$ by Lemma 11.
  - $\varphi_2 = \mathbf{c}$
    - $\varphi_1 = \mathbf{a}$ by (28) $\chi \geq \mathbf{e}$ and therefore (27) fails.

In all other cases (29) holds. ∎

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Lemma 16 *Given* (4) *and* (5), *and if* $a_1 = 0$, $a_2 = 1$, $b_1 = 1$, $b_2 = 1$, *then the condition* (6) *holds.*

Proof: We need to show that

$$\varphi_2 \to (\chi \to \psi) = \mathbf{1}, \qquad (30)$$

$$\varphi_1 \to (\varphi_2 \to \chi) = \mathbf{1} \qquad (31)$$

imply

$$\varphi_1 \to (\varphi_2 \to (\varphi_2 \to \psi)) = \mathbf{1}. \qquad (32)$$

We prove it by cases:

- $\cdot\ \psi = \mathbf{g}$
  - $\cdot\ \varphi_2 = \mathbf{a}$
    - $\cdot\ \varphi_1 = \mathbf{a}, \mathbf{b}, \mathbf{c}$ by (31) $\chi \geq \mathbf{e}$ and so (30) fails.
  - $\cdot\ \varphi_2 = \mathbf{b}$
    - $\cdot\ \varphi_1 = \mathbf{a}$ by (31) $\chi \geq \mathbf{d}$ and therefore (30) fails.

In all other cases (32) holds. ■

Theorem 2 *There is a logic with IDT$_2$ but without IDT$_3$.*

In Tables 2 and 3 we spell all variants needed by Corollary 1 to show that the logic $\mathbf{L}$ given by $\mathbf{M}$ has IDT$_2$, which together with Lemma 3 completes the proof of the previous theorem. Consequently, we obtained the complete picture of hierarchy of Implicational Deduction Theorems

$$\mathrm{IDT}_0 \supsetneq \mathrm{IDT}_1 \supsetneq \mathrm{IDT}_2 \supsetneq \mathrm{IDT}_3 = \mathrm{IDT}_4 = \dots$$

a_1	a_2	b_1	b_2	Solution
0	0	0	0	Lemma 7
1	0	0	0	Lemma 7
0	1	0	0	Lemma 7
0	0	1	0	Lemma 7
0	0	0	1	Lemma 7
2	0	0	0	Lemma 7
0	2	0	0	Lemma 7
0	0	2	0	Lemma 7
0	0	0	2	Lemma 7
1	1	0	0	Lemma 7
1	0	1	0	Lemma 8
1	0	0	1	Lemma 8
0	1	1	0	Lemma 9
0	1	0	1	Lemma 8
0	0	1	1	Lemma 7

Table 2: Proof variants for $a_1 + a_2 + b_1 + b_2 < 3$.

a_1	a_2	b_1	b_2	Solution
3	0	0	0	Lemma 7
0	3	0	0	Lemma 7
0	0	3	0	Lemma 7
0	0	0	3	Lemma 7
2	1	0	0	Lemma 7
2	0	1	0	Lemma 8
2	0	0	1	Lemma 8
1	2	0	0	Lemma 7
0	2	1	0	Lemma 10
0	2	0	1	Lemma 8
1	0	2	0	Lemma 11
0	1	2	0	Lemma 12
0	0	2	1	Lemma 7
1	0	0	2	Lemma 13
0	1	0	2	Corollary 2
0	0	1	2	Lemma 7
1	1	1	0	Lemma 14
1	1	0	1	Lemma 8
1	0	1	1	Lemma 15
0	1	1	1	Lemma 16

Table 3: Proof variants for $a_1 + a_2 + b_1 + b_2 = 3$.

5. Summary

We presented a hierarchy of logics satisfying some Implicational Deduction Theorems. We know that any logic with IDT$_i$ has also IDT$_j$ for any $j \leq i$ and for any $0 \leq i \leq 1$ there is a logic with IDT$_i$ but without IDT$_{i+1}$. Our paper showed that there is also a logic with IDT$_2$ but without IDT$_3$. Moreover, any logic with IDT$_i$, for $i \geq 3$, has also IDT$_j$ for any $j \geq i$ and hence any j. This completes the picture of our hierarchy.

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