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Abstract:

This study deals with conflicts of belief functions. Internal conflicts of belief functions and conflicts between belief functions are described and analyzed here. Differences of belief functions are distinguished from conflicts between them. Three new different approaches to conflicts are presented: combinational, plausibility and comparative. The presented approaches to conflicts are compared to Liu's interpretation of conflicts.

Keywords:
Belief functions, Dempster-Shafer theory, internal conflict, conflict between belief functions, combinational conflict, plausibility conflict, comparative conflict.

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1 Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing that enable representation of incomplete and uncertain knowledge, belief updating and combination of evidence. They were originally introduced as a principal notion of the Dempster-Shafer Theory or the Mathematical Theory of Evidence [9].

When combining belief functions (BFs) by the conjunctive rules of combination, conflicts often appear which are assigned to $\emptyset$ by non-normalized conjunctive rule $\odot$ or normalized by Dempster's rule of combination $\oplus$. Combination of conflicting BFs and interpretation of conflicts is often questionable in real applications, thus a series of alternative combination rules was suggested and a series of papers on conflicting belief functions was published, e.g. [2, 5, 8, 11].

This study introduces new ideas to interpretation, definition and measuring of conflicts of BFs. Three new approaches to interpretation and computation of conflicts are presented here.

The first one, the combinational approach is a modification of commonly used interpretation of conflict of BFs. An internal conflict within individual BFs is distinguished here from a conflict between two BFs which are combined (Section 3).

The second one, the plausibility approach also distinguishes internal conflict and conflict between BFs. This approach uses the normalized plausibility transformation and is based on support / opposition of elements of Ω by the BFs in question. Differences of BFs are distinguished from conflicts between them in this approach; as relatively highly different BFs are not necessarily mutually conflicting (Section 4).

The third approach, the comparative one, is based on a specification of bbms of focal elements to smaller ones and on measuring difference between such more specified BFs (Section 5).

After the presentation of new ideas, the presented approaches are compared and a series of open problems is suggested.

2 Preliminaries

Let us assume an exhaustive finite *frame of discernment* $\Omega = \{\omega_1, ..., \omega_n\}$, whose elements are mutually exclusive.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$, such that $\sum_{A\subseteq\Omega} m(A) = 1$, $m(\emptyset) = 0$; the values of bba are called *basic belief masses (bbm).* [3] A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$; let us further recall a *plausibility function* $Pl(A) = \sum_{\emptyset\neq A\cap X} m(X)$; bba m, belief function Bel and plausibility Pl uniquely correspond each to others. $\mathcal{P}(\Omega)$ is often denoted by 2^Ω.

A *focal element* is a subset X of the frame of discernment, such that $m(X) > 0$. If all the focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function* (BBF), it is a probability distribution on Ω in fact. Let us denote U_n the *uniform Bayesian belief function*[4] on n-element frame $\Omega_n = \{\omega_1, ..., \omega_n\}$, i.e. the uniform probability distribution on Ω_n. The belief function with the only focal element $m(\Omega) = 1$ is called the *vacuous belief function (VBF)*, a belief function with the only focal element $m(A) = 1$ is called *categorical* (or logical [1]) *belief function*, a belief function with two focal elements $m(A) = A$ and $m(\Omega) = 1 - A$ is called a *simple support function*, a belief function which focal elements are nested is called a *consonant belief function*.

The *normalized plausibility of Bel* is the BBF (a probability distrib.) $(Pl\_P(m))(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega\in\Omega} Pl(\{\omega\})}$. The *pignistic probability of Bel* is the following probability distribution $BetP(\omega_i) = \sum_{\omega_i\in X\subseteq\Omega} \frac{m(X)}{|X|}$.

Dempster's (conjunctive) rule of combination $\oplus$ is given as $(m_1 \oplus m_2)(A) = \sum_{X\cap Y=A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X\cap Y=\emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [9]; putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\odot$, see e. g. [10].

[3] $m(\emptyset) = 0$ is often assumed in accordance with Shafer's definition [9]. A classical counter example is Smets' Transferable Belief Model (TBM) which admits $m(\emptyset) \geq 0$.

[4] U_n which is idempotent w.r.t. Dempster's rule $\oplus$, and moreover neutral on the set of all BBFs, is denoted as ${}_{nD}0'$ in [6]; specially U_2 is denoted as ${}_{2D}0'$ there or simply as $0'$ (see also [4]), $0'$ comes from studies by Hájek & Valdes (e.g. [7]).

Any BF on $\Omega_2 = \{\omega_1, \omega_2\}$ is uniquely specified by two bbms $m(\{\omega_1\}), m(\{\omega_2\})$ as $m(\{\omega_1, \omega_2\}) = 1-(m(\{\omega_1\})+m(\{\omega_2\}))$. Hence we can simply represent any m on Ω_2 as a pair $m = (m(\{\omega_1\}), m(\{\omega_1\}))$. The normalized plausibility of m is also very simple in this case: $Pl\_P(m) = (Pl\_P(m)(\omega_1), 1 - Pl\_P(m)(\omega_1)) = (\frac{1-m(\{\omega_2\})}{2-(m(\{\omega_1\})+m(\{\omega_2\})}, \frac{1-m(\{\omega_1\})}{2-(m(\{\omega_1\})+m(\{\omega_2\})}) = m \oplus 0'$, where $0' = U_2 = (\frac{1}{2}, \frac{1}{2})$, VBF is $0 = (0, 0)$ on Ω_2, for detail see [4, 6].

3 Combinational conflicts of belief functions

3.1 Internal conflict of belief functions

When combining two belief functions Bel_1, Bel_2 given by bbms m_1 and m_2 conflicting masses $m_1(X) > 0$, $m_2(Y) > 0$ for $X \cap Y = \emptyset$ often appear. The sum of all pair-wise products of such belief masses corresponds to $m(\emptyset)$ when non-normalized conjunctive rule of combination is applied and $m = m_1 \circledcirc m_2$. This sum is called *weight of conflict between belief functions Bel_1 and Bel_2* in [9], and it is commonly used when dealing with conflicting belief functions. Unfortunately, the name and interpretation of this notion does not correctly correspond to reality in general. We often obtain positive sum of conflicting belief masses even if two numerically same belief functions are combined, see e.g. Example 1 [1], analogical example for $n = 5$ is discussed in [8]. As BBFs are used in Example 1, we present also Example 2 with general BFs.

Example 1 *Let us assume two BFs expressing that a six-sided die is fair.* $\Omega_6 = \{\omega_1, ..., \omega_6\} = \{1, 2, 3, 4, 5, 6\}$, $m_j(\{\omega_i\}) = 1/6$ *for* $i = 1, ..., 6$, $j = 1, 2$, $m_j(X) = 0$ *otherwise. Let* $m = m_1 \circledcirc m_2$. *We obtain* $m(\{\omega_i\}) = 1/36$ *for* $i = 1, ..., 6$, $m(\emptyset) = 5/6$, $m(X) = 0$ *otherwise.*

If we generalize Almond's and Liu's examples to uniform Bayesian belief functions on n-element frame $\Omega_n = \{\omega_1, ..., \omega_n\}$, *we have* $m_j(\{\omega_i\}) = 1/n$ *for* $i = 1, ..., n$, $j = 3, 4$, $m_j(X) = 0$ *otherwise, i.e.,* $m_j = U_n$. *Let* $m = m_3 \circledcirc m_4$. *We obtain resulting* $m(\{\omega_i\}) = 1/n^2$ *for* $i = 1, ..., n$, $m(\emptyset) = (n-1)/n$, $m(X) = 0$ *otherwise. We can notice that* $m(\emptyset)$ *increases with increasing* n.

Example 2 *Let us suppose for simplicity* $\Omega_2 = \{\omega_1, \omega_2\}$ *now. Let* $m_j(\{\omega_1\}) = 0.5$, $m_j(\{\omega_2\}) = 0.4$, $m_j(\{\omega_1, \omega_2\}) = 0.1$ *for* $j = 5, 6$, $m_j(X) = 0$ *otherwise. Let* $m = m_5 \circledcirc m_6$ *now. We obtain* $m(\{\omega_1\}) = 0.35$, $m(\{\omega_2\}) = 0.24$, $m(\{\omega_1, \omega_2\}) = 0.01$, $m(\{\omega_1\}) = 0.4$ $m(\emptyset) = 0.4$, $m(X) = 0$ *otherwise.*

Almond mentions that $m(\emptyset)$ is hardly interpretable as conflict between BFs in such a case [1]. Liu correctly says in [8], that $m(\emptyset)$ cannot be always interpreted as a degree of conflict between belief functions. On the other hand many of particular couples of belief masses are really in conflict with each other. From this we can see that the sum of all products of conflicting belief masses, what we call *total combinational conflict*, somehow includes also a conflict which is included within the individual belief functions, which are combined. We will call this *internal conflict*[5]. It is not known whether the internal conflicts are included in total conflict partially or entirely. On the other hand, a source of total combinational conflict $TotC$ arises either from internal combinational conflicts of individual BFs or from their mutual conflicting interrelations. Thus, we can describe this as

$$TotC(m_1, m_2) \leq IntC(m_1) + IntC(m_2) + C(m_1, m_2).$$

In the special case when two identical belief functions are combined we obtain $TotC(m, m) \leq IntC(m) + IntC(m)$, as we expect no conflict between two same pieces of evidence, because they fully agree with each other thus they are not in any mutual conflict. We further suppose $IntC(m) \leq TotC(m, m)$, thus we have

$$IntC(m) \leq TotC(m, m) \leq IntC(m) + IntC(m)$$

and $\frac{1}{2}TotC(m, m)) \leq IntC(m) \leq TotC(m, m)$.

Unfortunately, we have no precise formula how to precisely compute conflict $C(m_1, m_2)$ between BFs m_1 and m_2. Nevertheless we assume, that it is less than total conflict $TotC(m_1, m_2)$, and the

[5] We have to note, that Smets uses the name 'internal conflict' for $m(\emptyset)$ within individual non-normalized BFs [2]; nevertheless there are also other interpretations of $m(\emptyset)$ in non-normalized BFs. However, in our situation internal conflicts appear in classic BFs each satisfying $m(\emptyset) = 0$, see Examples 1, 2 and other examples in this report.

above inequality. Note again that internal conflict of one of the combined BFs can be greater than total conflict, when two different BFs are combined (e.g. $TotC(m, VBF) = 0$ and $IntC(VBF) = 0$, $IntC(m) \geq 0$, $C(m, VBF) = 0$). We can summarize this as it follows:

$$TotC(m_1, m_2) - (IntC(m_1)+IntC(m_2)) \leq C(m_1, m_2) \leq TotC(m_1, m_2).$$

3.2 Belief functions without internal conflict

There are many BFs without any internal conflicts: all categorical and all simple support BFs have no internal conflict as $A \cap A \neq \emptyset$, $A \cap \Omega = \Omega \cap A \neq \emptyset$ and $\Omega \cap \Omega \neq \emptyset$ for any $A \subseteq \Omega = \{\omega_1, ..., \omega_n\},\ n \geq 1$; further all consonant BFs have no internal conflict as the least focal element is a subset of or equal to intersection of any pair of focal elements; finally all BFs, whose all focal elements have non-empty intersection, have no internal conflict, i.e., all BFs such that there exist $X \subseteq \Omega,\ Pl(X) = 1$.

Example 3 *Let us suppose* $\Omega_4 = \{\omega_1, ..., \omega_4\}$ *and the following simple internally non-conflicting BFs:* $m_7(\{\omega_1, \omega_2\}) = 0.7$, $m_7(\Omega_4) = 0.3$; $m_8(\{\omega_1\}) = 0.1$, $m_8(\{\omega_1, \omega_2\}) = 0.3$, $m_8(\{\omega_1, \omega_2, \omega_3\}) = 0.2$, $m_8(\{\omega_1, \omega_3, \omega_4\}) = 0.1$, $m_8(\{\omega_1, \omega_2, \omega_3, \omega_4\}) = 0.3$.

3.3 Belief functions with internal conflict

As an example of BFs with internal conflict we can refer BFs from both Examples 1 and 2. Let us introduce another interesting example now.

Example 4 *Let us suppose* $\Omega_2 = \{\omega_1, \omega_2\}$ *and bba* $m_k = (\frac{1}{k}, \frac{k-1}{k})$ *for* $k > 1$.
$m_k \oplus m_k = (\frac{1}{1+(k-1)^2}, \frac{(k-1)^2}{1+(k-1)^2})$, $(m_k \circledcirc m_k)(\emptyset) = 2\frac{k-1}{k^2}$, $(m_k \circledcirc m_k)(\emptyset) \longrightarrow 0$ *for increasing* k. *Similarly* $(m_l \circledcirc m_l)(\emptyset) \longrightarrow 0$ *for* $m_l = (\frac{l-1}{l}, \frac{1}{l})$ *and increasing* l.

The greatest internal conflict among the BFs from the previous example has $0' = (\frac{1}{2}, \frac{1}{2}) = U_2$. Let us further note, that the maximal internal conflict of all BBFs on Ω_2 arises for $0' = U_2$, $(0' \circledcirc 0')(\emptyset) = \frac{1}{2}$. And for general BFs on Ω_2 the following observation holds true.

Observation 1 $0' = (\frac{1}{2}, \frac{1}{2}) = U_2$ *is the BF with the greatest internal conflict on* Ω_2.

Proof: Let us compute the total combinational conflict between BFs (a, b) and (a, b), which is an upper bound for internal conflict of BF (a, b); $((a, b) \circledcirc (a, b))(\emptyset) = ab + ba = 2ab$. $2ab$ such that $a, b \geq 0, a + b \leq 1$ is maximized iff $a = b$ is maximized iff $a = b = \frac{1}{2}$. □

3.4 Couples of totally non-conflicting belief functions

If there is neither any internal conflict of both members of a couple of BFs m_i, m_j nor a conflict between them, we say that m_i and m_j form a pair of *totally non-conflicting BFs*. This happens whenever all focal elements of both BFs have common non-empty intersection, i.e. whenever both BFs have non-empty intersections $I = \bigcap_{m_i(X)>0} X \neq \emptyset,\ J = \bigcap_{m_j(X)>0} X \neq \emptyset$ and $I \cap J \neq \emptyset$.

Example 5 *Totaly non-conflicting are e.g.* m_7 *and* m_8 *from the Example 3, or the following BFs on* Ω_6: $m_9(\{\omega_1, \omega_2, \omega_3, \omega_4\}) = 0.4$, $m_9(\{\omega_2, \omega_3, \omega_4\}) = 0.3$, $m_9(\Omega_6) = 0.3$;
$m_{10}(\{\omega_2, \omega_3, \omega_5\}) = 0.6$, $m_{10}(\{\omega_2, \omega_3, \omega_6\}) = 0.1$, $m_{10}(\{\omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}) = 0.2$, $m_{10}(\Omega_6) = 0.1$;
$I = \bigcap_{m_9(X)>0} X = \{\omega_2, \omega_3, \omega_4\}$, $J = \bigcap_{m_{10}(X)>0} X = \{\omega_2, \omega_3\}$ *and* $I \cap J = \{\omega_2, \omega_3\} \neq \emptyset$.
$(m_9 \circledcirc m_{10})(\emptyset) = 0$.

3.5 Combination of belief functions with the uniform Bayesian belief function U_n

When combining U_n with a simple support function (simple support (belief) function) $m(A) = a,\ m(\Omega) = 1 - a$ we obtain $m(\emptyset) = \frac{n-|A|}{n} m(A)$, specially $\frac{1}{2} m(A)$ for $\Omega_2 = \{\omega_1, \omega_2\}$. When combining U_n with a Bayesian belief function we always obtain $m(\emptyset) = \frac{n-1}{n}$, specially $m(\emptyset) = \frac{1}{2}$ for BBFs on Ω_2. Thus we have $IntC(0') + IntC(BBF) + C(0', BBF) \geq \frac{1}{2}$ and $\frac{1}{4} \leq IntC(0') \leq \frac{1}{2}$ on Ω_2, hence

$0 \leq C(0', BBF) \leq \frac{1}{4}$. It corresponds with the fact that $0'$ should be mutually non-conflicting with any BBF: 1) because $0'$ is neutral for Dempster's combination with BBFs, 2) Pl_P commutes with $\oplus$, thus all BFs with the same Pl_P should be mutually non-conflicting, VBF is mutually non-conflicting with any BF, what should hold true also for all BFs m' with the same $Pl\_P(m') = Pl\_P(VBF) = 0'$ (including $0'$ itself), more generally U_n for Ω_n. Let us look at the following simple example.

Example 6 *Let us suppose* $\Omega_2 = \{\omega_1, \omega_2\}$ *and bbas* $m_1 = (\frac{k-1}{k}, 0)$, $m_2 = (\frac{1}{k}, 0)$, $Pl\_P(m_1) = (\frac{k}{k+1}, \frac{1}{k+1})$, $Pl\_P(m_2) = (\frac{k}{2k-1}, \frac{k-1}{2k-1})$.
$(Pl\_P(m_1) \circledcirc m_1)(\emptyset) = \frac{k-1}{k(k+1)}$ *.... it is decreasing for increasing* k,
$(Pl\_P(m_1) \circledcirc Pl\_P(m_1))(\emptyset) = \frac{2k}{(k+1)^2}$ *.... it is decreasing for increasing* k,
as both m_1 *and* $Pl\_P(m_1)$ *go to* $\top = (1, 0)$ *for increasing* k;
$(Pl\_P(m_2) \circledcirc m_2)(\emptyset) = \frac{k-1}{k(2k-1)}$ *.... it is decreasing for increasing* k,
$(Pl\_P(m_2) \circledcirc Pl\_P(m_2))(\emptyset) = \frac{2k(k-1)}{(2k-1)^2}$ *.... it goes to* $\frac{1}{2}$ *for increasing* k,
as m_2 *goes to* $0 = (0, 0)$ *and* $Pl\_P(m_2)$ *to* $0' = (\frac{1}{2}, \frac{1}{2})$ *for increasing* k.

Note that the following holds true: $(a, b) \oplus 0' = (\frac{1-b}{2-a-b}, \frac{1-a}{2-a-b}) = Pl\_P((a, b))$ on Ω_2 and $m \oplus U_n = Pl\_P(m)$ on general frame Ω_n, see [6]. As the general case is not explicitly proved in [6], we can formulate is as the following proposition.

Proposition 2 *For normalized plausibility of singletons the following holds true:*

$$Pl\_P(m) = m \oplus U_n.$$

Proof: This simply follows results from [6], unfortunately not explicitly stated there. $Pl\_P(m) = Pl\_P(m) \oplus U_n$ (neutrality of U_n for BBFs) $= Pl\_P(m) \oplus Pl\_P(U_n) = Pl\_P(m \oplus U_n)$ (commutativity of Pl_P with $\oplus$) $= m \oplus U_n$ (as $m \oplus U_n$ is BBF, thus it is a fix point of Pl_P).
Or alternatively: $m \oplus U_n$ is normalization of $m \circledcirc U_n$, there is $(m \oplus U_n)(\{\omega_i\}) = \sum_{\omega_i \in A} \frac{1}{n} m(A) = \frac{1}{n} Pl(\{\omega_i\})$, and after normalization we obtain $(Pl\_P(m))(\{\omega_i\})$. □

4 Plausibility conflicts of belief functions

As in the previous section, we will further distinct internal conflicts of individual BFs from a mutual conflict between them. Let us first discuss what should belief functions really mean.

There is an unknown element $\omega_0 \in \Omega$ and we have only a partial uncertain evidence about the fact which one is it. This evidence is represented by a BF or by its corresponding bba. If all pieces of our evidence are correct and fully compatible with the situation, all focal elements should contain the unknown element ω_0 and there is now conflict within the corresponding BF. The more precise our evidence is the smaller should be the focal elements. In the extreme limit case of correct complete certain evidence there is the only focal element $\{\omega_0\}$, such that $m(\{\omega_0\}) = 1$. When obtaining new correct fully compatible pieces of evidence represented by BFs, their focal elements should also contain ω_0 and new BFs should be both internally and mutually non-conflicting. When combining such BFs their focal elements are decreasing keeping ω_0 as their element. Unfortunately real pieces of evidence often contain some conflicts or they are mutually conflicting or the situation itself may be (internally) conflicting. Hence we obtain internally and/or mutually conflicting BFs.

How is it possible that $m_1, m_2 = U_6$ from Example 1 have the high internal conflict? Let us notice that BF U_6, which was used in the example for description of behaviour of a fair die, does not express any belief about the fact which side of the die is up. It express a meta-information about the die, the information which is necessary within a decision making for redistribution of bbms of focal elements among their singletons. It express nothing about an uncertain case of the die. It is rather related to the betting/pignistic level than to the credal level of beliefs.

4.1 Internal plausibility conflict of belief functions

Element ω_0 should be element of all focal elements in correct non-conflicting cases, thus $Pl(\{\omega_0\})$ should be equal to 1. When $Pl(\{\omega_0\}) < 1$ there is some focal element X which does not include ω_0, thus $m(X)$ cannot be simply transferred to any $Y \subseteq X$ which includes ω_0. Such a BF is conflicting and it is often mutually conflicting with other BFs. On the other hand there can be more focal elements with plausibility 1 in less informative cases.

Let us define *internal plausibility conflict of belief function Bel* as

$$Pl\text{-}IntC(Bel) = 1 - max_{\omega\in\Omega} Pl(\{\omega\}),$$

where Pl is the plausibility equivalent to Bel. This definition is in accordance with the assumption from Section 3 that a BF is internally non-conflicting (BF has no internal conflict) whenever there exist $X \subseteq \Omega,\ Pl(X) = 1$. Maximal internal (plausibility) conflict has U_n: $Pl\text{-}IntC(U_n) = 1 - \frac{1}{n} = \frac{n-1}{n}$ as all elements ω_i have the same plausibility $\frac{1}{n}$ in the case of U_n and any change of belief masses increases plausibility of some $\omega \in \Omega$, hence internal plausibility conflict is decreased.

4.2 Plausibility conflict between belief functions on two-element frame of discernment Ω_2

For simplicity, let us suppose two-element frame of discernment $\Omega_2 = \{\omega_1, \omega_2\}$ in this subsection.

VBF is $0 = (0,0)$ on Ω_2. VBF is usually assumed to be neutral with respect to belief combination. This really holds for $0 = (0,0)$ and Dempter's rule of combination $\oplus$, i.e., $(0,0) \oplus (a,b) = (a,b)$ for any BF (a,b) on Ω_2, see [4]. $Pl\_P((0,0)) = (\frac{1}{2}, \frac{1}{2}) = 0'$ which is neutral when BBFs are combined with Demspter's rule: $(a, 1-a) \oplus (\frac{1}{2}, \frac{1}{2}) = (a, 1-a)$.
0, $0'$ and all BFs (a,a) do not support any of elements of Ω_2 because both of them have the same bbms and also the same Pl_P masses $\frac{1}{2}$. On the other hand all other BFs support one of ω_1, ω_2; ω_1 is *supported* by all BFs (a,b) for $a > b$ (where $Pl\_P((a,b))(\omega_1) > \frac{1}{2}$ [6]), thus ω_2 is *opposed* by these BFs. ω_1 is *fully (categorically) supported* and ω_2 is *fully opposed* by the categorical BF $\top = (1,0)$, where $Pl\_P((1,0)) = (1,0)$. In the other words, ω_1 is *confirmed* and ω_2 is *excluded* or *rejected* by $\top = (1,0)$. Analogically ω_1 is opposed and ω_2 supported by any BF (a,b) where $a < b$.

VBF $0 = (0,0)$ should be neutral thus it must be non-conflicting with any other BFs. Similarly for $0' = (\frac{1}{2}, \frac{1}{2})$ and all BFs (a,a) which do not support or reject any other BF.

Let us assume two BBFs $m_1 = (0.6, 0.4)$ and $m_2 = (0.8, 0.2)$. There is a relatively high difference between them and $(m_1 \circledcirc m_2)(\emptyset) = 0.44$, but both of them support ω_1 thus m_1 and m_2 should not be in mutual conflict. m_1 and m_2 are different but non-conflicting. Let us suppose $m_1 = (0.6, 0.4)$ and $m_3 = (0.45, 0.55)$ now. There is a less difference between them, and $(m_1 \circledcirc m_3)(\emptyset) = 0.51$ is higher. m_1 and m_3 support different ω_i thus they should be in a mutual conflict. Let us suppose $m_1 = (0.6, 0.4)$ and $m_4 = (0.40, 0.45)$ now. There is less difference between m_1 and m_4, than between m_1 and m_2, and $(m_1 \circledcirc m_4)(\emptyset) = 0.43$ is also smaller; but m_1 and m_4 support different ω_i thus they should be in mutual conflict, despite of mutually non-conflicting m_1 and m_2 which have both greater difference and greater mutual $m(\emptyset)$.

Similarly, all BFs which support ω_1 (i.e., (a,b) such that $a > b$, $Pl\_P((a,b)) > \frac{1}{2}$, i.e. $(a,b) > 0'$) should not be in mutual conflict. On the other hand, there is a conflict between any two BFs which support different ω_i (i.e., $(a,b), (c,d)$ such that $a > b, c < d$ or $a < b, c > d$).

Let us define: *two BFs Bel_1, Bel_2 on Ω_2 are mutually conflicting* whenever $(Pl\_P(Bel_1)(\omega_1) - \frac{1}{2})(Pl\_P(Bel_2)(\omega_1) - \frac{1}{2}) < 0$; they are *mutually non-conflicting* otherwise.

We would like to define the *plausibility conflict between two mutually conflicting BFs* $(a,b), (c,d)$ on Ω_2 as

$$Pl\text{-}C_0((a,b),(c,d)) = |Pl\_P((a,b))(\omega_1) - Pl\_P((c,d))(\omega_1)|.$$

Plausibility conflict should be 0 between any two mutually non-conflicting bbas.

Let us define *difference between two BFs Bel_1, Bel_2 on Ω* represented by m_1, m_2 as $Diff(m_1, m_2) = \sum_{X\subset\Omega} \frac{1}{2}|m_1(X) - m_2(X)|$, i.e., $Diff((a,b),(c,d)) = \frac{1}{2}(|a-c| + |b-d|)$.

Let us further define *Pl-difference between two BFs Bel_1, Bel_2*:
$Pl\text{-}Diff(m_1, m_2) = Diff(Pl\_P(m_1), Pl\_P(m_2))$ which is more related to a support/opposition of elements ω_i by m_i and to their plausibility conflictness.

Example 7 $m_1 = (0.4, 0.4)$, *$Pl\text{-}IntC(m_1) = 0.4$,* $m_2 = (0.9, 0.1)$, *$Pl\text{-}IntC(m_1) = 0.1$, $Diff(m_1, m_2) = Pl\text{-}Diff(m_1, m_2) = 0.4$, $Pl\text{-}C(m_1, m_2) = 0$.*

Observation 3 *The following holds true for Pl-C_0 on Ω_2:*
$Pl\text{-}C_0((a,b),(c,d)) = Pl\text{-}Diff((a,b),(c,d))$ *iff* $(a-b)(c-d) \leq 0$,
$Pl\text{-}C_0((a,b),(c,d)) = 0$ *iff* $(a-b)(c-d) \geq 0$.

4.3 Plausibility conflict between belief functions on general Ω_n

Plausibility conflict between belief functions is based on normalized plausibility of elements of Ω. It is computed separately for all elements of the frame of discernment Ω. VBF is usually assumed to be neutral when belief functions are combined. Normalized plausibility masses (see e.g. [3, 6]) of all $\omega \in \Omega$ are $Pl\_P(VBF)(\omega) = \frac{1}{n}$ in the case of VBF. Entire normalized plausibility of VBF is $Pl\_P(VBF) = U_n$ (which is idempotent and neutral w.r.t. combination $\oplus$ of BBFs).

Let us suppose a decision with respect to a given BF Bel: Whenever normalized plausibility $Pl\_P(Bel)(\omega)$ is greater than $\frac{1}{n}$, ω is *supported* by the BF in question. $\omega \in \Omega$ is *confirmed* when $Pl\_P(Bel)(\omega) = 1$, i.e., when $Bel(\{\omega\}) = 1$ (when $Pl(\Omega \setminus \{\omega\}) = 0$). On the other hand, ω is *opposed* when $Pl\_P(Bel)(\omega) < \frac{1}{n}$. ω is fully opposed (*excluded* or *rejected*) when $Pl\_P(Bel)(\omega) = 0$ as bbms of all X ($\omega \in X$) are zeros ($Pl(\{\omega\}) = 0$) and all positive bbms are assigned only to focal elements Y such that $\omega \notin Y$ ($Bel(\Omega \setminus \{\omega\}) = 1$).

If *normalized plausibility masses* $Pl\_P(Bel_1)(\omega)$, $Pl\_P(Bel_2)(\omega)$ are both $\geq \frac{1}{n}$ or both $\leq \frac{1}{n}$ we will say that they *are non-conflicting* (Bel_1 and Bel_2 are non-conflicting w.r.t. ω). We will say that Bel_1 and Bel_2 are non-conflicting when they are non-conflicting w.r.t. all $\omega \in \Omega$. It seems that these normalized plausibility masses are conflicting whenever one of them is $> \frac{1}{n}$ and the other $< \frac{1}{n}$. Let us denote the set of all elements which have not non-conflicting normalized plausibility masses by $\Omega_{PlC}(Bel_1, Bel_2) = \{\omega \in \Omega \mid (Pl\_P(Bel_1)(\omega) - 1/n)(Pl\_P(Bel_2)(\omega) - 1/n) < 0\}$.

We want to define *plausibility conflict between belief functions* Bel_1, Bel_2 (represented by bbas m_1 and m_2) as the sum of differences of conflicting normalized plausibility masses by the following formula

$$Pl\text{-}C_0(Bel_1, Bel_2) \quad = \sum_{\omega \in \Omega_{PlC}(Bel_1, Bel_2)} \frac{1}{2} \mid Pl\_P(Bel_1)(\omega) - Pl\_P(Bel_2)(\omega) \mid$$

Unfortunately this expression produces/classifies conflicts even in some cases of simple internally non-conflicting BFs, see Example 8. It is caused because $\omega \in \Omega_{PlC}(Bel_1, Bel_2)$ has frequently really conflicting normalized plausibility masses $Pl\_P(Bel_1)(\omega)$ and $Pl\_P(Bel_2)(\omega)$, but these $Pl\_P(Bel_1)(\omega)$ and $Pl\_P(Bel_2)(\omega)$ are only *potentially conflicting* in general. $Pl\text{-}C_0(Bel_1, Bel_2)$ is usually less than $m(\emptyset)$ in general examples, nevertheless in the case similar to those from Example 8 we have to use the following modified definition:

$$Pl\text{-}C(Bel_1, Bel_2) = min(Pl\text{-}C_0(Bel_1, Bel_2),\ (m_1 \circledcirc m_2)(\emptyset)).$$

We will say that Bel_1 *and* Bel_2 *are non-conflicting* whenever they are non-conflicting w.r.t. all $\omega \in \Omega$ or if $(m_1 \circledcirc m_2)(\emptyset) = 0$, i.e., whenever $Pl\text{-}C(Bel_1, Bel_2) = 0$.

Example 8 *Let us suppose* Ω_6 *now. Let* $m_1(\{\omega_1\}) = 1$, $m_2(\{\omega_1, \omega_2, \omega_3, \omega_4\}) = 1$. *We obtain* $Pl\_P(m_1)(\omega_1) = 1 > \frac{1}{6}$, $Pl\_P(m_1)(\omega_i) = 0 < \frac{1}{6}$ *for* $i > 1$; $Pl\_P(m_2)(\omega_i) = \frac{1}{4} > \frac{1}{6}$ *for* $i = 1, 2, 3, 4$, $Pl\_P(m_2)(\omega_i) = 0 < \frac{1}{6}$ *for* $i = 5, 6$; *normalized plausibility masses are conflicting for* $\omega_2, \omega_3, \omega_4$, *thus* $Pl\text{-}C_0(m_1, m_2) = \frac{1}{2}(\frac{1}{4} + \frac{1}{4} + \frac{1}{4}) = \frac{3}{8}$. *Nevertheless* m_1 *and* m_2 *seem to be intuitively non-conflicting.*

Example 8 (cont.) $(m_1 \circledcirc m_2)(\{\omega_1\}) = 1$, $(m_1 \circledcirc m_2)(X) = 0$ *otherwise, specially also* $(m_1 \circledcirc m_2)(\emptyset) = 0$, *thus* m_1 *and* m_2 *are combinationaly non-conflicting. When computing* $Pl\text{-}C(m_1, m_2)$ *we obtain* $Pl\text{-}C(m_1, m_2) = min(\frac{3}{8}, 0) = 0$, *hence there is neither any plausibility conflict between* m_1 *and* m_2. *Hence* m_1 *and* m_2 *are really non-conflicting.*

5 Comparative conflict between belief functions

Thirdly, let us suggest another idea of conflictness / non-conflictness between belief functions which is motivated by interpretation of BFs and their corresponding bbas. We know that our belief on a specific situation can be usually specified by obtaining of a new evidence tending to decreasing size of focal elements. The idea of comparative conflictness / non-conflictness is a specification of bbms to smaller focal elements such that fit to focal elements of the other BF as much as possible. *The comparative conflict between BFs Bel_1 and Bel_2* is defined as the least difference of such more specified bbms derived from the input m_1 and m_2.

Example 9 *Let us start with a simple example on Ω_2. Let $m_1 = (0.4, 0)$, $m_2 = (0, 0.4)$, $m_3 = (0.6, 0)$, $m_4 = (0, 0.6)$.*

All considered BFs are simple support functions, thus they have no internal conflicts either combinational or plausibility one. On the other hand BFs m_1 and m_2 are mutually conflicting in the previous sense, there are both combinational and plausibility conflicts between them, similarly for BFs m_3 and m_4. $(m_1 \odot m_2)(\emptyset) = 0.16$, $(m_3 \odot m_4)(\emptyset) = 0.36$, $Pl\_P(m_1) = (\frac{10}{16}, \frac{6}{16})$, $Pl\_P(m_2) = (\frac{6}{16}, \frac{10}{16})$, $Pl\_P(m_3) = (\frac{10}{14}, \frac{4}{14})$, $Pl\_P(m_4) = (\frac{4}{14}, \frac{10}{14})$, and $Pl\text{-}C_0(m_1, m_2) = \frac{6}{16} = 0.375$, $Pl\text{-}C_0(m_3, m_4) = \frac{6}{14} = 0.42857$, thus $Pl\text{-}C(m_1, m_2) = (m_1 \odot m_2)(\emptyset) = 0.16$, $Pl\text{-}C(m_3, m_4) = (m_3 \odot m_4)(\emptyset) = 0.36$.

In the first case (of m_1 and m_2) we can specify part of bbms $m_i(\Omega)$ to singletons to obtain numerically same, thus mutually non-conflicting BFs: $m_1' = (0.4, 0.4)$, $m_2' = (0.4, 0.4)$, thus $Diff(m_1', m_2') = 0$ and cp-$C(m_1, m_2) = 0$, i.e. m_1 and m_2 are comparatively non-conflicting. Note that, $m_1'(\Omega_2) > 0$, thus there are many other possibilities of non-conflicting bbm specifications of m_1 and m_2 in this case: e.g. $m_1'' = (0.6, 0.4)$, $m_2'' = (0.6, 0.4)$, $m_1''' = (0.5, 0.4)$, $m_2''' = (0.5, 0.4)$, $m_1'''' = (0.45, 0.50)$, $m_2'''' = (0.45, 0.50)$, etc.

In the second case (of m_3 and m_4) either specification of entire $m_i(\Omega)$ does not produce non-conflicting BFs: $m_3' = (0.6, 0.4)$, $m_4' = (0.4, 0.6)$, thus $Diff(m_3', m_4') = cp\text{-}C(m_3, m_4) = 0.2$, because there is no possibility to assign bbm 0.6 both to ω_1 and ω_2 (and reverse relocation of some bbm from ω_1 and/or ω_2 to entire Ω_2 is not a specifiation of bbms). Note that other specifications of bbms have greater difference than m_3' and m_4' have; e.g. $m_3'' = (0.7, 0.3)$, $m_4'' = (0.3, 0.7)$, $m_3''' = (0.6, 0.3)$, $m_4''' = (0.3, 0.6)$, $m_3'''' = (0.8, 0.2)$, $m_4'''' = (0.4, 0.6)$, where $Diff(m_3'', m_4'') = 0.4$, $Diff(m_3''', m_4'') = 0.3$, $Diff(m_3'''', m_4'''') = 0.4$, etc.

The comparative result is qualitatively different from combinational and plausibility approaches in the case of comparatively non-conflicting m_1 and m_2 which are both plausibility and combinationally mutually conflicting: $Pl\text{-}C(m_1, m_2) = min(0.25, 0.16) = 0.16$ ($Pl\text{-}IntC(m_1) = Pl\text{-}IntC(m_2) = 0$) and $C(m_1, m_2) = 0.16$ ($IntC(m_1) = IntC(m_2) = 0$). There is no qualitative difference in the case of comparatively conflicting m_3 and m_4, but in accordance with the first case the comparative conflict 0.2 is less than mutual plausibility and combinational conflicts are: $Pl\text{-}C(m_3, m_4) = min(0.2857, 0.36) = 0.2857$ ($Pl\text{-}IntC(m_3) = Pl\text{-}IntC(m_4) = 0$) and $C(m_3, m_4) = 0.36$ ($IntC(m_1) = IntC(m_2) = 0$). The comparative result for the couple of BFs m_3, m_4 is not qualitatively different from combinational and plausibility approaches, however also both combinational and plausibility mutual conflicts between these BFs are greater than those between comparatively non-conflicting m_1 and m_2: $C(m_3, m_4) = 0.36 > 0.16 = C(m_1, m_2)$ and $Pl\text{-}C(m_3, m_4) = 0.2857 > 0.16 = Pl\text{-}C(m_1, m_2)$.

Example 10 *Let us assume more general example on Ω_3 now. Let $m_5(\{\omega_1\}) = 0.3$, $m_5(\{\omega_1, \omega_2\}) = 0.6$, $m_5(\{\omega_1, \omega_2, \omega_3\}) = 0.1$, $m_6(\{\omega_2\}) = 0.3$, $m_6(\{\omega_3\}) = 0.1$, $m_6(\{\omega_1, \omega_3\}) = 0.5$, $m_6(\{\omega_2, \omega_3\}) = 0.1$. There is neither combinational nor plausibility internal conflict in m_5, there is $0.18 \leq IntC(m_6) \leq 0.36$, $Pl\text{-}IntC(m_6) = 0.3$, $(m_5 \odot m_6)(\emptyset) = 0.21$, there are the following normalized plausibilities $Pl\_P(m_5) = (\frac{10}{18} > \frac{1}{3}, \frac{7}{18} > \frac{1}{3}, \frac{1}{18} < \frac{1}{3})$, $Pl\_P(m_6) = (\frac{5}{16} < \frac{1}{3}, \frac{4}{16} < \frac{1}{3}, \frac{7}{16} > \frac{1}{3})$, all the elements ω_i supported by m_5 are opposed by m_6 and vice versa, thus there is both combinational and plausibility conflict between m_5 and m_6.*

We can specify bbms of focal element to smaller ones (uniquely to singletons in this case) as it follows: $m_5'(\{\omega_1\}) = 0.5$, $m_5'(\{\omega_2\}) = 0.4$, $m_5'(\{\omega_3\}) = 0.1$, $m_6'(\{\omega_1\}) = 0.5$, $m_6'(\{\omega_2\}) = 0.4$, $m_6'(\{\omega_3\}) = 0.1$. We have obtained the numerically same BFs m_5', m_6', thus $Diff(m_5', m_6') = 0$ and m_5 and m_6 are comparatively non-conflicting.

The comparative approach to conflicts classifies less conflicting BFs than the previous two approaches do. Unfortunately no algorithm for specification of bbms to smaller focal elements has been yet created. Thus this new approach can be applied only to simple illuminative examples now. An elaboration of this approach remains as an open problem for future.

6 Comparison of the presented approaches

Let us compare the presented approaches and Liu's two-dimensional measure of conflict cf on three examples. cf is defined as $cf(m_i, m_j) = (m_\oplus(\emptyset), difBetP_{m_i}^{m_j})$ in [8], where $m_\oplus(\emptyset)$ should be rather $m_{\circledcirc}(\emptyset)$ (more precisely $(m_i \circledcirc m_j)(\emptyset)$) in fact, and the second component $difBetP_{m_i}^{m_j}$ is defined as $difBetP_{m_i}^{m_j} = max_{A\subseteq\Omega}(|BetP_{m_i}(A) - BetP_{m_j}(A)|)$, we can simplify this using $Diff(BetP_{m_i}, BetP_{m_j})$ according the following observation.

Observation 4 *For $difBetP$, any bbms m_i, m_j and the corresponding pignistic probabilities $BetP_i$, $BetP_j$ the following holds true:*

$$difBetP_{m_i}^{m_j} = Diff(BetP_i, BetP_j).$$

Proof: $difBetP_{m_i}^{m_j} = |BetP_{m_i}(A_m) - BetP_{m_j}(A_m)|$ for some $A_m \subset \Omega$ such that, $|BetP_{m_i}(A_m) - BetP_{m_j}(A_m)| \geq |BetP_{m_i}(A) - BetP_{m_j}(A)|$ for all $A \subseteq \Omega$.

i) Let us suppose $difBetP_{m_i}^{m_j} = BetP_{m_i}(A_m) - BetP_{m_j}(A_m)$, i.e., $BetP_{m_i}(A_m) \geq BetP_{m_j}(A_m)$ now. Thus there is $BetP_{m_i}(\{\omega\}) \geq BetP_{m_j}(\{\omega\})$ for all $\omega \in A_m$ and $BetP_{m_i}(\{\omega\}) \leq BetP_{m_j}(\{\omega\})$ for all $\omega \notin A_m$.

If it does not hold true and there is $\omega_x \in A_m$ such that $BetP_{m_i}(\{\omega_x\}) < BetP_{m_j}(\{\omega_x\})$ or there is $\omega_y \in \Omega \setminus A_m$ such that $BetP_{m_i}(\{\omega_y\}) > BetP_{m_j}(\{\omega_y\})$. For ω_x we obtain $BetP_{m_i}(A_m \setminus \{\omega_x\}) - BetP_{m_j}(A_m \setminus \{\omega_x\}) = BetP_{m_i}(A_m) - BetP_{m_i}(\{\omega_x\}) - BetP_{m_j}(A_m) + BetP_{m_j}(\{\omega_x\}) = BetP_{m_i}(A_m) - BetP_{m_j}(A_m) + BetP_{m_j}(\{\omega_x\} - BetP_{m_i}(\{\omega_x\}) > BetP_{m_i}(A_m) - BetP_{m_j}(A_m)$. Thus existence of ω_x is in contradiction with maximality of difference of $BetP$ for A_m.

Analogically for ω_y we obtain $BetP_{m_i}(A_m \cup \{\omega_y\}) - BetP_{m_j}(A_m \cup \{\omega_y\}) = BetP_{m_i}(A_m) + BetP_{m_i}(\{\omega_y\}) - BetP_{m_j}(A_m) - BetP_{m_j}(\{\omega_y\}) = BetP_{m_i}(A_m) - BetP_{m_j}(A_m) + BetP_{m_i}(\{\omega_y\} - BetP_{m_j}(\{\omega_y\}) > BetP_{m_i}(A_m) - BetP_{m_j}(A_m)$. Thus existence of ω_y is in contradiction with maximality for A_m again.

Hence $difBetP_{m_i}^{m_j} = \sum_{BetP_{m_i}(\{\omega\})>BetP_{m_j}(\{\omega\})} BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\})$.

ii) Analogically, in the case of $difBetP_{m_i}^{m_j} = -(BetP_{m_i}(A_m) - BetP_{m_j}(A_m))$, there is $BetP_{m_i}(\{\omega\}) \leq BetP_{m_j}(\{\omega\})$ for all $\omega \in A_m$ and $BetP_{m_i}(\{\omega\}) \geq BetP_{m_j}(\{\omega\})$ for all $\omega \notin A_m$. Thus there is $difBetP_{m_i}^{m_j} = -\sum_{BetP_{m_i}(\{\omega\})<BetP_{m_j}(\{\omega\})} BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\}) = \sum_{BetP_{m_i}(\{\omega\})>BetP_{m_j}(\{\omega\})} BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\})$, because $\sum_{\omega\in\Omega} BetP_{m_i}(\{\omega\}) = 1 = \sum_{\omega\in\Omega} BetP_{m_j}(\{\omega\})$, and $\sum_{\omega\in\Omega} BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\}) = 0$.

Thus we have $difBetP_{m_i}^{m_j} = \sum_{BetP_{m_i}(\{\omega\})>BetP_{m_j}(\{\omega\})} BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\}) =$
$\sum_{BetP_{m_i}(\{\omega\})<BetP_{m_j}(\{\omega\})} BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\}) =$
$\frac{1}{2}[\sum_{BetP_{m_i}(\{\omega\})>BetP_{m_j}(\{\omega\})} BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\})$
$+\sum_{BetP_{m_i}(\{\omega\})<BetP_{m_j}(\{\omega\})} BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\})] =$
$= \frac{1}{2}[\sum_{BetP_{m_i}(\{\omega\})>BetP_{m_j}(\{\omega\})} BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\}) + \sum_{BetP_{m_i}(\{\omega\})<BetP_{m_j}(\{\omega\})}(BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\})) + \sum_{BetP_{m_i}(\{\omega\})=BetP_{m_j}(\{\omega\})} BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\})] =$
$\frac{1}{2}\sum_{\omega\in\Omega} BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\}) = Diff(BetP_{m_i}, BetP_{m_j}) = Diff(BetP_i, BetP_j)$, because $BetP_i, BetP_j$, are pignistic probabilities corresponding to m_i, m_j. □

Example 11 *Let us suppose $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ now.*
$m_1(\{\omega_1\}) = 0.2$, $m_1(\{\omega_2\}) = 0.1$, $m_1(\{\omega_1, \omega_2\}) = 0.3$, $m_1(\{\omega_1, \omega_3\}) = 0.1$, $m_1(\Omega_3) = 0.3$,
$m_2(\{\omega_1\}) = 0.3$, $m_2(\{\omega_2\}) = 0.1$, $m_2(\{\omega_1, \omega_2\}) = 0.1$, $m_2(\{\omega_2, \omega_3\}) = 0.1$, $m_2(\Omega_3) = 0.4$,
$Pl\_P(m_1)(\omega_1) = 0.45$, $Pl\_P(m_1)(\omega_2) = 0.35$, $Pl\_P(m_1)(\omega_3) = 0.20$,
$Pl\_P(m_2)(\omega_1) = 0.40$, $Pl\_P(m_1)(\omega_2) = 0.35$, $Pl\_P(m_1)(\omega_3) = 0.25$,
$BetP_1(\omega_1) = 0.50$, $BetP_1(\omega_2) = 0.35$, $BetP_1(\omega_3) = 0.15$,
$Diff(m_1, m_2) = 0.25$, $Pl\text{-}Diff(m_1, m_2) = 0.05$, $difBetP(m_1, m_2) = 0.033$,
$Diff(m_1, m_2) = 0.25$, $Pl\text{-}Diff(m_1, m_2) = 0.05$, $Diff(BetP_1, BetP_2) = 0.033$,
$(m_1 \circledcirc m_1)(\emptyset) = 0.06$, $(m_2 \circledcirc m_2)(\emptyset) = 0.12$, $(m_1 \circledcirc m_2)(\emptyset) = 0.08$,
$0.03 \leq IntC(m_1) \leq 0.06$, $0.06 \leq IntC(m_2) \leq 0.12$, $0 \leq C(m_1, m_2) \leq 0.08$, $TotC(m_1, m_2) = 0.08$,

$Pl\text{-}IntC(m_1) = 0.1$, $Pl\text{-}IntC(m_2) = 0.2$, $Pl\text{-}C(m_1, m_2) = 0$,
$m_1'(\{\omega_1\}) = 0.3$, $m_1'(\{\omega_2\}) = 0.1$, $m_1'(\{\omega_1, \omega_2\}) = 0.2$, $m_1'(\{\omega_1, \omega_3\}) = 0.1$, $m_1'(\{\omega_2, \omega_3\}) = 0.1$, $m_1'(\Omega_3) = 0.2$,
$m_2'(\{\omega_1\}) = 0.3$, $m_2'(\{\omega_2\}) = 0.1$, $m_2'(\{\omega_1, \omega_2\}) = 0.2$, $m_2'(\{\omega_1, \omega_3\}) = 0.1$, $m_2'(\{\omega_2, \omega_3\}) = 0.1$, $m_2'(\Omega_3) = 0.2$,
($m_1'(\Omega_3) = m_2'(\Omega_3) > 0$, thus there are many of such more specified bbms, similarly to Example 9)
$Diff(m_1', m_2') = 0$, $cp\text{-}C(m_1, m_2) = 0$,
$cf(m_1, m_2) = (m_\oplus(\emptyset), difBetP_{m_1}^{m_2}) = (0.08, 0.033)$.

Example 12 *Let us suppose $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ again and compare m_1 to m_3.*
$m_3(\{\omega_2\}) = 0.1$, $m_3(\{\omega_3\}) = 0.3$, $m_3(\{\omega_1, \omega_2\}) = 0.1$, $m_3(\{\omega_2, \omega_3\}) = 0.1$, $m_3(\Omega_3) = 0.4$,
$Pl\_P(m_3)(\omega_1) = 0.25$, $Pl\_P(m_3)(\omega_2) = 0.35$, $Pl\_P(m_3)(\omega_3) = 0.40$,
$BetP_3(\omega_1) = 0.183$, $BetP_3(\omega_2) = 0.333$, $BetP_3(\omega_3) = 0.483$,
$Diff(m_1, m_3) = 0.45$, $Pl\text{-}Diff(m_1, m_3) = 0.2$, $Diff(BetP_1, BetP_3) = 0.333$,
$(m_1 \circledcirc m_1)(\emptyset) = 0.06$, $(m_3 \circledcirc m_3)(\emptyset) = 0.12$, $(m_1 \circledcirc m_3)(\emptyset) = 0.23$,
$0.03 \leq IntC(m_1) \leq 0.06$, $0.06 \leq IntC(m_3) \leq 0.12$, $0.14 \leq C(m_1, m_3) \leq 0.23$, $TotC(m_1, m_3) = 0.23$,
$Pl\text{-}IntC(m_1) = 0.1$, $Pl\text{-}IntC(m_3) = 0.2$, $Pl\text{-}C(m_1, m_3) = 0.2$,
$m_1''(\{\omega_1\}) = 0.2$, $m_1''(\{\omega_2\}) = 0.1$, $m_1''(\{\omega_2\}) = 0.3$, $m_1''(\{\omega_1, \omega_2\}) = 0.3$, $m_1''(\{\omega_1, \omega_3\}) = 0.0$, $m_1''(\{\omega_2, \omega_3\}) = 0.1$, $m_1''(\Omega_3) = 0.0$,
$m_3''(\{\omega_1\}) = 0.2$, $m_3''(\{\omega_2\}) = 0.1$, $m_3''(\{\omega_2\}) = 0.3$, $m_3''(\{\omega_1, \omega_2\}) = 0.3$, $m_3''(\{\omega_1, \omega_3\}) = 0.0$, $m_3''(\{\omega_2, \omega_3\}) = 0.1$, $m_3''(\Omega_3) = 0.0$,
($m_1''(\Omega_3) = m_3''(\Omega_3) = 0$ this time, but m_1'', m_3'' are not BBFs as $m_i''(\{\omega_1, \omega_2\}), m_i''(\{\omega_3, \omega_2\}) > 0$, thus there are again many of such more specified bbms)
$Diff(m_1'', m_3'') = 0$, $cp\text{-}C(m_1, m_3) = 0$,
$cf(m_1, m_3) = (m_\oplus(\emptyset), difBetP_{m_1}^{m_3}) = (0.23, 0.333)$.

Example 13 *Let us suppose very conflicting Zadeh's example on $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ now.*
$m_1(\{\omega_1\}) = 0.9$, $m_1(\{\omega_2\}) = 0.0$, $m_1(\{\omega_3\}) = 0.1$, $m_1(\Omega_3) = 0.0$,
$m_2(\{\omega_1\}) = 0.0$, $m_2(\{\omega_2\}) = 0.9$, $m_2(\{\omega_3\}) = 0.1$, $m_2(\Omega_3) = 0.0$,
$Pl\_P(m_1)(\omega_1) = 0.9$, $Pl\_P(m_1)(\omega_2) = 0.0$, $Pl\_P(m_1)(\omega_3) = 0.1$,
$Pl\_P(m_2)(\omega_1) = 0.0$, $Pl\_P(m_1)(\omega_2) = 0.9$, $Pl\_P(m_1)(\omega_3) = 0.1$,
$BetP_1(\omega_1) = 0.9$, $BetP_1(\omega_2) = 0.0$, $BetP_1(\omega_3) = 0.1$,
$BetP_2(\omega_1) = 0.0$, $BetP_2(\omega_2) = 0.9$, $BetP_2(\omega_3) = 0.1$,
$Diff(m_1, m_2) = 0.9$, $Pl\text{-}Diff(m_1, m_2) = 0.9$, $Diff(BetP_1, BetP_2) = 0.9$,
$(m_1 \circledcirc m_1)(\emptyset) = 0.18$, $(m_2 \circledcirc m_2)(\emptyset) = 0.18$, $(m_1 \circledcirc m_2)(\emptyset) = 0.99$,
$0.09 \leq IntC(m_1) \leq 0.18$, $0.09 \leq IntC(m_2) \leq 0.18$, $0.81 \leq C(m_1, m_2) \leq 0.99$, $TotC(m_1, m_2) = 0.99$,
$Pl\text{-}IntC(m_1) = 0.1$, $Pl\text{-}IntC(m_2) = 0.1$, $Pl\text{-}C(m_1, m_2) = 0.9$,
$m_1' = m_1$, $m_2' = m_2$ *because everything is already focused to singletons,*
$cp\text{-}C(m_1, m_2) = 0.9$, $cf(m_1, m_2) = (m_\oplus(\emptyset), difBetP_{m_1}^{m_2}) = (0.99, 0.9)$.

We can easily notice that all the approaches agree with the high conflictness of the Zadeh's example, the common feature of their results is, that the commonly used $m(\emptyset) = TotC$ is the most conflicting and that the combinational conflict between m_is is not precise (as its precise definition is still missing).

All the approaches have similar results in the first least conflicting Example 11. The most important difference in this example is the fact, that there is no plausibility nor comparative conflict between m_1 and m_2.

The greatest differences among the results are in the most general Example 12. There is again no comparative conflict between m_1 and m_3 (as there exist non-conflicting common specification of both bbms), but there is the plausibility conflict between them. If we assume that combinational conflict is somewhere close to the middle of its interval, the highest conflict is classified by the common $m(\emptyset)$ and Liu's approaches. It reflects that there is no internal conflicts considered in these approaches. There is neither any internal conflict in comparative approach, but this approach more reflects the individual input bbms and usually produces the least conflict.

Both Liu's and plausibility approaches use a probabilistic transformation for computing of conflict, pignistic and normalized plausibility. Thus Liu's conflict is more related to decisional pignistic level, while the plausibility more to credal combinational level (especially when Dempster's rule or the non-normalized conjunctive rule is used), because normalized plausibility transformation commutes with Dempster's rule [3, 6]. Nevertheless the main difference between these two approaches is not in different pignistic transformations but in the fact that Liu does not distinguish differences from conflicts. Hence any two different BFs supporting and opposing the same element(s) of Ω are conflicting in Liu's interpretation whenever there is any internal conflict there (whenever does not exist $X \subseteq \Omega$ such that $Pl(X) = 1$). But such BFs are never mutually conflicting in the plausibility approach.

7 Open problems

The ideas presented in this report are brand new, thus they open a lot of questions and open problems. The principal ones are the following:

- to find more precise specification of combinational conflict $C(Bel_1, Bel_2)$ between BFs Bel_1 and Bel_2;
- elaboration of plausibility approach to conflicts;
- to create an algorithm for common belief mass specification needed for exclusion of comparative conflict;
- to create an algorithm for belief mass specifications of Bel_1 and Bel_2 with the least difference, which is necessary for comparative conflict computation;
- to study mathematical properties of defined measures of conflicts;
- make a detail comparison of the presented combinational, plausibility and comparative approaches;
- make a more detail comparison of the new approaches to the classic $m(\emptyset)$ and to Liu's approach [8].

8 Conclusion

This theoretical study introduces three new approaches to conflicts of belief functions: new approach to combinational conflicts, plausibility approach and comparative approach. It distinguishes internal conflict of individual belief functions from their mutual conflict between them. Important distinctness of differences of belief functions from their mutual conflicts is introduced and underlined. On the other hand, the important role of $m(\emptyset)$ for conflict measurement was strenghtened, see combinational and plausibility conflicts.

The presented ideas enable new deeper understanding of conflicts of belief functions. They can be applied to studies of belief combination and fusion of beliefs. The series of open problems may be challenging for a future research. The ideas presented in this report are here to open new scientific discussions about this interesting and complex topic.

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