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Rohn, Jiří
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Jiří Rohn

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Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789, e-mail:rohn@cs.cas.cz

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Abstract:

Results on the inverse interval matrix, both theoretical and computational, are surveyed. Described are, among others, formulae for the inverse interval matrix, NP-hardness of its computation, various classes of interval matrices for which the inverse can be given explicitly, and closed-form formulae for an enclosure of the inverse.

Keywords:
Interval matrix, inverse interval matrix, NP-hardness, enclosure, unit midpoint, inverse sign stability, nonnegative invertibility, absolute value equation, algorithm.

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# 1 Introduction

In our recent paper [24]we presented a survey of forty necessary and sufficient conditions for regularity of interval matrices. It is now followed by a survey of properties of the inverse interval matrix which is closely related to the previous topic because the inverse interval matrix is defined for regular interval matrices only.

After some preliminaries in Sections 2 and 3, the inverse interval matrix is defined in Section 4. Next we introduce matrices $B_y$ defined for each $\pm 1$-vector $y$ and demonstrate their use for inverse matrix representation (Theorem 5) and for establishing finite formulae for the inverse interval matrix (Theorem 7). Then we present Coxson's result [5] showing that computing the inverse interval matrix is NP-hard. In the next Section 10 we show that for an interval matrix with unit midpoint the inverse interval matrix can be given explicitly by simple formulae (Theorem 12). Explicit formulae for an enclosure of the inverse of a strongly regular interval matrix are given in Section 11. In the next four sections we give explicit formulae for the interval inverse of interval matrices that are either inverse sign stable (Section 12), or are of inverse sign pattern (Section 13), or are nonnegative invertible (Section 14), or have uniform width (Section 15). In the last Section 16 we describe available software for computing the inverse interval matrix or its enclosure. The Appendix contains a MATLAB-like description of an algorithm for solving an absolute value equation which is used in Section 6 for computation of the matrices $B_y$.

# 2 Notations

We use the following notations. $A_{ij}$ denotes the $ij$th entry, $A_{i\bullet}$ the $i$th row and $A_{\bullet j}$ the $j$th column of a matrix $A$. Matrix inequalities, as $A \leq B$ or $A < B$, are understood componentwise. $A \circ B$ denotes the Hadamard (entrywise) product of $A, B \in \mathbb{R}^{m\times n}$, i.e., $(A \circ B)_{ij} = A_{ij}B_{ij}$ for each $i, j$. Minimum (or maximum) matrix of a compact (in particular, finite) set of matrices $X$ is defined componentwise, i.e.,

$$(\min\{\, A \mid A \in X \,\})_{ij} = \min\{\, A_{ij} \mid A \in X \,\},$$

$$(\max\{\, A \mid A \in X \,\})_{ij} = \max\{\, A_{ij} \mid A \in X \,\}$$

for each $i, j$. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. For each matrix $A$ we define its sign matrix $\operatorname{sgn}(A)$ by

$$(\operatorname{sgn}(A))_{ij} = \begin{cases} 1 & \text{if } A_{ij} \geq 0, \\ -1 & \text{if } A_{ij} < 0 \end{cases}$$

for each $i, j$. The same notations also apply to vectors that are considered one-column matrices. $I$ is the unit matrix, $e_j$ is the $j$th column of $I$, $e = (1, \ldots, 1)^T$ is the vector of all ones, and $E = ee^T$ is the matrix of all ones. $Y_n = \{y \mid |y| = e\}$ is the set of all

$\pm 1$-vectors in $\mathbb{R}^n$, so that its cardinality is $2^n$. For each $y \in \mathbb{R}^n$ we denote

$$T_y = \operatorname{diag}(y_1, \dots, y_n) = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_n \end{pmatrix},$$

and $\varrho(A)$ is the spectral radius of $A$.

# 3 Interval matrices

Given two $n \times n$ matrices $A_c$ and $\Delta$, $\Delta \geq 0$, the set of matrices

$$\mathbf{A} = \{A \mid |A - A_c| \leq \Delta\}$$

is called a (square) interval matrix with midpoint matrix $A_c$ and radius matrix $\Delta$. Since the inequality $|A - A_c| \leq \Delta$ is equivalent to $A_c - \Delta \leq A \leq A_c + \Delta$, we can also write

$$\mathbf{A} = \{A \mid \underline{A} \leq A \leq \overline{A}\} = [\underline{A}, \overline{A}],$$

where $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$ are called the bounds of $\mathbf{A}$.

Given an $n \times n$ interval matrix $\mathbf{A}$, we define matrices

$$A_{yz} = A_c - T_y \Delta T_z \tag{3.1}$$

for each $y, z \in Y_n$. The definition implies that

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i \Delta_{ij} z_j = \begin{cases} \overline{A}_{ij} & \text{if } y_i z_j = -1, \\ \underline{A}_{ij} & \text{if } y_i z_j = 1 \end{cases} \qquad (i, j = 1, \dots, n),$$

so that $A_{yz} \in \mathbf{A}$ for each $y, z \in Y_n$. Since the cardinality of $Y_n$ is $2^n$, the cardinality of the set of matrices $\{A_{yz} \mid y, z \in Y_n\}$ is at most $2^{2n}$.

# 4 Definition of the inverse interval matrix

A square interval matrix $\mathbf{A}$ is called *regular* if each $A \in \mathbf{A}$ is nonsingular, and it is said to be *singular* otherwise (i.e., if it contains a singular matrix). In particular, an interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ with

$$\varrho(|A_c^{-1}|\Delta) < 1 \tag{4.1}$$

is regular (Beeck [3]); interval matrices satisfying (4.1) are called *strongly regular*. Inverse interval matrix is defined for regular interval matrices only.

**Definition.** For a regular interval matrix $\mathbf{A}$ we define its inverse interval matrix $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ by

$$\begin{aligned} \underline{B} &= \min\{ A^{-1} \mid A \in \mathbf{A} \}, \\ \overline{B} &= \max\{ A^{-1} \mid A \in \mathbf{A} \} \end{aligned}$$

(componentwise).

*Comment.* This means that

$$\underline{B}_{ij} = \min\{ (A^{-1})_{ij} \mid A \in \mathbf{A} \}, \tag{4.2}$$

$$\overline{B}_{ij} = \max\{ (A^{-1})_{ij} \mid A \in \mathbf{A} \} \qquad (i, j = 1, \ldots, n). \tag{4.3}$$

Since $\mathbf{A}$ is regular, the mapping $A \mapsto A^{-1}$ is continuous in $\mathbf{A}$ and all the minima and maxima in (4.2), (4.3) are attained. Thus, $\mathbf{A}^{-1}$ is the narrowest interval matrix enclosing the set of matrices $\{ A^{-1} \mid A \in \mathbf{A} \}$. Instead of "inverse interval matrix", we sometimes say simply "interval inverse".

# 5 The matrices $B_y$

First we show that regularity of an $n \times n$ interval matrix implies existence of $2^n$ uniquely determined matrices.

**Theorem 1.** *[19, Thm. 5.1, (A3)] For a square interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$, the following assertions are equivalent:*

*(i)* $\mathbf{A}$ *is regular,*

*(ii) for each $y \in Y_n$ the matrix equation*

$$A_c B - T_y \Delta |B| = I \tag{5.1}$$

*has a unique matrix solution $B_y$,*

*(iii) for each $y \in Y_n$ the matrix equation (5.1) has a solution.*

The main message here is the implication "(i)⇒(ii)"; (iii) is added for completeness.

It is useful to formulate the equation (5.1) columnwise.

**Theorem 2.** *Let $\mathbf{A}$ be regular. Then for each $y \in Y_n$ and for each $j \in \{1, \ldots, n\}$ we have*

$$(B_y)_{\bullet j} = x_{yj},$$

*where $x_{yj}$ is the unique solution of the equation*

$$A_c x - T_y \Delta |x| = e_j. \tag{5.2}$$

This theorem forms the basis of an algorithm for computing the $B_y$'s presented in the next section. We have still another expression for the $j$th column of $B_y$ by means of the matrices $A_{yz}$ introduced in (3.1).

**Theorem 3.** *Let* $\mathbf{A}$ *be regular. Then for each* $y \in Y_n$ *and for each* $j \in \{1, \ldots, n\}$ *we have*

$$(B_y)_{\bullet j} = \left(A^{-1}_{yz(j)}\right)_{\bullet j}, \tag{5.3}$$

*where*

$$z(j) = \operatorname{sgn}((B_y)_{\bullet j}).$$

Since $z(j)$ depends on $j$, we cannot generally state that $B_y = A^{-1}_{yz}$ for some $z$. It may even be $B_y^{-1} \notin \mathbf{A}$. As a consequence of (5.3) we obtain that

$$(B_y)_{ij} = \left(A^{-1}_{yz(j)}\right)_{ij} \tag{5.4}$$

for each $y, i, j$. Of course, (5.3) and (5.4) cannot be directly used for computation of $(B_y)_{\bullet j}$ since they contain $z(j)$, the sign vector of the result.

## 6 Computation of the $B_y$'s

Theorem 2 shows us a way how to compute the matrix $B_y$ column-by-column provided we are able to solve an equation of the type

$$Ax + B|x| = b, \tag{6.1}$$

called an *absolute value equation*. This can be done by a finite algorithm **signaccord** from [23] whose detailed MATLAB-like description is given in the Appendix. Its syntax is

$$[x, S, flag] = \textbf{signaccord}(A, B, b),$$

where $A, B, b$ is the data of (6.1), $x$ is a solution of (6.1) (if found), $S$ is a singular matrix in the interval matrix $[A - |B|, A + |B|]$ (if found), and $flag$ is a verbal description of the output (*'solution'* or *'singular'*). The behavior of the algorithm is described in Theorem 25. Its important feature is that for a regular interval matrix $[A-|B|, A+|B|]$ it always finds a solution to (6.1) (in infinite precision arithmetic), which in this case is unique [23]. As reported in [23], the algorithm takes an average number of steps (passes through the **while** loop) about $0.11 \cdot n$, where $n$ is the matrix size.

Solving the equations (5.2) for $j = 1, \ldots, n$, we obtain an algorithm (Fig. 6.1) for computing the matrix $B_y$ for a given $y$.

The following theorem (unpublished) follows directly from Theorems 2 and 25.

**Theorem 4.** *For each square interval matrix* $\mathbf{A}$ *and for each* $y \in Y_n$ *the algorithm (Fig. 6.1) in a finite number of steps either finds a matrix* $B_y$ *satisfying (5.1), or issues an empty matrix* $B_y$ *in which case* $\mathbf{A}$ *is singular.*

It should be noted that success in computation of a *single* matrix $B_y$ does not guarantee regularity; it is the existence of solutions of *all* the equations (5.1), $y \in Y_n$ that implies regularity of $\mathbf{A}$ (Theorem 1, (iii)).

```
function B_y = bymatrix(A, y)
for j = 1 : n
    [x, S, flag] = signaccord(A_c, −T_y Δ, e_j);
    if flag = 'singular', B_y = []; return
    end
    (B_y)_•j = x;
end
```

Figure 6.1: An algorithm for computing $B_y$.

# 7 Inverse matrix representation theorem

The following theorem, which is of independent interest, brings us closer to the formulae for the inverse interval matrix to be given in the next section.

**Theorem 5.** *[19, Thm. 6.1] Let* $\mathbf{A}$ *be regular. Then for each* $A \in \mathbf{A}$ *there exist nonnegative diagonal matrices* $L_y$, $y \in Y_n$, *satisfying* $\sum_{y \in Y_n} L_y = I$ *such that*

$$A^{-1} = \sum_{y \in Y_n} B_y L_y \tag{7.1}$$

*holds.*

The formula (7.1) implies that for each $i, j$ we have

$$(A^{-1})_{ij} = \sum_{y \in Y_n} (B_y)_{ij} (L_y)_{jj} \tag{7.2}$$

where all the $(L_y)_{jj}$'s are nonnegative and $\sum_{y \in Y_n} (L_y)_{jj} = I_{jj} = 1$, hence $(A^{-1})_{ij}$ is a convex combination of the values $(B_y)_{ij}$ over all $y \in Y_n$.

Using the formula (5.3), we can reformulate the representation theorem in terms of the matrices $A_{yz}$ defined in (3.1).

**Theorem 6.** *[21, Thm. 1.1] Let* $\mathbf{A}$ *be regular. Then for each* $A \in \mathbf{A}$ *there exist nonnegative diagonal matrices* $L_{yz}$, $y, z \in Y_n$, *satisfying* $\sum_{y,z \in Y_n} L_{yz} = I$ *such that*

$$A^{-1} = \sum_{y,z \in Y_n} A_{yz}^{-1} L_{yz} \tag{7.3}$$

*holds.*

Obviously, the convex combination property again holds accordingly here. The expansion (7.3) is perhaps more clear than (7.1) because it employs explicitly expressed matrices $A_{yz}^{-1}$ instead of rather obscure matrices $B_y$, but the number of matrices $A_{yz}^{-1}$ is $2^{2n}$ compared to "only" $2^n$ matrices $B_y$.

# 8 Formulae for the inverse interval matrix

Finally, using (7.2) and (5.4), we obtain the following simply formulated, but important result.

**Theorem 7.** *[19, Thm. 6.2] Let* $\mathbf{A}$ *be regular. Then its inverse* $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ *is given by*

$$\underline{B} = \min_{y \in Y_n} B_y,$$

$$\overline{B} = \max_{y \in Y_n} B_y.$$

Similarly, from Theorem 6 we can derive an analogous result.

**Theorem 8.** *[21, (1.3), (1.4)] Let* $\mathbf{A}$ *be regular. Then its inverse* $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ *is given by*

$$\underline{B} = \min_{y,z \in Y_n} A_{yz}^{-1},$$

$$\overline{B} = \max_{y,z \in Y_n} A_{yz}^{-1}.$$

The formulation of Theorem 8 is advantageous in that it leads us to some clues about matrices at which bounds of the inverse interval matrix are attained.

**Theorem 9.** *[21, Thm. 1.2] Let* $\mathbf{A}$ *be regular and let* $i, j \in \{1, \ldots, n\}$*. Then we have:*

*(i)* $\underline{B}_{ij} = (A_{yz}^{-1})_{ij}$ *for some* $y, z \in Y_n$ *satisfying*

$$y^T \circ (A_{yz}^{-1})_{i\bullet} \le 0^T, \tag{8.1}$$

$$z \circ (A_{yz}^{-1})_{\bullet j} \ge 0, \tag{8.2}$$

*(ii)* $\overline{B}_{ij} = (A_{yz}^{-1})_{ij}$ *for some* $y, z \in Y_n$ *satisfying*

$$y^T \circ (A_{yz}^{-1})_{i\bullet} \ge 0^T,$$

$$z \circ (A_{yz}^{-1})_{\bullet j} \ge 0.$$

For instance, the Hadamard product inequalities (8.1), (8.2) are equivalent to

$$y_k (A_{yz}^{-1})_{ik} \le 0 \qquad (k = 1, \ldots, n), \tag{8.3}$$

$$z_h (A_{yz}^{-1})_{hj} \ge 0 \qquad (h = 1, \ldots, n).$$

Thus, if we know in advance that e.g. $\underline{B}_{ik} > 0$, then $(A_{yz}^{-1})_{ik} > 0$ for each $y, z \in Y_n$ and (8.3) implies that $y_k = -1$; similarly, if $\overline{B}_{ik} < 0$, then (8.3) gives $y_k = 1$. Hence, preliminary knowledge of the signs of the bounds may lead us to reduction, sometimes significant, of the number of matrices $A_{yz}$ to be inverted. We shall explore these ideas further in Section 12.

# 9 NP-hardness

The formulae given for the inverse interval matrix in Theorems 7 and 8 are inherently exponential. The question whether essentially simpler formulae may be found was answered in the negative by Coxson [5] who proved that computation of the inverse interval matrix is NP-hard.

**Theorem 10.** *[5] The following problem is NP-hard:*
Instance. *A strongly regular interval matrix* $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ *with symmetric rational* $A_c$ *and* $\Delta$*.*
Question. *Is* $\overline{B}_{11} \geq 1$*, where* $[\underline{B}, \overline{B}] = \mathbf{A}^{-1}$ *?*

Hence, if the famous conjecture "P≠NP" is true, then there does not exist a polynomial-time algorithm for computing the interval inverse. In view of this fact, in what follows we shall concentrate on special classes of interval matrices for which the inverse can be computed by simpler means.

# 10 Inverse of an interval matrix with unit midpoint

The first such a class is formed by interval matrices with unit midpoint, i.e., of the form $\mathbf{A} = [I - \Delta, I + \Delta]$. Such matrices are regular if and only if $\varrho(\Delta) < 1$ holds [16, Prop. 4.1], which is equivalent to

$$M := (I - \Delta)^{-1} \geq 0. \tag{10.1}$$

Hence, we assume that $\varrho(\Delta) < 1$ throughout this section. The main point here consists in the fact all the matrices $B_y$, $y \in Y_n$ can be described explicitly. The following theorem gives a general matrix formula (10.2) as well as three different componentwise formulae (10.3), (10.4), and (10.5). We use $M = (m_{ij})$ given by (10.1) and $\mu = (\mu_j)$ defined by

$$\mu_j = \frac{m_{jj}}{2m_{jj} - 1} \qquad (j = 1, \ldots, n).$$

**Theorem 11.** *[16, Thm. 4.2] Let* $\varrho(\Delta) < 1$*. Then for each* $y \in Y_n$ *the unique solution of the matrix equation*[2]

$$B - T_y \Delta |B| = I$$

*is given by*

$$B_y = T_y M T_y + T_y (M - I) T_\mu (I - T_y), \tag{10.2}$$

*i.e. componentwise*

$$(B_y)_{ij} = y_i y_j m_{ij} + y_i (1 - y_j)(m_{ij} - I_{ij})\mu_j, \tag{10.3}$$

[2] This is the equation (5.1) with $A_c = I$.

*or*

$$(B_y)_{ij} = \begin{cases} y_i m_{ij} & \text{if } y_j = 1, \\ y_i(2\mu_j - 1)m_{ij} & \text{if } y_j = -1 \text{ and } i \neq j, \\ \mu_j & \text{if } y_j = -1 \text{ and } i = j, \end{cases} \tag{10.4}$$

*or*

$$(B_y)_{ij} = \frac{(y_i + (1 - y_i)I_{ij})m_{ij}}{y_j + (1 - y_j)m_{jj}} \tag{10.5}$$

$(i, j = 1, \ldots, n)$.

Using Theorem 7, we obtain simple formulae for the interval inverse in this case.

**Theorem 12.** *[16, Thm. 4.3] Let* $\mathbf{A} = [I - \Delta, I + \Delta]$ *with* $\varrho(\Delta) < 1$. *Then the inverse interval matrix* $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ *is given by*

$$\begin{aligned} \underline{B} &= -M + T_\kappa, \\ \overline{B} &= M, \end{aligned} \tag{10.6}$$

*where*

$$\kappa_j = \frac{2m_{jj}^2}{2m_{jj} - 1} \qquad (j = 1, \ldots, n),$$

*or componentwise*

$$\begin{aligned} \underline{B}_{ij} &= \begin{cases} -m_{ij} & \text{if } i \neq j, \\ \mu_j & \text{if } i = j, \end{cases} \\ \overline{B}_{ij} &= m_{ij} \end{aligned}$$

$(i, j = 1, \ldots, n)$.

In particular, we have this consequence.

**Theorem 13.** *[16, Cor. 4.4] If* $\varrho(\Delta) < 1$, *then the inverse interval matrix* $[I - \Delta, I + \Delta]^{-1} = [\underline{B}, \overline{B}]$ *satisfies*

$$\tfrac{1}{2} \leq \underline{B}_{jj} \leq 1 \leq \overline{B}_{jj}$$

*for each* $j$.

According to (10.6), $\overline{B} = (I - \Delta)^{-1}$. The last theorem of this section reveals at what matrices the entries of $\underline{B}$ are attained.

**Theorem 14.** *[16, Thm. 5.1] For each* $i, j$ *we have:*

*(i) if* $i \neq j$, *then*

$$\underline{B}_{ij} = (I - T_y \Delta T_y)^{-1}_{ij}$$

*for each* $y \in Y$ *satisfying* $y_i y_j = -1$,

*(ii) if* $i = j$, *then*

$$\underline{B}_{jj} = (I - T_y \Delta T_z)^{-1}_{jj}$$

*for each* $y \in Y$ *satisfying* $y_j = -1$ *and* $z = y + 2e_j$.

## 11 Enclosure of the inverse interval matrix

An interval matrix $\mathbf{C}$ is called an *enclosure* of $\mathbf{A}^{-1}$ if $\mathbf{A}^{-1} \subseteq \mathbf{C}$ holds. Computation of an enclosure of the inverse of a strongly regular interval matrix can be performed in polynomial time, as shown in the following theorem which is a follow-up of previous results by Hansen [10], Bliek [4] and Rohn [20] on interval linear equations.

**Theorem 15.** *[6, Thm. 2.40] Let $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ be strongly regular. Then we have*

$$\mathbf{A}^{-1} \subseteq [\underline{\underline{B}}, \overline{\overline{B}}],$$

*where*

$$\begin{aligned}
M &= (I - |A_c^{-1}|\Delta)^{-1}, \\
\mu &= (M_{11}, \ldots, M_{nn})^T, \\
T_\nu &= (2T_\mu - I)^{-1}, \\
\underset{\sim}{B} &= -M|A_c^{-1}| + T_\mu(A_c^{-1} + |A_c^{-1}|), \\
\widetilde{B} &= \quad M|A_c^{-1}| + T_\mu(A_c^{-1} - |A_c^{-1}|), \\
\underline{\underline{B}} &= \min\{\underset{\sim}{B}, T_\nu \underset{\sim}{B}\}, \\
\overline{\overline{B}} &= \max\{\widetilde{B}, T_\nu \widetilde{B}\}.
\end{aligned}$$

Other types of enclosures were studied by Hansen [8], Hansen and Smith [9], Herzberger and Bethke [13], and Herzberger [11], [12].

Preliminary knowledge of an enclosure may make computation of the interval inverse easier, see Theorem 17 below.

## 12 Inverse sign stability

Let $Z$ be a matrix satisfying $|Z| = E$, i.e., a $\pm 1$-matrix. We say that a regular interval matrix $\mathbf{A}$ is *inverse $Z$-stable* if

$$Z \circ A^{-1} > 0$$

holds for each $A \in \mathbf{A}$. This means that for each $i, j$, either $(A^{-1})_{ij} < 0$ for each $A \in \mathbf{A}$ (if $Z_{ij} = -1$), or $(A^{-1})_{ij} > 0$ for each $A \in \mathbf{A}$ (if $Z_{ij} = 1$). We say simply that $\mathbf{A}$ is inverse sign stable if it is inverse $Z$-stable for some $Z$.

We have the following finite characterization.

**Theorem 16.** *[21, Thm. 2.1] $\mathbf{A}$ is inverse $Z$-stable if and only if each $A_{yz}$ is nonsingular and*

$$Z \circ A_{yz}^{-1} > 0 \tag{12.1}$$

*holds for each $y, z \in Y_n$.*

Notice that regularity of $\mathbf{A}$ is not assumed; it follows from (12.1). The next theorem gives a sufficient inverse $Z$-stability condition verifiable in polynomial time.

**Theorem 17.** *[improved version of [21], Thm. 2.2] If $\mathbf{A}$ is strongly regular and if*

$$\underline{\underline{B}} \circ \overline{\overline{B}} > 0$$

*holds, where $\underline{\underline{B}}, \overline{\overline{B}}$ are as in Theorem 15, then $\mathbf{A}$ is inverse $Z$-stable, where $Z = \operatorname{sgn}(\underline{\underline{B}})$.*

The main reason for introducing inverse $Z$-stable matrices is the following theorem which gives explicit componentwise formulae for entries of the bounds of the inverse interval matrix. It is an easy consequence of Theorem 9.

**Theorem 18.** *[21, Thm. 2.3] Let $\mathbf{A}$ be inverse $Z$-stable. Then the bounds of its inverse $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ are given by the explicit formulae*

$$\begin{aligned} \underline{B}_{ij} &= (A^{-1}_{-y(i),z(j)})_{ij} \\ \overline{B}_{ij} &= (A^{-1}_{y(i)z(j)})_{ij} \qquad (i,j = 1,\dots,n), \end{aligned}$$

*where $y(i) = \operatorname{sgn}((Z_{i\bullet})^T)$ and $z(j) = \operatorname{sgn}(Z_{\bullet j})$ for each $i, j$.*

## 13 Inverse sign pattern

Let $\mathbf{A}$ be regular. If there exist (fixed) $z, y \in Y_n$ such that

$$(zy^T) \circ A^{-1} \geq 0 \tag{13.1}$$

holds for each $A \in \mathbf{A}$, then $\mathbf{A}$ is said to be of the *inverse sign pattern* $(z, y)$. In other words, for each $i, j$ we have $z_i y_j (A^{-1})_{ij} \geq 0$ for each $A \in \mathbf{A}$, so that $z_i y_j$ prescribes the sign of $(A^{-1})_{ij}$. If strict inequality holds in (13.1), then $\mathbf{A}$ is inverse $zy^T$-stable. The property (13.1) can be succinctly reformulated as

$$T_z A^{-1} T_y \geq 0$$

for each $A \in \mathbf{A}$. It is a rather surprising fact that for both the characterization and the explicit form of interval inverse we need only two matrices in this case, namely $A^{-1}_{yz}$ and $A^{-1}_{-y,z}$.

**Theorem 19.** *[19, Thm. 4.6] $\mathbf{A}$ is of the inverse sign pattern $(z, y)$ if and only if $A_{yz}$ and $A_{-y,z}$ are nonsingular and*

$$T_z A^{-1}_{yz} T_y \geq 0, \tag{13.2}$$

$$T_z A^{-1}_{-y,z} T_y \geq 0 \tag{13.3}$$

*hold*[3]*.*

[3]Which implicitly asserts that the two conditions (13.2) and (13.3) imply regularity of $\mathbf{A}$.

The following theorem has not been published so far.

**Theorem 20.** *If* $\mathbf{A}$ *is of the inverse sign pattern* $(z, y)$*, then its inverse interval matrix is given by*

$$\mathbf{A}^{-1} = [\min\{A_{yz}^{-1}, A_{-y,z}^{-1}\}, \max\{A_{yz}^{-1}, A_{-y,z}^{-1}\}]. \tag{13.4}$$

See Garloff [7] for the special case of $y = z = (1, -1, 1, -1, \dots, (-1)^{n-1})^T$.

# 14 Nonnegative invertibility

An interval matrix $\mathbf{A}$ is said to be *nonnegative invertible* if it is of the inverse sign pattern $(e, e)$, i.e., if

$$A^{-1} \geq 0$$

holds for each $A \in \mathbf{A}$. As immediate consequences of Theorems 19 and 20 we obtain the following two results.

**Theorem 21.** *[14]* $\mathbf{A}$ *is nonnegative invertible if and only if* $\underline{A}^{-1} \geq 0$ *and* $\overline{A}^{-1} \geq 0$.

**Theorem 22.** *[15] If* $\mathbf{A} = [\underline{A}, \overline{A}]$ *is nonnegative invertible, then*

$$\mathbf{A}^{-1} = [\overline{A}^{-1}, \underline{A}^{-1}]. \tag{14.1}$$

The last formula follows from the fact that $\underline{A}^{-1} - \overline{A}^{-1} = \underline{A}^{-1}(\overline{A} - \underline{A})\overline{A}^{-1} \geq 0$ which gives $\underline{A}^{-1} \geq \overline{A}^{-1}$, hence (13.4) implies (14.1). Finally, we have the following inverse expansion theorem.

**Theorem 23.** *[18, Thm. 2] If* $\mathbf{A}$ *is inverse nonnegative, then for each* $A \in \mathbf{A}$ *there holds*

$$A^{-1} = \left(\sum_{j=0}^{\infty} (\overline{A}^{-1}(\overline{A} - A))^j\right)\overline{A}^{-1}.$$

# 15 Uniform width

An interval matrix $\mathbf{A}$ is said to be of *uniform width* if it is of the form

$$\mathbf{A} = [A_c - \alpha E, A_c + \alpha E] \tag{15.1}$$

for some $\alpha \geq 0$. For sufficiently small $\alpha$, its inverse can be again expressed explicitly. Let us denote

$$\begin{aligned} c &= |A_c^{-1}|e, \\ d &= |A_c^{-1}|^T e. \end{aligned}$$

**Theorem 24.** *[17, Thm. 2] Let $A_c$ be nonsingular and let $\alpha \geq 0$ satisfy*

$$\alpha(cd^T + \|c\|_1 |A_c^{-1}|) < |A_c^{-1}|. \tag{15.2}$$

*Then for the interval inverse $[\underline{B}, \overline{B}]$ of (15.1) we have*

$$\begin{aligned}\underline{B}_{ij} &= (A_c^{-1})_{ij} - \frac{\alpha c_i d_j}{1 + \alpha z(j)^T A_c^{-1} y(i)},\\ \overline{B}_{ij} &= (A_c^{-1})_{ij} + \frac{\alpha c_i d_j}{1 - \alpha z(j)^T A_c^{-1} y(i)} \qquad (i, j = 1, \ldots, n),\end{aligned}$$

*where*

$$\begin{aligned} y(i) &= \operatorname{sgn}(((A_c^{-1})_{i\bullet})^T),\\ z(j) &= \operatorname{sgn}((A_c^{-1})_{\bullet j}).\end{aligned}$$

The condition (15.2) provides for both strong regularity and inverse sign stability of **A**.

## 16 Software

The freely available verification software package VERSOFT [2] written in INTLAB [25], [26], a toolbox of MATLAB, contains a file VERINVERSE.M [1] for computing a verified inverse of a square interval matrix. Its syntax is

```
[B,S]=verinverse(A)
```

where $A$ is an interval matrix, $B$ is its verified interval inverse (if found), and $S$ is a very tight interval matrix which is a part of $A$ and is verified to contain a singular matrix in $A$ (if found). $B$ and $S$ are never assigned numerical values simultaneously; at least one of them is a matrix of *NaN*'s as the two options - regularity and singularity - exclude each other. The interval matrix $B$, if computed, is verified to contain the interval inverse of $A$ and the overestimation is solely due to the outward rounding committed; in infinite precision arithmetic it would compute the exact interval inverse. It is based on a not-a-priori-exponential algorithm **hull** for solving interval linear equations described in [22]; its theoretical basis and implementation details have not been published. Nevertheless, the computation may occasionally last long as the problem is NP-hard (Theorem 10). In such cases we recommend computation of a polynomial-time enclosure described in Theorem 15. This enclosure has not been included into VERSOFT. INTLAB users may employ the function INV.M adapted for an interval argument by S. M. Rump [26].

# 17 Appendix: An algorithm for solving the absolute value equation

This appendix contains a MATLAB-like description of an algorithm for solving the absolute value equation accompanied by a finite termination theorem. Both these results were referred to in Section 6.

**Theorem 25.** *[23, Thm. 3.1] For each $A, B \in \mathbb{R}^{n\times n}$ and each $b \in \mathbb{R}^n$, the sign accord algorithm (Fig. 17.1) in a finite number of steps either finds a solution of the equation*

$$Ax + B|x| = b,$$

*or states singularity of the interval matrix $[A - |B|, A + |B|]$ (and, in most cases, also finds a singular matrix $S \in [A - |B|, A + |B|]$).*

```
function [x, S, flag] = signaccord (A, B, b)
% Finds a solution to Ax + B|x| = b or states
% singularity of [A − |B|, A + |B|].
x = [ ]; S = [ ]; flag = 'singular';
if A is singular, S = A; return, end
p = 0 ∈ ℝ^n;
z = sgn(A^{-1}b);
if A + BT_z is singular, S = A + BT_z; return, end
x = (A + BT_z)^{-1}b;
C = −(A + BT_z)^{-1}B;
while z_j x_j < 0 for some j
    k = min{j | z_j x_j < 0};
    if 1 + 2z_k C_kk ≤ 0
        S = A + B(T_z + (1/C_kk)e_k e_k^T);
        x = [ ];
        return
    end
    p_k = p_k + 1;
    if log_2 p_k > n − k, x = [ ]; return, end
    z_k = −z_k;
    α = 2z_k/(1 − 2z_k C_kk);
    x = x + α x_k C_•k;
    C = C + α C_•k C_k•;
end
flag = 'solution';
```

Figure 17.1: The sign accord algorithm [23].

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