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A Characterization of Strong Regularity of Interval Matrices

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Abstract:

As the main result of this paper it is proved that an interval matrix $[A_c - \Delta, A_c + \Delta]$ is strongly regular if and only if the matrix inequality $M(I - |I - RA_c| - |R|\Delta) \geq I$ has a solution $M \geq 0$ and R (I is the identity matrix). Several consequences of this result are drawn.

Keywords:
Interval matrix, strong regularity, spectral radius, matrix inequality, solvability.

1 Introduction

An $n \times n$ interval matrix

$$\mathbf{A} = [A_c - \Delta, A_c + \Delta] = \{ A \mid |A - A_c| \leq \Delta \}$$

is called *strongly regular* [3] if A_c is nonsingular and

$$\varrho(|A_c^{-1}|\Delta) < 1 \qquad (1.1)$$

holds. In view of well-known properties of nonnegative matrices (see the equivalence of (i) and (ii) in Theorem 1 below), the condition (1.1) can be equivalently written as

$$(I - |A_c^{-1}|\Delta)^{-1} \geq 0. \qquad (1.2)$$

Evaluation of the left-hand side of (1.2) requires computation of two exact inverses. In this paper we show that the condition (1.2) can be equivalently replaced by the condition of solvability of the matrix inequality

$$M(I - |I - RA_c| - |R|\Delta) \geq I, \qquad (1.3)$$

where $M, R \in \mathbb{R}^{n \times n}$ and M is required to be nonnegative. In the preliminary announcement [6] from 1994, which was never followed by a full paper, this author mentioned that "it can be shown that matrices R and $M \geq 0$ satisfying (1.3) exist if and only if (1.1) holds". Now, almost exactly 16 years after writing [6], the author has returned to the task of filling in this gap. This is done in Theorem 3 below whose consequence (Theorem 4) also states that if the inequality (1.3) has a solution $M \geq 0$ and R, then it also has a particular solution $M_0 = (I - |A_c^{-1}|\Delta)^{-1} \geq 0$ and $R_0 = A_c^{-1}$. Thus, M and R can be viewed as approximations of the exact inverses M_0 and R_0. In the last Theorem 5 we sum up some consequences of solvability of (1.3). In particular, we prove that if $M \geq 0$, R solve (1.3), then

$$|A^{-1} - R| \leq (M - I)|R|$$

holds for each $A \in \mathbf{A}$. We do not give any applications of these results here, having in mind writing another paper about these issues.

We use the following notations: I is the unit matrix, $\varrho(A)$ denotes the spectral radius of A, for $A = (a_{ij})$ we denote its absolute value by $|A| = (|a_{ij}|)$, and matrix inequalities are understood componentwise.

2 Auxiliary results

In the proofs to follow, we shall essentially use spectral properties of nonnegative matrices that are summed up in the following two theorems. The proofs can be found in Horn and Johnson [4] or Meyer [5].

Theorem 1. *For a nonnegative square matrix G, the following assertions are equivalent:*

(i) $\varrho(G) < 1$,
(ii) $I - G$ *is nonsingular and* $(I - G)^{-1} \geq 0$,
(iii) $G^j \to 0$ *as* $j \to \infty$,
(iv) $Gx < x$ *for some* $x > 0$.

Moreover, if any of these conditions is met, then $(I - G)^{-1} = \sum_{j=0}^{\infty} G^j$.

Theorem 2. *If $A, B \in \mathbb{R}^{n \times n}$ satisfy $|A| \leq B$, then $\varrho(A) \leq \varrho(|A|) \leq \varrho(B)$.*

3   Characterization

In this section we show that strong regularity of $[A_c - \Delta, A_c + \Delta]$ is equivalent to solvability of the matrix inequality

$$M(I - |I - RA_c| - |R|\Delta) \geq I, \tag{3.1}$$

where $M, R \in \mathbb{R}^{n \times n}$, and M is required to be nonnegative. Alternatively, we can also write (3.1) as

$$M(I - G) \geq I, \tag{3.2}$$

where

$$G = |I - RA_c| + |R|\Delta \tag{3.3}$$

is a nonnegative matrix. We have this result.

Theorem 3. *An interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ is strongly regular if and only if the inequality (3.1) has a solution M, R, where $M \geq 0$.*

Proof. Let (3.1) have a solution $M \geq 0$ and R. Then for G given by (3.3) we have (3.2), which can be written as

$$I + MG \leq M.$$

Postmultiplying this inequality by G and adding I to both sides we obtain

$$I + G + MG^2 \leq I + MG \leq M$$

and by induction

$$\sum_{j=0}^{k} G^j + MG^{k+1} \leq M$$

for $k = 0, 1, 2, \ldots$. In view of nonnegativity of M, this shows that the nonnegative matrix series $\sum_{j=0}^{\infty} G^j$ satisfies

$$\sum_{j=0}^{\infty} G^j \leq M, \tag{3.4}$$

hence it is convergent, so that $G^j \to 0$ and consequently

$$\varrho(G) < 1 \tag{3.5}$$

(Theorem 1). Now we have

$$I - RA_c \le |I - RA_c| \le G,$$

hence by Theorem 2,

$$\varrho(I - RA_c) \le \varrho(|I - RA_c|) \le \varrho(G) < 1. \tag{3.6}$$

Since $\varrho(I - RA_c) < 1$, the matrix

$$RA_c = I - (I - RA_c) \tag{3.7}$$

is nonsingular, which gives that both A_c and R are nonsingular. Moreover, (3.7) implies that

$$A_c^{-1} R^{-1} = (RA_c)^{-1} = \sum_{j=0}^{\infty} (I - RA_c)^j$$

(Theorem 1), hence

$$A_c^{-1} = \sum_{j=0}^{\infty} (I - RA_c)^j R,$$

and thus also

$$|A_c^{-1}| \le \sum_{j=0}^{\infty} |I - RA_c|^j |R| = (I - |I - RA_c|)^{-1} |R|$$

(because $\sum_{j=0}^{\infty} |I - RA_c|^j$ is again convergent by (3.6)), and

$$|A_c^{-1}| \Delta \le (I - |I - RA_c|)^{-1} |R| \Delta. \tag{3.8}$$

Since $\varrho(G) < 1$, Theorem 1 implies existence of an $x > 0$ satisfying $Gx < x$, i.e.,

$$|I - RA_c| x + |R| \Delta x < x,$$

hence

$$|R| \Delta x < (I - |I - RA_c|) x$$

and

$$(I - |I - RA_c|)^{-1} |R| \Delta x < x \tag{3.9}$$

in view of (3.6). Now, from (3.8) and (3.9) we finally obtain

$$|A_c^{-1}| \Delta x \le (I - |I - RA_c|)^{-1} |R| \Delta x < x,$$

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where $x > 0$, hence $\varrho(|A_c^{-1}|\Delta) < 1$ by Theorem 1, which proves that $\mathbf{A}$ is strongly regular. Conversely, if $\mathbf{A}$ is strongly regular, then $(I - |A_c^{-1}|\Delta)^{-1} \geq 0$ again by Theorem 1 and

$$
\begin{aligned}
M_0 &= (I - |A_c^{-1}|\Delta)^{-1} \geq 0, & (3.10) \\
R_0 &= A_c^{-1} & (3.11)
\end{aligned}
$$

satisfy (3.1) as an equation. □

Thus we have also proved the following result.

Theorem 4. *If the inequality (3.1) has a solution $M \geq 0$ and R, then it also has a particular solution $M_0 \geq 0$ and R_0 given by (3.10), (3.11).*

4   Consequences

Below, we list some consequences of solvability of the inequality (3.1).

Theorem 5. *Let (3.1) have a solution $M \geq 0$ and R. Then we have:*

(a) $\varrho(|I - RA_c| + |R|\Delta) < 1$,
(b) $(I - |I - RA_c| - |R|\Delta)^{-1} \leq M$,
(c) $\varrho(I - RA_c) < 1$,
(d) R *is nonsingular,*
(e) $[R^{-1} - \Delta, R^{-1} + \Delta]$ *is strongly regular,*
(f) *each $A \in [A_c - \Delta, A_c + \Delta]$ is nonsingular and $A^{-1} = \sum_{j=0}^{\infty}(I - RA)^j R$,*
(g) $|A^{-1} - R| \leq (M - I)|R|$ *for each $A \in \mathbf{A}$.*

Proof. The assertions (a), (c) and (d) have been proved in the proof of Theorem 3, and (b) is a consequence of (3.4). From (a) it follows that $\varrho(|R|\Delta) \leq \varrho(|I - RA_c| + |R|\Delta) < 1$, which means that $[R^{-1} - \Delta, R^{-1} + \Delta]$ is strongly regular by definition, thus proving (e). To prove the remaining two assertions, take an $A \in \mathbf{A}$. Then it satisfies the identity

$$
RA = I - (I - RA) \tag{4.1}
$$

and since

$$
|I - RA| = |I - RA_c + R(A_c - A)| \leq |I - RA_c| + |R|\Delta = G, \tag{4.2}
$$

there holds

$$
\varrho(I - RA) \leq \varrho(G) < 1
$$

by Theorem 2 and (3.5), so that (4.1) shows that RA is nonsingular, hence A is nonsingular. Next, from (4.1) it follows

$$
A^{-1}R^{-1} = \sum_{j=0}^{\infty}(I - RA)^j,
$$

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hence

$$A^{-1} = \sum_{j=0}^{\infty} (I - RA)^j R, \qquad (4.3)$$

which proves (f). Finally, from (4.3), (4.2) and (3.4) we have that

$$|A^{-1} - R| \leq \sum_{j=1}^{\infty} (|I - RA_c| + |R|\Delta)^j |R| = (\sum_{j=0}^{\infty} G^j - I)|R| \leq (M - I)|R|,$$

which concludes the proof. □

A square interval matrix $\mathbf{A}$ is called *regular* if each $A \in \mathbf{A}$ is nonsingular; a strongly regular interval matrix is regular (Beeck [1]). For a regular interval matrix $\mathbf{A}$, its inverse interval matrix $\mathbf{A}^{-1}$ is defined as the narrowest interval matrix containing the set $\{ A^{-1} \mid A \in \mathbf{A} \}$. Computing the inverse interval matrix is NP-hard (Coxson [2]). Therefore in practical computations we usually resort to *enclosures* of the inverse, i.e., to interval matrices containing $\mathbf{A}^{-1}$, but not necessarily the minimal ones with respect to inclusion. The assertion (g) of Theorem 5 says that if $M \geq 0$ and R solve (3.1), then

$$[R - (M - I)|R|, \, R + (M - I)|R|]$$

is an enclosure of $[A_c - \Delta, A_c + \Delta]^{-1}$.

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