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Rohn, Jiří

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Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789, e-mail:rohn@cs.cas.cz

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Abstract:

Explicit formulae for the inverse of an interval matrix of the form $[I - \Delta, I + \Delta]$ (where I is the unit matrix) are proved via finding explicit solutions of certain nonlinear matrix equations.

Keywords:
Interval matrix, unit midpoint, inverse interval matrix, regularity.

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1 Introduction

In this paper we study the inverse of an interval matrix of a special form

$$\mathbf{A} = [I - \Delta, I + \Delta] \tag{1.1}$$

(i.e., having the unit midpoint). Computing the inverse interval matrix (defined in Section 3) is NP-hard in general (Coxson [1]). Yet it was shown in [7], Theorem 2, that in the special case of an interval matrix of the form (1.1) the inverse interval matrix can be expressed by simple formulae in terms of the matrix

$$M = (I - \Delta)^{-1}$$

(Theorem 5 below). The result was proved there as an application of a very special assertion on interval linear equations. In this paper we give another proof of this theorem making use of a general result (Theorem 2) according to which the inverse of an $n \times n$ interval matrix can be computed from unique solutions of 2^n nonlinear matrix equations. As the main result of this paper we show in Theorem 4 that for interval matrices of the form (1.1) the unique solution of each of these 2^n nonlinear equations can be expressed explicitly; this, in turn, makes it possible to express the inverse of (1.1) explicitly, as showed in the proof of Theorem 5. Moreover, this approach also allows us to specify the matrices in $\mathbf{A}$ at whose inverses the componentwise bounds on $\mathbf{A}^{-1}$ are attained (Theorem 7).

The paper is organized as follows. In Section 2 we sum up the notations used. In Section 3 the inverse interval matrix is defined and a general (finite, but exponential) method for its computation is given. The explicit solutions of the respective nonlinear equations are described in Section 4 and are then used for deriving explicit formulae for the inverse of (1.1). Finally in Section 5 matrices are described at whose inverses the componentwise bounds on the interval inverse are attained.

2 Notations

We use the following notations. A_{ij} denotes the ijth entry and $A_{\bullet j}$ the jth column of A. Matrix inequalities, as $A \leq B$ or $A < B$, are understood componentwise. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. The same notations also apply to vectors that are considered one-column matrices. I is the unit matrix, e_j denotes its jth column, and $e = (1, \ldots, 1)^T$ is the vector of all ones. $Y_n = \{y \mid |y| = e\}$ is the set of all ± 1-vectors in $\mathbb{R}^n$, so that its cardinality is 2^n. For each $y \in \mathbb{R}^n$ we denote

$$T_y = \operatorname{diag}(y_1, \ldots, y_n) = \begin{pmatrix} y_1 & 0 & \ldots & 0 \\ 0 & y_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & y_n \end{pmatrix}.$$

Finally, we introduce the *real* spectral radius of a square matrix A by

$$\varrho_0(A) = \max\{|\lambda| \mid \lambda \text{ is a real eigenvalue of } A\}, \tag{2.1}$$

and we set $\varrho_0(A) = 0$ if no real eigenvalue exists; $\varrho(A)$ is the usual spectral radius of A.

3 Inverse interval matrix

Given two $n \times n$ matrices A_c and Δ, $\Delta \geq 0$, the set of matrices

$$\mathbf{A} = \{A \mid |A - A_c| \leq \Delta\}$$

is called a (square) interval matrix with midpoint matrix A_c and radius matrix Δ. Since the inequality $|A - A_c| \leq \Delta$ is equivalent to $A_c - \Delta \leq A \leq A_c + \Delta$, we can also write

$$\mathbf{A} = \{A \mid \underline{A} \leq A \leq \overline{A}\} = [\underline{A}, \overline{A}],$$

where $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$ are called the bounds of $\mathbf{A}$.

Definition. A square interval matrix $\mathbf{A}$ is called regular if each $A \in \mathbf{A}$ is nonsingular, and it is said to be singular otherwise (i.e., if it contains a singular matrix).

Many necessary and sufficient regularity conditions are known (the paper [9] surveys forty of them). We shall use here the following one (condition (xxxiv) in [9]; see (2.1) for the definition of ϱ_0).

Proposition 1. *A square interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ is regular if and only if A_c is nonsingular and*

$$\max_{y,z \in Y_n} \varrho_0(A_c^{-1} T_y \Delta T_z) < 1$$

holds.

Definition. For a regular interval matrix $\mathbf{A}$ we define its inverse interval matrix $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ by

$$\begin{aligned} \underline{B} &= \min\{\, A^{-1} \mid A \in \mathbf{A}\,\}, \\ \overline{B} &= \max\{\, A^{-1} \mid A \in \mathbf{A}\,\} \end{aligned}$$

(componentwise).

Comment 3.1. This means that

$$\underline{B}_{ij} = \min\{\,(A^{-1})_{ij} \mid A \in \mathbf{A}\,\}, \tag{3.1}$$

$$\overline{B}_{ij} = \max\{\,(A^{-1})_{ij} \mid A \in \mathbf{A}\,\} \qquad (i, j = 1, \ldots, n). \tag{3.2}$$

Since $\mathbf{A}$ is regular, the mapping $A \mapsto A^{-1}$ is continuous in $\mathbf{A}$ and all the minima and maxima in (3.1), (3.2) are attained. Thus, $\mathbf{A}^{-1}$ is the narrowest interval matrix enclosing the set of matrices $\{ A^{-1} \mid A \in \mathbf{A} \}$. For more results on the inverse interval matrix see Hansen [2], Hansen and Smith [3], Herzberger and Bethke [4], and Rohn [6], [8]. Computing the inverse interval matrix is NP-hard (Coxson [1]).

We have the following general result ([6], Theorem 5.1, assertion (A3), and Theorem 6.2).

Theorem 2. *Let $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ be regular. Then for each $y \in Y$ the matrix equation*

$$A_c B - T_y \Delta |B| = I$$

has a unique matrix solution B_y and for the inverse interval matrix $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ we have

$$\begin{aligned} \underline{B} &= \min\{ B_y \mid y \in Y \}, \\ \overline{B} &= \max\{ B_y \mid y \in Y \} \end{aligned}$$

(componentwise).

Thus, in contrast to the definition, only a finite number of matrices B_y, $y \in Y$ (albeit 2^n of them) are needed to compute the inverse interval matrix. In the next section we shall show that in case of $A_c = I$ all the matrices B_y can be expressed explicitly.

4 Inverse interval matrix with unit midpoint

From now on, we shall consider interval matrices of the form

$$\mathbf{A} = [I - \Delta, I + \Delta], \tag{4.1}$$

i.e., with unit midpoint I. First, we shall resolve the question of regularity of (4.1).

Proposition 3. *An interval matrix (4.1) is regular if and only if*

$$\varrho(\Delta) < 1 \tag{4.2}$$

holds.

Proof. For each $y, z \in Y$ we have

$$\varrho_0(T_y \Delta T_z) \le \varrho(T_y \Delta T_z) \le \varrho(|T_y \Delta T_z|) = \varrho(\Delta) = \varrho_0(\Delta) = \varrho_0(T_e \Delta T_e)$$

(the equation $\varrho(\Delta) = \varrho_0(\Delta)$ being a consequence of the Perron-Frobenius theorem [5]), hence

$$\max_{y,z \in Y_n} \varrho_0(T_y \Delta T_z) = \varrho(\Delta)$$

and the assertion follows from Proposition 1. $\square$

As is well known, the condition $\varrho(\Delta) < 1$ implies

$$(I-\Delta)^{-1} = \sum_{j=0}^{\infty} \Delta^j \geq I \geq 0$$

(because Δ is nonnegative). Put

$$M = (I-\Delta)^{-1} = (m_{ij})$$

and

$$\mu = (\mu_j),$$

where

$$\mu_j = \frac{m_{jj}}{2m_{jj}-1} \qquad (j = 1, \ldots, n). \tag{4.3}$$

Then we obviously have

$$\begin{aligned}
m_{ij} &\geq 0, && (4.4)\\
m_{jj} &\geq 1, && (4.5)\\
2m_{jj}-1 &\geq 1, && (4.6)\\
2\mu_j - 1 &\in (0,1], && (4.7)\\
\mu_j &\in (\tfrac{1}{2}, 1], && (4.8)\\
\mu_j &\leq m_{jj}, && (4.9)\\
(2\mu_j - 1)m_{jj} &= \mu_j && (4.10)
\end{aligned}$$

$(i, j = 1, \ldots, n)$, and also

$$M\Delta = \Delta M = \sum_{j=1}^{\infty} \Delta^j = M - I. \tag{4.11}$$

These simple facts will be utilized in the proofs to follow.

Under the assumption (4.2), the interval matrix (4.1) is regular, hence by Theorem 2 the equation

$$B - T_y \Delta |B| = I$$

has a unique solution B_y for each $y \in Y_n$. We shall now show that this B_y can be expressed explicitly.

Theorem 4. *Let $\varrho(\Delta) < 1$. Then for each $y \in Y$ the unique solution of the matrix equation*

$$B - T_y\Delta|B| = I \tag{4.12}$$

is given by

$$B_y = T_yMT_y + T_y(M - I)T_\mu(I - T_y), \tag{4.13}$$

i.e. componentwise

$$(B_y)_{ij} = y_iy_jm_{ij} + y_i(1 - y_j)(m_{ij} - I_{ij})\mu_j, \tag{4.14}$$

or

$$(B_y)_{ij} = \begin{cases} y_im_{ij} & \text{if } y_j = 1, \\ y_i(2\mu_j - 1)m_{ij} & \text{if } y_j = -1 \text{ and } i \neq j, \\ \mu_j & \text{if } y_j = -1 \text{ and } i = j, \end{cases} \tag{4.15}$$

or

$$(B_y)_{ij} = \frac{(y_i + (1 - y_i)I_{ij})m_{ij}}{y_j + (1 - y_j)m_{jj}} \tag{4.16}$$

$(i, j = 1, \ldots, n)$.

Comment 4.1. We give two proofs of this theorem. The first one shows how the formulae (4.13)-(4.16) can be derived. The second one demonstrates that once they are known, it is relatively simple to prove that B_y given by them is indeed a solution to (4.12). As it can be expected, the first proof is essentially longer, but more informative.

Proof. Under the assumption (4.2) it follows from Theorem 2 that the equation (4.12) has a unique solution B_y. Fix a $j \in \{1, \ldots, n\}$ and put

$$x = T_y(B_y)_{\bullet j} \tag{4.17}$$

(where $(B_y)_{\bullet j}$ is the jth column of B_y), then from (4.12), if written in the form

$$T_yB - \Delta|T_yB| = T_y$$

(because $|T_yB| = |B|$), it follows that x satisfies the equation

$$x - \Delta|x| = y_je_j. \tag{4.18}$$

If $y_j = 1$, then $x = \Delta|x| + e_j \geq 0$, hence $|x| = x$ and from (4.18) we have simply $x = (I - \Delta)^{-1}e_j = Me_j$, hence

$$x_i = m_{ij} \tag{4.19}$$

for each i. Now, let $y_j = -1$. Then from (4.18) it follows that $x_i \geq 0$ for each $i \neq j$, so that we can write

$$|x| = (x_1, \ldots, x_{j-1}, |x_j|, x_{j+1}, \ldots, x_n)^T = x + (|x_j| - x_j)e_j,$$

and from (4.18) we obtain

$$(I - \Delta)x = -e_j + (|x_j| - x_j)\Delta e_j,$$

hence premultiplying this equation by the nonnegative matrix $M = (I - \Delta)^{-1}$ gives

$$x = -Me_j + (|x_j| - x_j)M\Delta e_j = -Me_j + (|x_j| - x_j)(M - I)e_j \tag{4.20}$$

(using (4.11)) and consequently

$$x_j = -m_{jj} + (|x_j| - x_j)(m_{jj} - 1). \tag{4.21}$$

Assuming $x_j \geq 0$, we would have $x_j = -m_{jj} \leq -1 < 0$ by (4.5), a contradiction. This shows that $x_j < 0$, hence $|x_j| = -x_j$, and (4.21) yields

$$x_j = -\frac{m_{jj}}{2m_{jj} - 1} = -\mu_j \tag{4.22}$$

(see (4.3)). Hence

$$|x_j| - x_j = -2x_j = 2\mu_j,$$

and substituting into (4.20) gives

$$x = -Me_j + 2\mu_j(M - I)e_j,$$

so that

$$x_i = -m_{ij} + 2\mu_j m_{ij} = (2\mu_j - 1)m_{ij} \tag{4.23}$$

for each $i \neq j$. Hence from (4.19), (4.23) and (4.22) we obtain that

$$x_i = \begin{cases} m_{ij} & \text{if } y_j = 1, \\ (2\mu_j - 1)m_{ij} & \text{if } y_j = -1 \text{ and } i \neq j, \\ -\mu_j & \text{if } y_j = -1 \text{ and } i = j \end{cases}$$

for each i. Since $(B_y)_{\bullet j} = T_y x$ by (4.17), this means that

$$(B_y)_{ij} = y_i x_i = \begin{cases} y_i m_{ij} & \text{if } y_j = 1, \\ y_i(2\mu_j - 1)m_{ij} & \text{if } y_j = -1 \text{ and } i \neq j, \\ \mu_j & \text{if } y_j = -1 \text{ and } i = j, \end{cases}$$

which is (4.15). Hence we can see that $(B_y)_{ij}$, aside from m_{ij} and μ_j, depends on y_i and y_j only. The values of $(B_y)_{ij}$ for all possible combinations of y_i and y_j are summed up in Fig. 4.1. Validity of (4.14), (4.16) can be checked simply by assigning $y_i = \pm 1$, $y_j = \pm 1$ into their right-hand sides and verifying that the results obtained correspond to those in Fig. 4.1. Finally, rewriting (4.14) in the equivalent form

$$(B_y)_{ij} = y_i m_{ij} y_j + y_i(m_{ij} - I_{ij})\mu_j(1 - y_j),$$

we can see that this is the componentwise version of (4.13) (taking into account that all three matrices T_y, T_μ, I are diagonal). □

y_i	y_j	$(B_y)_{ij}$
1	1	m_{ij}
-1	1	$-m_{ij}$
1	-1	$(2\mu_j - 1)m_{ij}$
-1	-1	$-(2\mu_j - 1)m_{ij} + 2\mu_j I_{ij}$

Figure 4.1: Dependence of $(B_y)_{ij}$ on y_i, y_j.

Proof. Equivalence of (4.13), (4.14), (4.15) and (4.16) has been established in the previous proof. From (4.15), (4.7) and (4.10) we have

$$|B_y|_{ij} = \begin{cases} m_{ij} & \text{if } y_j = 1, \\ (2\mu_j - 1)m_{ij} & \text{if } y_j = -1 \end{cases}$$

for each i, j, but also

$$(M(T_y + T_\mu(I - T_y)))_{ij} = m_{ij}(y_j + \mu_j(1 - y_j)) = \begin{cases} m_{ij} & \text{if } y_j = 1, \\ (2\mu_j - 1)m_{ij} & \text{if } y_j = -1 \end{cases}$$

for each i, j, which shows that

$$|B_y| = M(T_y + T_\mu(I - T_y)).$$

Then

$$\begin{aligned} B_y - I &= T_y(M - I)T_y + T_y(M - I)T_\mu(I - T_y) = T_y(M - I)(T_y + T_\mu(I - T_y)) \\ &= T_y \Delta M(T_y + T_\mu(I - T_y)) = T_y \Delta |B_y| \end{aligned}$$

(because of (4.11) and of the fact that $T_y^2 = I$), hence

$$B_y - T_y \Delta |B_y| = I,$$

which means that B_y is a solution to (4.12) which, according to Theorem 2, is unique. □

Now we shall apply this result to the inverse interval matrix.

Theorem 5. *Let* $\mathbf{A} = [I - \Delta, I + \Delta]$ *with* $\varrho(\Delta) < 1$. *Then the inverse interval matrix* $\mathbf{A}^{-1} = [\underline{B}, \overline{B}]$ *is given by*

$$\underline{B} = -M + T_\kappa, \tag{4.24}$$

$$\overline{B} = M, \tag{4.25}$$

where

$$\kappa_j = \frac{2m_{jj}^2}{2m_{jj} - 1} \qquad (j = 1, \dots, n), \tag{4.26}$$

or componentwise

$$\underline{B}_{ij} = \begin{cases} -m_{ij} & \text{if } i \neq j, \\ \mu_j & \text{if } i = j, \end{cases} \tag{4.27}$$

$$\overline{B}_{ij} = m_{ij} \tag{4.28}$$

$(i, j = 1, \ldots, n)$.

Proof. For each $i \neq j$, Theorem 4 in view of Fig. 4.1, (4.4) and (4.7) gives

$$\begin{aligned} \underline{B}_{ij} &= \min_{y \in Y} (B_y)_{ij} = \min\{m_{ij}, -m_{ij}, (2\mu_j - 1)m_{ij}, -(2\mu_j - 1)m_{ij}\} \\ &= \min\{-m_{ij}, -(2\mu_j - 1)m_{ij}\} = -m_{ij} \end{aligned}$$

and similarly

$$\overline{B}_{ij} = \max\{m_{ij}, (2\mu_j - 1)m_{ij}\} = m_{ij}.$$

If $i = j$, then it must be $y_i = y_j$, hence only the first and the last row of Fig. 4.1 apply, giving

$$\underline{B}_{jj} = \min_{y \in Y} (B_y)_{jj} = \min\{m_{jj}, -(2\mu_j - 1)m_{jj} + 2\mu_j\} = \min\{m_{jj}, \mu_j\} = \mu_j$$

due to (4.10) and (4.9), and similarly

$$\overline{B}_{jj} = \max\{m_{jj}, \mu_j\} = m_{jj}.$$

This proves (4.27), (4.28) and thus also (4.24), (4.25) in view of the fact that κ_j defined by (4.26) satisfies

$$-m_{jj} + \kappa_j = \mu_j$$

for each j. □

In particular, we have the following result.

Corollary 6 *If $\varrho(\Delta) < 1$, then the inverse interval matrix $[I - \Delta, I + \Delta]^{-1} = [\underline{B}, \overline{B}]$ satisfies*

$$\tfrac{1}{2} \leq \underline{B}_{jj} \leq 1 \leq \overline{B}_{jj}$$

for each j.

Proof. This is a consequence of Theorem 5 and of (4.5), (4.8). □

5 Attainment

According to the definition of the inverse interval matrix

$$[I-\Delta, I+\Delta]^{-1} = [\underline{B}, \overline{B}],$$

for each i, j there exists a matrix, say A^{ij}, such that $|A^{ij}| \le \Delta$ and

$$\underline{B}_{ij} = (I - A^{ij})^{-1}_{ij}$$

holds (we write $(I - A^{ij})^{-1}_{ij}$ instead of $((I - A^{ij})^{-1})_{ij}$), and an analogue holds for $\overline{B}_{ij}$. In the last section we give an explicit expression of such an A^{ij} for each i, j.

First of all, the situation is quite evident for $\overline{B}$ because from (4.25) we have

$$\overline{B} = M = (I-\Delta)^{-1},$$

so that all the componentwise upper bounds are attained at the inverse of $I-\Delta$. But the case of $\underline{B}$ is more involved.

Theorem 7. *For each i, j we have:*

(i) if $i \ne j$, then

$$\underline{B}_{ij} = (I - T_y \Delta T_y)^{-1}_{ij}$$

for each $y \in Y$ satisfying $y_i y_j = -1$,

(ii) if $i = j$, then

$$\underline{B}_{jj} = (I - T_y \Delta T_z)^{-1}_{jj}$$

for each $y \in Y$ satisfying $y_j = -1$ and $z = y + 2e_j$.

Comment 5.1. Notice that $I - T_y \Delta T_z \in [I-\Delta, I+\Delta]$ for each $y, z \in Y$.

Proof. Let $i, j \in \{1, \dots, n\}$.

(i) For each $y \in Y_n$ there holds

$$(I - T_y \Delta T_y)^{-1} = (T_y (I-\Delta) T_y)^{-1} = T_y M T_y,$$

hence if $y_i y_j = -1$, then

$$(I - T_y \Delta T_y)^{-1}_{ij} = y_i y_j m_{ij} = -m_{ij} = \underline{B}_{ij}.$$

(ii) We have

$$\begin{aligned} I - T_y \Delta T_z &= I - T_y \Delta (T_y + 2e_j e_j^T) \\ &= (I - T_y \Delta T_y)(I - (I - T_y \Delta T_y)^{-1} 2T_y \Delta e_j e_j^T) \\ &= (I - T_y \Delta T_y)(I - 2T_y M T_y T_y \Delta e_j e_j^T) \\ &= (I - T_y \Delta T_y)(I - 2T_y M \Delta e_j e_j^T), \end{aligned}$$

so that by the Sherman-Morrison formula [10] applied to the matrix in the last parentheses,

$$\begin{aligned}(I - T_y \Delta T_z)^{-1} &= (I - 2T_y M \Delta e_j e_j^T)^{-1}(I - T_y \Delta T_y)^{-1} \\ &= \left(I + \frac{2T_y M \Delta e_j e_j^T}{1 - 2e_j^T T_y M \Delta e_j}\right) T_y M T_y \\ &= T_y M T_y + \frac{2T_y M \Delta e_j e_j^T T_y M T_y}{1 + 2(M\Delta)_{jj}}\end{aligned}$$

(because $y_j = -1$) and consequently

$$\begin{aligned}(I - T_y \Delta T_z)^{-1}_{jj} &= m_{jj} + \frac{2(T_y M \Delta)_{jj}(T_y M T_y)_{jj}}{1 + 2(M - I)_{jj}} \\ &= m_{jj} - \frac{2(M - I)_{jj} m_{jj}}{1 + 2(M - I)_{jj}} \\ &= m_{jj} - \frac{2(m_{jj} - 1)m_{jj}}{2m_{jj} - 1} \\ &= \frac{m_{jj}}{2m_{jj} - 1} = \mu_j = \underline{B}_{jj}.\end{aligned}$$

□

Hence, the $\underline{B}_{ij}$'s are attained at inverses of many matrices in $[I - \Delta, I + \Delta]$. But the results can be essentially simplified if we use the particular set of vectors

$$y(j) = e - 2e_j = (1, \dots, 1, -1, 1, \dots, 1)^T \qquad (j = 1, \dots, n).$$

Corollary 8 *For each i, j we have:*

(i) $\underline{B}_{ij} = (I - T_{y(j)} \Delta T_{y(j)})^{-1}_{ij}$ *if* $i \neq j$,

(ii) $\underline{B}_{jj} = (I - T_{y(j)} \Delta)^{-1}_{jj}$.

Proof. The results are immediate consequences of Theorem 7 since $y(j)_j = -1$, $y(j)_i y(j)_j = -1$ for each $i \neq j$ and $z = y(j) + 2e_j = e$. □

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