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**Institute of Computer Science**  
**Academy of Sciences of the Czech Republic**

## **An Improvement of the Bauer-Skeel Bounds**

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Technical report No. V-1065

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## **An Improvement of the Bauer-Skeel Bounds**

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Abstract:

It is shown that the Hansen-Bliek-Rohn bounds are never worse, and “almost always” better, than the classical Bauer-Skeel bounds. Formulae for overestimation of the Hansen-Bliek-Rohn bounds are given.

Keywords:

Bauer-Skeel bounds, Hansen-Bliek-Rohn bounds, comparison, overestimation.

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# 1 Introduction

This is transcription of slides of a talk I delivered at the Bergische Universität Wuppertal, Germany on Dec. 9, 2003. It has never been made a regular paper. So after six years, I decided to make it at least a report written in a terse “slide-like” style. Theorems 3 through 6 are new.

## 2 Notations

- $A \leq B$ ,
- $|A|$ ,
- $\min\{A, B\}, \max\{A, B\}$

understood componentwise.

Especially,

$$|A - B| \leq C$$

is equivalent to

$$B - C \leq A \leq B + C.$$

## 3 The problem

Given a fixed system

$$A_c x_c = b_c$$

with  $A_c$  nonsingular, and a perturbed system

$$Ax = b$$

such that

$$\begin{aligned} |A - A_c| &\leq \Delta, \\ |b - b_c| &\leq \delta, \end{aligned}$$

estimate

$$\dots \leq x \leq \dots$$

in terms of  $x_c, A_c^{-1}, \Delta$  and  $\delta$ .

## 4 Assumption

We shall assume throughout that the data satisfy

$$\varrho(|A_c^{-1}|\Delta) < 1.$$

Under this *spectral condition* we have

$$M := (I - |A_c^{-1}|\Delta)^{-1} = \sum_{j=0}^{\infty} (|A_c^{-1}|\Delta)^j \geq I \geq 0.$$

In particular,

$$M_{ii} \geq 1$$

for each  $i$  (a property which will turn out extremely important).

## 5 The Bauer-Skeel bounds

**Theorem 1. (Bauer 1966, Skeel 1979)** *If*

$$\varrho(|A_c^{-1}|\Delta) < 1,$$

*then for each  $A, b$  such that  $|A - A_c| \leq \Delta$  and  $|b - b_c| \leq \delta$ ,  $A$  is nonsingular and the solution of*

$$Ax = b$$

*satisfies*

$$-x^* + x_c + |x_c| \leq x \leq x^* + x_c - |x_c|,$$

*where*

$$\begin{aligned} M &= (I - |A_c^{-1}|\Delta)^{-1}, \\ x^* &= M(|x_c| + |A_c^{-1}|\delta). \end{aligned}$$

**Note.** Usually presented as  $|x - x_c| \leq x^* - |x_c|$ , with  $\delta = 0$  or in normwise setting. Two inversions needed.

## 6 Proof

We have

$$A_c^{-1}A = I - A_c^{-1}(A_c - A),$$

where

$$\varrho(A_c^{-1}(A_c - A)) \leq \varrho(|A_c^{-1}(A_c - A)|) \leq \varrho(|A_c^{-1}|\Delta) < 1,$$

hence  $A_c^{-1}A$  is nonsingular and thus also  $A$ . If  $Ax = b$ , then

$$\begin{aligned} |x - x_c| &= |A_c^{-1}A_c(x - x_c)| \\ &\leq |A_c^{-1}| \cdot |(A_c - A)x + (b - b_c)| \\ &\leq |A_c^{-1}|(\Delta|x| + \delta). \end{aligned}$$

[Attention: This is the bifurcation point of the two proofs.]

$$\begin{aligned} |x - x_c| &\leq |A_c^{-1}|(\Delta|x| + \delta) = |A_c^{-1}|(\Delta|x - x_c + x_c| + \delta) \\ &\leq |A_c^{-1}|\Delta|x - x_c| + |A_c^{-1}|(\Delta|x_c| + \delta), \end{aligned}$$

hence

$$(I - |A_c^{-1}|\Delta)|x - x_c| \leq |A_c^{-1}|(\Delta|x_c| + \delta).$$

Premultiplying by  $M = (I - |A_c^{-1}|\Delta)^{-1} \geq 0$ :

$$\begin{aligned} |x - x_c| &\leq M|A_c^{-1}|(\Delta|x_c| + \delta) \\ &= (M - I)|x_c| + M|A_c^{-1}|\delta \\ &= x^* - |x_c| \end{aligned}$$

and equivalently

$$-x^* + x_c + |x_c| \leq x \leq x^* + x_c - |x_c|. \quad \square$$

## 7 The HBR bounds

**Theorem 2. (Hansen 1992, Bliok 1992, R. 1993)** *Under the same assumption*

$$\rho(|A_c^{-1}|\Delta) < 1$$

we have

$$\min\{\underline{x}, T\underline{x}\} \leq x \leq \max\{\tilde{x}, T\tilde{x}\},$$

where

$$\begin{aligned} M &= (I - |A_c^{-1}|\Delta)^{-1}, \\ D &= \text{diag}(M_{11}, \dots, M_{nn}), \\ T &= (2D - I)^{-1}, \\ x^* &= M(|x_c| + |A_c^{-1}|\delta), \\ \underline{x} &= -x^* + D(x_c + |x_c|), \\ \tilde{x} &= x^* + D(x_c - |x_c|). \end{aligned}$$

## 8 Proof

As in the proof of the Bauer-Skeel bounds we proceed up to the “bifurcation point”

$$|x - x_c| \leq |A_c^{-1}|(\Delta|x| + \delta),$$

but then we continue in another way: we have on one hand

$$x - x_c \leq |x - x_c| \leq |A_c^{-1}|(\Delta|x| + \delta) \quad (8.1)$$

and on the other hand

$$|x| - |x_c| \leq |x - x_c| \leq |A_c^{-1}|(\Delta|x| + \delta). \quad (8.2)$$

**For  $i$  fixed, take the  $i$ th inequality from (8.1) and for  $j \neq i$  from (8.2):**

$$x_i \leq (x_c)_i + (|A_c^{-1}|(\Delta|x| + \delta))_i$$

$$|x_j| \leq |x_c|_j + (|A_c^{-1}|(\Delta|x| + \delta))_j, \quad j \neq i.$$

Since  $x_i = |x_i| + (x_i - |x_i|)$  and the same holds for  $(x_c)_i$ , we can put them together as

$$|x| + (x_i - |x_i|)e_i \leq |x_c| + ((x_c)_i - |x_c|_i)e_i + |A_c^{-1}|(\Delta|x| + \delta),$$

which implies

$$(I - |A_c^{-1}|\Delta)|x| + (x_i - |x_i|)e_i \leq |x_c| + ((x_c)_i - |x_c|_i)e_i + |A_c^{-1}|\delta.$$

Again premultiplying by  $M = (I - |A_c^{-1}|\Delta)^{-1} \geq 0$ :

$$|x| + (x_i - |x_i|)Me_i \leq x^* + ((x_c)_i - |x_c|_i)Me_i$$

and taking the  $i$ th inequality we get

$$|x_i| + (x_i - |x_i|)M_{ii} \leq x_i^* + ((x_c)_i - |x_c|_i)M_{ii} = \tilde{x}_i,$$

an inequality containing  $x_i$  only. If  $x_i \geq 0$ , then this inequality becomes

$$x_i \leq \tilde{x}_i,$$

and if  $x_i < 0$ , then it turns into

$$x_i \leq \tilde{x}_i / (2M_{ii} - 1) = T_{ii}\tilde{x}_i,$$

in both cases

$$x_i \leq \max\{\tilde{x}_i, T_{ii}\tilde{x}_i\}.$$

Since  $i$  was arbitrary, we conclude that

$$x \leq \max\{\tilde{x}, T\tilde{x}\},$$

which is the upper bound. Similarly for the lower one.  $\square$

## 9 Comparison: preliminaries

For comparison, denote the Bauer-Skeel bounds by

$$\underline{x} \leq x \leq \bar{x}$$

and the HBR bounds by

$$\underline{\underline{x}} \leq x \leq \bar{\bar{x}},$$

i.e.

$$\begin{aligned} \underline{x} &= -x^* + x_c + |x_c|, \\ \bar{x} &= x^* + x_c - |x_c|, \\ \underline{\underline{x}} &= \min\{\underline{x}, T\underline{x}\}, \\ \bar{\bar{x}} &= \max\{\bar{x}, T\bar{x}\}. \end{aligned}$$

It turns out that crucial for the comparison is the fact that

$$M_{ii} \geq 1 \text{ for each } i.$$

## 10 Main result

**Theorem 3.** *Under the common assumption  $\rho(|A_c^{-1}|\Delta) < 1$ , for each  $i$  we have*

$$\begin{aligned} \bar{x}_i - \bar{\bar{x}}_i &\geq \min \left\{ (M_{ii} - 1)(|x_c|_i - (x_c)_i), \frac{2(M_{ii}-1)}{2M_{ii}-1}(x_i^* - |x_c|_i) \right\} \geq 0, \\ \underline{\underline{x}}_i - \underline{x}_i &\geq \min \left\{ (M_{ii} - 1)(|x_c|_i + (x_c)_i), \frac{2(M_{ii}-1)}{2M_{ii}-1}(x_i^* - |x_c|_i) \right\} \geq 0. \end{aligned}$$

In particular,

$$\underline{x} \leq \underline{\underline{x}} \leq \bar{\bar{x}} \leq \bar{x},$$

i.e. the HBR bounds are *never worse* than the Bauer-Skeel bounds.

**Remark.** Nonnegativity follows from the facts that  $M \geq I$  and  $x^* = M(|x_c| + |A_c^{-1}|\delta) \geq |x_c|$ .

## 11 Refinement

**Theorem 4.** *Let the spectral condition hold. Then for each  $i$  such that  $M_{ii} > 1$  and  $(x_c)_i \neq 0$  we have*

$$(\bar{x}_i - \underline{x}_i) - (\bar{\bar{x}}_i - \underline{\underline{x}}_i) \geq \frac{2(M_{ii} - 1)^2}{2M_{ii} - 1} |x_c|_i > 0,$$

hence

$$\bar{\bar{x}}_i - \underline{\underline{x}}_i < \bar{x}_i - \underline{x}_i,$$

i.e., the  $i$ th HBR bound is better than the Bauer-Skeel bound.

**Remark.** Recall that  $M = (I - |A_c^{-1}|\Delta)^{-1} = \sum_{j=0}^{\infty} (|A_c^{-1}|\Delta)^j \geq I$ . Hence  $M_{ii} > 1$  e.g. if  $(|A_c^{-1}|\Delta)_{ii} > 0$ .



## 12 Partial conclusion

We can conclude that the HBR bounds are “almost always” better than the Bauer-Skeel bounds. Still, how good are the HBR bounds themselves?

## 13 Exact bounds

For each  $i$  define

$$\begin{aligned} x_i^e &= \min\{x_i; Ax = b, |A - A_c| \leq \Delta, |b - b_c| \leq \delta\}, \\ x_i^E &= \max\{x_i; Ax = b, |A - A_c| \leq \Delta, |b - b_c| \leq \delta\}. \end{aligned}$$

Obviously,  $x^e$  and  $x^E$  are **exact** componentwise bounds, so that they satisfy

$$\underline{x} \leq x^e \leq x^E \leq \bar{x}$$

( $x^e, x^E$  are NP-hard to compute). Now, what is the amount of overestimation?

## 14 Overestimation of the HBR bounds

**Theorem 5. (2000, not yet published)** *Let the spectral condition hold. Then for each  $i \in \{1, \dots, n\}$  we have*

$$\begin{aligned} \underline{x}_i &\leq x_i^e \leq \underline{x}_i + \underline{d}_i, \\ \bar{x}_i - \bar{d}_i &\leq x_i^E \leq \bar{x}_i, \end{aligned}$$

where

$$\begin{aligned} \underline{d}_i &= (M|(\text{diag}(\underline{z})A_c^{-1}\text{diag}(\underline{z}) - |A_c^{-1}|)(\underline{\xi}_i \Delta M e_i + \Delta x^* + \delta)|)_i, \\ \bar{d}_i &= (M|(\text{diag}(\bar{z})A_c^{-1}\text{diag}(\bar{z}) - |A_c^{-1}|)(\bar{\xi}_i \Delta M e_i + \Delta x^* + \delta)|)_i, \\ \underline{\xi}_i &= (|\underline{x}| + \underline{x} - x_c - |x_c|)_i, \\ \bar{\xi}_i &= (|\bar{x}| - \bar{x} + x_c - |x_c|)_i \end{aligned}$$

and  $\underline{z}, \bar{z}$  are given by

$$\underline{z}_j = \begin{cases} \text{sgn}(x_c)_j & \text{if } j \neq i, \\ -1 & \text{if } j = i, \end{cases}, \quad \bar{z}_j = \begin{cases} \text{sgn}(x_c)_j & \text{if } j \neq i, \\ 1 & \text{if } j = i \end{cases}, \quad (j = 1, \dots, n).$$

## 15 Example (J. Albrecht 1961)

Here  $A_c x = b_c$  reads

$$\begin{aligned} 4.33x_1 - 1.12x_2 - 1.08x_3 + 1.14x_4 &= 3.52 \\ -1.12x_1 + 4.33x_2 + 0.24x_3 - 1.22x_4 &= 1.57 \\ -1.08x_1 + 0.24x_2 + 7.21x_3 - 3.22x_4 &= 0.54 \\ 1.14x_1 - 1.22x_2 - 3.22x_3 + 5.43x_4 &= -1.09 \end{aligned}$$

and

$$\Delta_{ij} = \delta_i = 0.005$$

for each  $i, j$ ,

$$\varrho(|A_c^{-1}|\Delta) = 0.008.$$

## 16 Results

(rounded to four decimal digits)

$$[\underline{x}, \underline{x} + \underline{d}] = \begin{pmatrix} [1.0408, 1.0441] \\ [0.5567, 0.5593] \\ [0.1056, 0.1072] \\ [-0.2352, -0.2299] \end{pmatrix}, \quad [\bar{x} - \bar{d}, \bar{x}] = \begin{pmatrix} [1.0517, 1.0517] \\ [0.5670, 0.5689] \\ [0.1129, 0.1164] \\ [-0.2218, -0.2210] \end{pmatrix}.$$

## 17 Unsatisfactory result (1997)

Theorem 5 applied to the system

$$\begin{pmatrix} \varepsilon^2 & [-\varepsilon, \varepsilon] & [-\varepsilon, \varepsilon] & [-\varepsilon, \varepsilon] \\ 0 & 1.1 & 1 & 1 \\ 0 & 1 & 1.1 & 1 \\ 0 & 1 & 1 & 1.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ [-\varepsilon, \varepsilon] \\ [-\varepsilon, \varepsilon] \\ [-\varepsilon, \varepsilon] \end{pmatrix}$$

works for each  $\varepsilon > 0$  (since  $\varrho(|A_c^{-1}|\Delta) = 0$ ) and yields **independently of  $\varepsilon$**

$$[\bar{x}_1 - \bar{d}_1, \bar{x}_1] = \left[ \frac{30}{31}, \frac{1230}{31} \right] = [0.97, 39.68]$$

whereas  $x_1^E = \frac{830}{31} = 26.77$ , i.e.,  $\frac{\bar{x}_1 - x_1^E}{x_1^E} = \frac{40}{83} = 0.48$  (rounded to two decimal digits).

## 18 Zero overestimation cases

**Theorem 6.** *Let the spectral condition hold. Then we have:*

- (i)  $x^e = \underline{x}$ ,  $x^E = \bar{x}$  if  $A_c$  is a diagonal matrix with positive diagonal entries,
- (ii)  $x^e = \underline{x}$  if  $A_c^{-1} \geq 0$  and  $A_c^{-1}b_c \leq 0$ ,
- (iii)  $x^E = \bar{x}$  if  $A_c^{-1} \geq 0$  and  $A_c^{-1}b_c \geq 0$ .

## 19 Conclusions

- both the Bauer-Skeel bounds and the HBR bounds require solving  $A_c x = b_c$  and computing  $A_c^{-1}$  and  $(I - |A_c^{-1}| \Delta)^{-1}$ ,
- the HBR bounds are never worse, and “almost always” better, than the Bauer-Skeel bounds,
- overestimation of the HBR bounds can be computed at almost no additional cost.

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