On the Computation of Relaxed Pessimistic Solutions to MPECs

Institute of Computer Science
Academy of Sciences of the Czech Republic

On the computation of relaxed pessimistic solutions to MPECs

M. Červinka, C. Matonoha, J. Outrata

Technical report No. 1057

December 2009

Pod Vodárenskou věží 2, 182 07 Prague 8 phone: +420 2 688 42 44, fax: +420 2 858 57 89,
e-mail:e-mail:ics@cs.cas.cz

Institute of Computer Science
Academy of Sciences of the Czech Republic

On the computation of relaxed pessimistic solutions to MPECs [1]

M. Červinka, C. Matonoha, J. Outrata [2]

Technical report No. 1057

December 2009

Abstract:

In this paper, we propose a new numerical method to compute approximate and the so-called relaxed pessimistic solutions to mathematical programs with equilibrium constraints (MPECs) where the solution map arising in the equilibrium constraint is generally not single-valued. This method combines two existing codes, BOBYQA for derivative-free optimization under box constraints, and a method for solving MPECs from the interactive system UFO. We comment on local convergence to approximate pessimistic solutions and report on numerical performance on several test problems.

Keywords:
MPEC, pessimistic solution, value function, local convergence.

---

[1]This work was supported by the Grant Agency of the Czech Republic, project No. 201/09/1957, and the institutional research plan No. AV0Z10300504.

[2]M. Červinka (cervinka@utia.cas.cz), J. Outrata (outrata@utia.cas.cz): Institute of Information Theory and Automation AS CR, Pod Vodárenskou věží 4, 182 08 Prague 8, Czech Republic; C. Matonoha (matonoha@cs.cas.cz): Institute of Computer Science AS CR, Pod Vodárenskou věží 2, 182 07 Prague 8, Czech Republic.

1 Introduction

In the last twenty years researchers have paid a lot of attention to a class of optimization problems where, among the constraints, there is a special one in the form of a variational inequality or a complementarity problem. One speaks about an *equilibrium constraint*, and the overall optimization problem coined the name *MPEC*. As an early version of an MPEC one can consider the Stackelberg game of two players ([17]), and we use the respective terminology very often also in the MPEC setting.

Let us consider an abstract MPEC in the form

$$\begin{aligned} &\underset{x}{\text{minimize}}\ f(x,y) \\ &\text{subject to} \\ &\qquad y \in S(x) \\ &\qquad x \in \omega. \end{aligned} \tag{1}$$

In (1), $x \in \mathbb{R}^n$ is the strategy of the dominant player called *Leader*, who acts first and aims to minimize his continuous objective f by using strategies from a closed set $\omega \subset \mathbb{R}^n$. The so-called *solution map* $S[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$, arising in the equilibrium constraint $(x, y) \in \text{Gph}\, S$, assigns x the set of possible responses of his opponent(s) called *Follower(s)*. So, $y \in \mathbb{R}^m$ stands for the cumulative strategy of all Followers and S describes their behavior. Unfortunately, problem (1) is not well-posed, whenever S is not single-valued on ω. Then, namely, the Leader can hardly optimize his choice of x, not knowing the response of his opponent(s).

To avoid this hurdle, one imposes in some situations an additional hypothesis specifying the response of the Follower(s) at those $x \in \omega$, where $S(x)$ is not a singleton. Usually we assume that he (they) behave(s) with respect to the Leader's objective either in a *cooperative* or in a *non-cooperative* way. In the former case one speaks about the *optimistic* solution concept in which the MPEC (1) is replaced by a hierarchical optimization problem where, on the upper level, one minimizes the value function

$$\mu(x) := \inf_{y \in S(x)} f(x, y)$$

over $x \in \omega$. This allows us (under mild assumptions) to convert (1) to the (well-defined) MPEC

$$\begin{aligned} &\underset{x,y}{\text{minimize}}\ f(x,y) \\ &\text{subject to} \\ &\qquad (x, y) \in \text{Gph} S \\ &\qquad x \in \omega. \end{aligned} \tag{2}$$

In (2) one minimizes f with respect to both variables x and y. A vast majority of the MPEC literature, including the monographs [13], [14] and [7], is devoted to problem (2) and its numerous variants.

To introduce its counterpart, the *pessimistic* solution concept, one usually employs the value function $\vartheta[\mathbb{R}^n \to \bar{\mathbb{R}}]$ defined by

$$\vartheta(x) := \sup_{y \in S(x)} f(x,y).$$

A pair $(\hat{x}, \hat{y}) \in \omega \times \mathbb{R}^m$ is declared a (local) *pessimistic* solution to (1), provided

$$\begin{aligned} &\vartheta(\hat{x}) = f(\hat{x}, \hat{y}) \\ &\vartheta(\hat{x}) \leq \vartheta(x) \text{ for all } x \in \mathcal{O} \cap \omega, \end{aligned} \tag{3}$$

where $\mathcal{O}$ is a neighborhood of $\hat{x}$.

Such a pair exists, however, only under special, rather restrictive assumptions on f and S. Therefore, especially in numerous papers by Loridan and Morgan (see eg. [8], [9], [10]), a lot of attention has been paid to various relaxations of condition (3), leading to more workable solution concepts for the non-cooperative case. Such an effort is very important because this type of behavior of the Follower(s) can frequently be observed in applications.

An incentive for this paper has been provided by a successful algorithm BOBYQA, developed by Powell for derivative-free minimization of (possibly discontinuous) functions ([15]). As mentioned by Dempe in [6], to find a pessimistic solution to (1) we have to minimize either a discontinuous, implicitly given value function which is generally not lower semicontinuous, or its special relaxation constructed via a modification of the equilibrium constraint. In this paper we address the first possibility.

The plan of the paper is as follows. In the next section we provide a preliminary analysis of the problem and address two "relaxed" solution schemes which are suitable and reasonable to consider when a local pessimistic solution to MPEC does not exist. Further, we give a brief description of our proposed numerical method, particularly its two already existing components, BOBYQA and UFO. We also comment on convergence rate of our method. In the final section we summarize our numerical experience on several test MPECs.

2 Problem formulation and numerical method

As we mentioned in the introduction, a (local) pessimistic solution to (1) exists only under restrictive assumptions on problem data. Therefore we consider the following relaxation of the pessimistic solution concept.

Definition 1. *(relaxed pessimistic solution to MPEC)*
The pair $(\hat{x}, \hat{y}) \in \omega \times \mathbb{R}^m$ *is called a* relaxed pessimistic *solution to (1), provided that* $\exists x_i \xrightarrow{\omega} \hat{x},\ y_i \to \hat{y},\ y_i \in S(x_i)$, *such that* $\vartheta(x_i) = f(x_i, y_i)$ *and* $\vartheta(x_i) \to \inf_{x \in \omega} \vartheta(x)$.

Clearly, possible accumulation points $\tilde{y}$ of $\{y_i\}$ do not generally fulfill the relation $\vartheta(\hat{x}) = f(\hat{x}, \hat{y})$ due to the possible lack of continuity of S.

For simplicity, throughout the whole sequel we impose the following assumptions.

Assumption 1. *$\omega := \{x \in \mathbb{R}^n | a_i \leq x_i \leq b_i\}$, where $a_i, b_i \in \bar{\mathbb{R}}, i = 1, \ldots, n$.*

Assumption 2. *S is nonempty and compact-valued over ω.*

Assumption 3. *S is outer semicontinuous over ω, cf. [16, Definition 5.4].*

By [2, Theorem 1.4.16], Assumptions 2 and 3 ensure that $\mu(x)$ is lower semicontinuous (lsc) over ω, ϑ is upper semicontinuous (usc) over ω and that for all $x \in \omega$ one has

$$\mu(x) = \min_{y \in S(x)} f(x, y),$$
$$\vartheta(x) = \max_{y \in S(x)} f(x, y).$$

Let us denote by $\hat{\vartheta}$ the lsc regularization of ϑ, i.e. the largest lsc minorant of ϑ. Then it is clear that $\hat{x}$ is a relaxed pessimistic solution to (1) if and only if it is a local minimum of $\hat{\vartheta}$ over ω. In this way we have lower estimates for all approximations of pessimistic solutions at our disposal. Further, this type of relaxed solution to (1) exists whenever ω is compact.

However, the relaxed pessimistic solutions are generally not feasible for MPEC (1). In that case, the Leader is usually forced to deviate slightly from his optimal behavior and has to be content with an approximate solution.

Definition 2. *((δ, ε)-pessimistic solution to MPEC)*
Let $\hat{x}$ be a relaxed pessimistic solution to (1) and $\delta, \varepsilon > 0$ be given. We say that $(\tilde{x}, \tilde{y})$ is a (δ, ε)-pessimistic *solution to (1), provided*

$$\begin{aligned}\hat{\vartheta}(\tilde{x}) &= \vartheta(\tilde{x}) = f(\tilde{x}, \tilde{y}) \\ \hat{\vartheta}(\tilde{x}) &\leq \vartheta(\hat{x}) + \varepsilon \\ ||\tilde{x} - \hat{x}|| &< \delta.\end{aligned}$$

This notion corresponds to so-called η-solutions by Loridan and Morgan ([8]) when $\delta = +\infty$. We include a parameter δ to this concept because in many cases the choice of δ directly corresponds to the trust-region radius in the numerical method described below.

Our main aim is to suggest a numerical procedure for the computation of relaxed pessimistic and (δ, ε)-pessimistic solutions to (1). To this end we split the pessimistically formulated MPEC into the outer and the inner optimization problems.

For solving the inner optimization problem

$$\begin{aligned}&\underset{y}{\text{maximize}}\ f(x, y) \\ &\text{subject to} \\ &\qquad y \in S(x)\end{aligned} \tag{4}$$

with a fixed x we use a suitable optimization method from the interactive system UFO. As explained in the previous sections, the optimal value function of this problem, $\vartheta(x)$, is a generally discontinuous function. Thus, for the outer optimization problem

$$\begin{array}{ll} \underset{x}{\text{minimize}} \ \vartheta(x) & \\ \text{subject to} & \\ & x \in \omega \end{array} \tag{5}$$

we use the code BOBYQA for derivative-free optimization, developed by Powell. To find pessimistic solutions to MPECs, we have joined these two algorithms into one code. A brief description of both algorithms follows.

UFO [12] is an interactive system for universal functional optimization that serves for solving both dense medium-size and sparse large-scale optimization problems. It can be used for formulation and solution of particular optimization problems, for preparation of specialized optimization routines and for designing and testing new optimization methods. We can generate a large number of modifications of a given method and find the most suitable implementation. The optimization methods can be implemented with various strategies for a step-size selection. It contains line-search methods, general trust-region methods, special trust-region methods for nonlinear least squares, Marquardt-type methods for nonlinear least squares and filter-type methods for nonlinear programming including Fletcher-Leyffer filters, barrier filters and Markov filters. The UFO system also contains many efficient methods for solving related subproblems, for example methods for solving systems of ordinary differential equations.

BOBYQA [15] is an algorithm for seeking a local minimum of a function F of several variables, constrained by lower and upper bounds on each variable. The function values of F can be specified by a "black box" and the information about its derivatives is not available. Hence the algorithm can be used also for discontinuous functions as it is in our case.

BOBYQA is based on finding interpolation points $u_1, \ldots, u_m$ and computing a quadratic approximations Q_k to F that satisfy $Q_k(u_i) = F(u_i),\ i = 1, \ldots, m$. At each iteration, a new point $x_{k+1} = x_k + d_k$ is computed and one of the interpolation points, say u_j, is replaced by x_{k+1}. Thus only one interpolation point is altered on each iteration. Direction vector d_k is chosen by minimizing $Q_k(x_k + d)$ subject to the prescribed bounds on variables under the condition $d \leq \Delta_k$, where Δ_k is the current trust-region radius. At each iteration, as a new point of a minimizing sequence x_k^* we take the point which minimizes F among all current interpolation points.

BOBYQA consists of a very accurate and efficient system of updating the approximation models and it maintains a "good" set of interpolation points. This makes BOBYQA numerically very stable and not sensitive to a reasonable level of computational errors in values of the objective. However, BOBYQA does not make use of the problem structure and the established local convergence rate is closer to linear then to quadratic. For this reason, the algorithm sometimes prefers the early termination, see [5, Section 1.3].

From the above discussion it is clear that our code (consisting of BOBYQA and UFO) is not very sensitive to possible computational errors of UFO and that the convergence rate depends solely on performance of BOBYQA. In cases when no early termination occurs, we are able to compute a sequence of (δ, ε)-pessimistic solutions converging to the relaxed pessimistic solution by choosing the final trust-region radius as δ. Note that ε depends on the local Lipschitz behavior of ϑ around the respective approximate pessimistic solution. These conclusions are supported by our numerical experience gained on the test problems below.

Our final note is about the special situations when the map S happens to be continuous over ω. Then ϑ is continuous over ω as well and the notions of relaxed pessimistic and (δ, ε)-pessimistic solutions become superfluous. Our proposed procedure will then generate pessimistic solutions in the sense of (3).

3 Numerical experiments

We have performed preliminary tests on several examples with small dimension by using the code BOBYQA for the outer problem and the UFO system for the inner problem as stated above. Examples 1 and 2 refer to the example in [7, Section 5.1], the former being the pessimistic and the latter being the optimistic formulation of the same problem. Example 3 is a simple mathematical program with complementarity constraints (MPCC) where the solution map is single-valued and continuous at each point from the interior of the feasible set. Example 4 can be considered as a generalization of Example 1 into more dimensions. By including Examples 2 and 3 in our test problems we intend to show that our proposed method could be used also for computation of optimistic solutions and solutions to (1) whenever S is single-valued, respectively. Now we briefly describe the examples.

Example 1. [7] An MPCC with relaxed pessimistic solution:

$$\min_{x\in[-2,2]} \max_{y\in S(x)} x^2 + y_1^2, \tag{6}$$

where

$$S(x) = \left\{ y \in \mathbb{R}^3_+ \;\middle|\; \begin{array}{l} y_1 \leq 1, \;\; y_2 - y_3 - x = 0, \\ y_1 y_3 = 0, \;\; y_2(y_1 - 1) = 0 \end{array} \right\}.$$

The solution of the inner problem for a fixed x has the form

- if $x < 0$ then $y^* = (0, 0, -x)^\top$ with $\vartheta(x) = x^2$;
- if $x \geq 0$ then $y^* = (1, x, 0)^\top$ with $\vartheta(x) = x^2 + 1$.

We can see that there is no global (nor local) solution of problem (6). However, $\hat{x} = 0$ is a relaxed pessimistic solution.

Example 2. [7] An MPCC with optimistic solution:

$$\min_{x\in[-2,2]} \min_{y\in S(x)} x^2 + y_1^2, \tag{7}$$

where S is the same multifunction as in Example 1. Now, the solution of the inner problem for a fixed x has the form

- if $x \le 0$ then $y^* = (0, 0, -x)^\top$ with $\mu(x) = x^2$;
- if $x > 0$ then $y^* = (1, x, 0)^\top$ with $\mu(x) = x^2 + 1$.

Thus the optimistic solution of (7) is attained at $x^* = 0, y^* = (0, 0, 0)^\top$.

Example 3. An MPCC with unique lower-level solution at each feasible point:

$$\min_{x \in [-2,0] \times [-2,0], y \in S(x)} x_1 + x_2 + 2(y_1 + y_2), \tag{8}$$

where

$$S(x) = \left\{ y \in \mathbb{R}^2_+ \,\middle|\, \begin{array}{l} y_1 - 2y_2 - x_1 \ge 0, \;\; y_1(y_1 - 2y_2 - x_1) = 0 \\ y_2 - 2y_1 - x_2 \ge 0, \;\; y_2(y_2 - 2y_1 - x_2) = 0 \end{array} \right\}.$$

The solution of (8) is attained at $x^* = (0, 0)^\top, y^* = (0, 0)^\top$.

Example 4. A 3-dimensional MPCC with relaxed pessimistic solution:

$$\min_{x \in \omega} \max_{y \in S(x)} \|y\|^2 \tag{9}$$

where

$$S(x) = \{y \in \mathbb{R}^3 | y = \arg\min_{y \in (x_1, x_2, x_1^2 + x_2^2)^\top + \mathbb{C}} (-x_1 y_1 - x_2 y_2 + y_3)\},$$

$\mathbb{C}$ is a unit cube in $\mathbb{R}^3$ and ω is a box in $\mathbb{R}^2$ with suitable lower and upper bounds. Using the first order necessary optimality conditions, problem (9) can be rewritten as

$$\begin{array}{l} \min_{x \in \omega} \max_{y \in \mathbb{R}^3, \sigma \in \mathbb{R}^3_+, \lambda \in \mathbb{R}^3_+} (y_1^2 + y_2^2 + y_3^2) \\ \qquad \text{subject to} \\ \qquad\qquad x_1 \le y_1 \le x_1 + 1, \\ \qquad\qquad x_2 \le y_2 \le x_2 + 1, \\ \qquad\qquad x_1^2 + x_2^2 \le y_3 \le x_1^2 + x_2^2 + 1, \\ \qquad\qquad \sigma_1 - \lambda_1 = x_1, \;\; \sigma_1(x_1 + 1 - y_1) = 0, \;\; \lambda_1(y_1 - x_1) = 0, \\ \qquad\qquad \sigma_2 - \lambda_2 = x_2, \;\; \sigma_2(x_2 + 1 - y_2) = 0, \;\; \lambda_2(y_2 - x_2) = 0, \\ \qquad\qquad \sigma_3 - \lambda_3 = -1, \;\; \sigma_3(x_1^2 + x_2^2 + 1 - y_3) = 0, \;\; \lambda_3(y_3 - x_1^2 - x_2^2) = 0. \end{array} \tag{10}$$

The solution of the inner maximization problem for a fixed x has the following form:

- if $x_1 \ge 0, \;\; x_2 \ge 0$ then $y^* = (x_1, x_2, x_1^2 + x_2^2)^\top + (1, 1, 0)^\top$ and

$$\vartheta(x) = (x_1 + 1)^2 + (x_2 + 1)^2 + (x_1^2 + x_2^2)^2;$$

- if $x_1 \ge 0, \;\; x_2 < 0$ then $y^* = (x_1, x_2, x_1^2 + x_2^2)^\top + (1, 0, 0)^\top$ and

$$\vartheta(x) = (x_1 + 1)^2 + x_2^2 + (x_1^2 + x_2^2)^2;$$

- if $x_1 < 0,\ x_2 \geq 0$ then $y^* = (x_1, x_2, x_1^2 + x_2^2)^\top + (0, 1, 0)^\top$ and

$$\vartheta(x) = x_1^2 + (x_2 + 1)^2 + (x_1^2 + x_2^2)^2;$$

- if $x_1 < 0,\ x_2 < 0$ then $y^* = (x_1, x_2, x_1^2 + x_2^2)^\top$ and

$$\vartheta(x) = x_1^2 + x_2^2 + (x_1^2 + x_2^2)^2.$$

Clearly, $\vartheta(x)$ is a discontinuous function. The relaxed pessimistic solution of (9) is $\hat{x} = (0, 0)^\top$ while (δ, ε)-pessimistic solution is $\tilde{x} = (-\delta, -\delta)^\top$ for an arbitrarily small $\delta > 0$ with $\varepsilon = 2\delta^2(1 + 2\delta^2)$.

The inner problem is an optimization problem with constraints which include complementarity conditions. Such an optimization problem is difficult to solve since none of the standard constraint qualifications are satisfied at any feasible point. To remove complications caused by complementarity constraints, special methods are being developed. The expressions themselves can either be taken as the complementarity pairs, one of the two constraints can be slacked, both of the expressions in the constraints can be slacked, zero complementarity condition can be replaced by a relaxation parameter driven to zero, or the complementarity constraint can be moved to the objective function in the form of an l_1-penalty term with a large enough penalty parameter. All such reformulations of the problem have the same optimal solution but they have different algorithmic performance. Preliminary results based on the interior-point approach using an exact penalty function to remove complementarity constraints can be found in paper [11] and references therein.

In our numerical experiments we have solved the inner problem as a standard nonlinear problem with complementarity equalities replaced by inequalities. Examples 1,2, and 3 are solved by methods for general nonlinear optimization. We construct a sequence of iterations $y^{i+1} = y^i + d^i$ converging to a local solution y^* for a fixed x. Direction vectors d^i are generated by a trust-region approach which can be used when the Hessian matrix of the problem is not positive definite (e.g. in Example 1). The structure of Example 4 enables to use methods for sum-of-squares minimization. For solving the KKT conditions, we have used the Newton method and the resulting system of linear equations is solved by a line-search approach. For more details concerning optimization methods, see [1], [3], [4].

Numerical results in all examples indicate that we have to be careful in choosing parameters of our numerical method. In particular, the choice of values of `RHOBEG` and `RHOEND`, the initial and final values of a trust-region radius for BOBYQA, and of initial points x^0, y^0 for UFO, are critical. In all our computations we have set the number of interpolation points of BOBYQA to $m = 2n + 1$, where n is the dimension of x. Since BOBYQA can be used only for problems with $n \geq 2$, in Examples 1 and 2 we have introduced an artificial variable which, however, does not enter the objectives.

Table 1 shows results for Example 1. We see that it is not easy to choose the right values of `RHOBEG` and `RHOEND` for which the algorithm stops at the desired optimum. A stagnation in function value occurs for certain values of `RHOEND`, moreover, lowering the values of `RHOBEG` can also deteriorate the result. The reason for this behavior might be the

		$x^0 = 1$		$x^0 = -1$	
RHOBEG	RHOEND	$\hat{x}$	$\vartheta(\hat{x})$	$\hat{x}$	$\vartheta(\hat{x})$
1E-01	1E-04	-1.799460E-03	3.238055E-06	-5.071861E-05	2.572376E-09
1E-01	1E-08	-6.957051E-05	4.840058E-09	-6.873385E-07	4.740431E-13
1E-02	1E-04	-3.249525E-02	1.055941E-03	-3.210313E-05	1.030611E-09
1E-02	1E-08	-1.716110E-04	2.945033E-08	-1.448631E-06	2.098531E-12
1E-04	1E-06	2.046750E-06	4.189306E-12	-7.062689E-07	4.988192E-13
1E-05	1E-06	-2.083862E-07	2.473150E-13	-7.993180E-06	6.389112E-11
1E-06	1E-08	-3.030543E-02	9.184189E-04	-1.605396E-02	2.577294E-04
1E-06	1E-14	-3.030279E-02	9.182592E-04	-1.605395E-02	2.577294E-04
1E-07	1E-08	3.905667E-07	9.999998E-01	-1.418746E-06	2.012842E-12
1E-07	1E-14	3.907758E-07	9.999998E-01	-1.418262E-06	2.011470E-12
1E-08	1E-10	-1.782062E-06	3.175747E-12	-6.808809E-07	4.636840E-13
1E-12	1E-14	-1.394823E-06	1.945534E-12	-1.716068E-04	2.944889E-08
1E-15	1E-16	-9.141234E-03	8.356216E-05	-5.758049E-03	3.315513E-05

Table 1: Results for Example 1.

slow linear convergence of the algorithm resulting in the early termination. As we expected, the choices of initial point x^0 influence significantly the computation of a solution.

Another situation arises in obtained results when solving Example 2. As this problem has a pessimistic solution, a suitably chosen initial x^0, e.g. $x^0 = -1$, results in computation of the correct solution for arbitrarily chosen values of `RHOBEG` and `RHOEND`. On the other hand, starting at $x^0 = 1$ gives considerably worse results, see Table 2.

A typical behavior of function $\vartheta(x)$ during the iteration process is shown on following figures. Figure 1 corresponds to Example 1 with $x^0 = 1$, `RHOBEG` $= 10^{-5}$, `RHOEND` $= 10^{-9}$ and $x_0 = -1$, `RHOBEG` $= 10^{-1}$, `RHOEND` $= 10^{-7}$, respectively. Figure 2 corresponds to Example 2 with $x^0 = 1$ and $x^0 = -1$, both with `RHOBEG` $= 10^{-5}$, `RHOEND` $= 10^{-6}$. To better reflect the convergence rate we have used function values in logarithmic scale.

		$x^0 = 1$		$x^0 = -1$	
RHOBEG	RHOEND	x^*	$\vartheta(x^*)$	x^*	$\vartheta(x^*)$
1E-01	1E-04	-4.464945E-04	1.993642E-07	2.095502E-11	7.101800E-14
1E-01	1E-08	1.049396E-08	1.755819E-14	5.344864E-09	6.921173E-14
1E-02	1E-04	3.619327E-14	7.103005E-14	0.000000E+00	7.103006E-14
1E-02	1E-08	9.926963E-09	6.881548E-14	9.630018E-09	6.882312E-14
1E-07	1E-08	1.551564E-01	9.603774E-01	9.425458E-09	6.882954E-14
1E-07	1E-14	1.551564E-01	9.603774E-01	1.034798E-08	1.753320E-14
1E-08	1E-10	-2.237152E-08	7.618187E-14	3.243859E-07	1.226983E-13
1E-10	1E-12	1.000004E-06	1.000508E-12	3.242208E-07	1.226003E-13
1E-15	1E-16	1.034802E-08	1.753320E-14	1.034798E-08	1.753320E-14

Table 2: Results for Example 2.

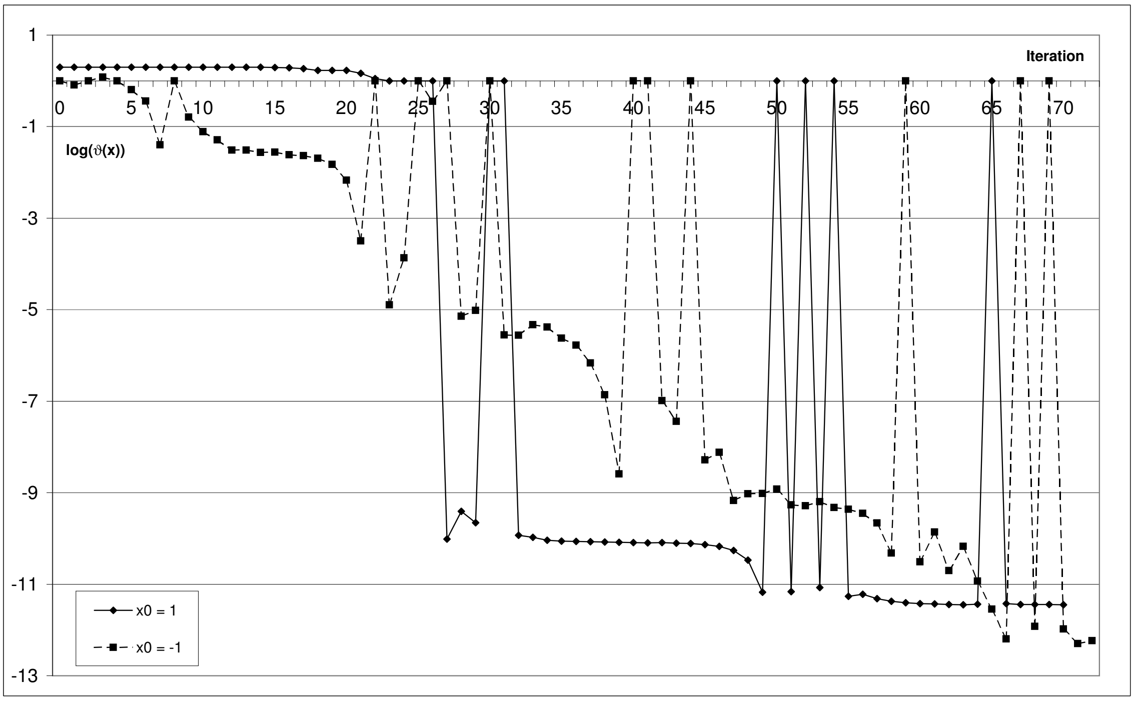

Figure 1: Function $\log(\vartheta(x))$ for Example 1.

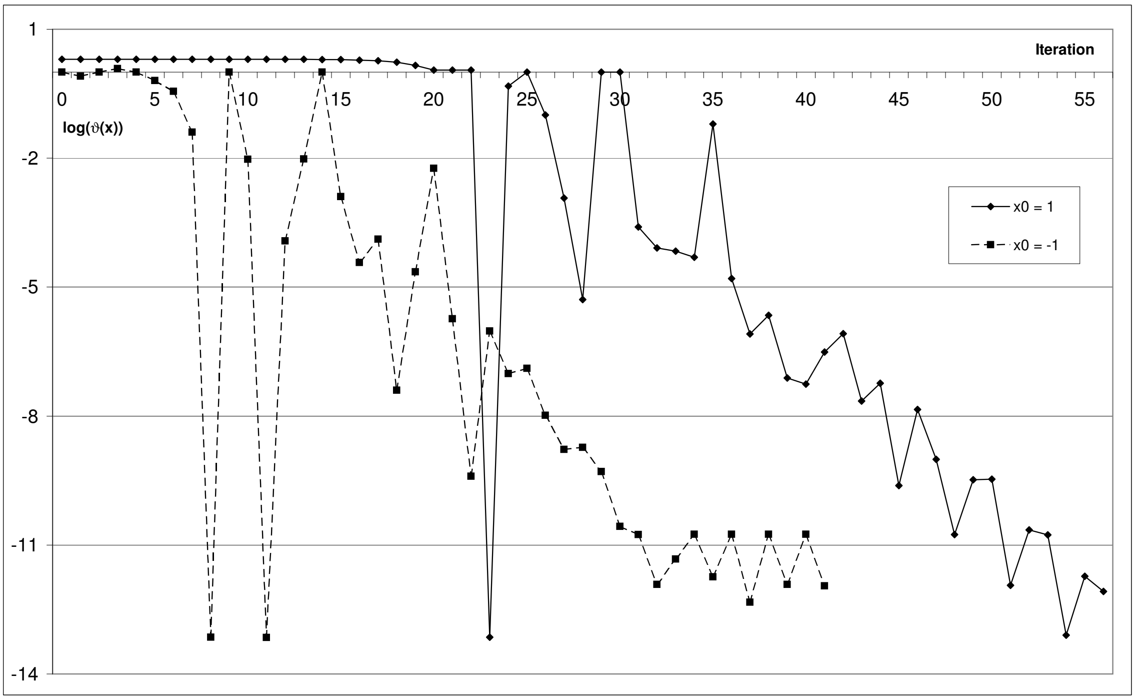

Figure 2: Function $\log(\vartheta(x))$ for Example 2.

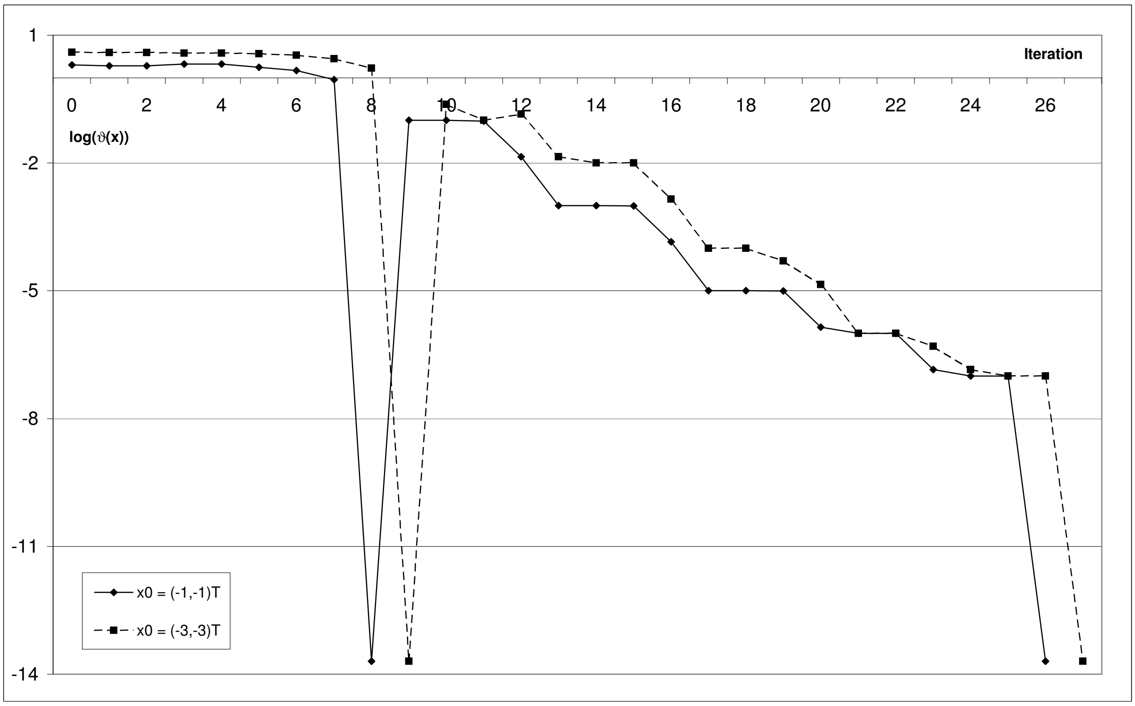

Figure 3: Function $\log(\vartheta(x))$ for Example 3.

Example 3 is rather simple and could be easily solved by conventional MPEC solvers. The results are very good for arbitrarily chosen values of RHOBEG and RHOEND for a feasible initial point $x^0 = (-1,-1)^\top$ as well as for an infeasible initial point $x^0 = (-3,-3)^\top$. In each computation we have obtained the true solution $x^* = (0,0)^\top$ with function value of order 10^{-14}. The behavior of a function $\vartheta(x)$ in logarithmic scale in each iteration of BOBYQA can be seen on Figure 3.

To solve a problem in Example 4, we have tested various strategies, see Table 3. We used small bounds as well as larger bounds for variables (parameters a, b), high as well as low desired precision for the solution (parameter RHOEND), and different starting points x^0, y^0.

The precision of the computed solution strongly depends on the choice of parameter RHOEND. As was mentioned above, the inner optimization problem is hard to solve, since no standard constraint qualification is satisfied at any feasible point. The last column of Table 3 (IF) shows the total number of inner failures caused either by generating direction vectors that were not descent or by exceeding the total number of iterations.

For this reason, we were unable to solve the inner problem for initial points y^0 stated in Table 3. Nevertheless, for larger values of RHOEND (e.g. 10^{-6}) we managed to successfully resolve this problem after choosing another starting point y^0 for the same fixed outer iteration x. On the contrary, when we desired higher precision (e.g. $\texttt{RHOEND} = 10^{-10}$) and the outer iteration x was very small (near the solution), we did not succeed in solving the

RHOBEG	RHOEND	$a,\ b$	x^0	y^0	$\tilde{x}_1$	$\tilde{x}_2$	$\vartheta(\tilde{x})$	NFV	IF
1E-01	1E-06	-2,2	$(1,1)^\top$	$(1,1,1)^\top$	-2,82E-06	-9,42E-07	8,82E-12	193	17
			$(1,1)^\top$	$-(1,1,1)^\top$	-2,81E-06	-9,41E-07	8,81E-12	193	46
			$-(1,1)^\top$	$(1,1,1)^\top$	-1,78E-05	-1,76E-05	6,28E-10	117	10
			$-(1,1)^\top$	$-(1,1,1)^\top$	-1,78E-05	-1,76E-05	6,28E-10	117	23
		-10,10	$(10,10)^\top$	$(1,1,1)^\top$	-2,37E-06	-2,32E-05	5,43E-10	202	73
			$(10,10)^\top$	$-(1,1,1)^\top$	-2,37E-06	-2,32E-05	5,43E-10	202	79
			$-(10,10)^\top$	$(1,1,1)^\top$	-1,80E-05	-9,27E-06	4,10E-10	205	21
			$-(10,10)^\top$	$-(1,1,1)^\top$	-4,34E-05	-2,66E-05	2,59E-09	234	46
1E-01	1E-10	-2,2	$(1,1)^\top$	$(1,1,1)^\top$	-1,05E-08	-2,26E-07	5,14E-14	277	18
			$(1,1)^\top$	$-(1,1,1)^\top$	-1,03E-09	-5,84E-09	3,50E-17	275	23
			$-(1,1)^\top$	$(1,1,1)^\top$	-1,03E-08	-2,21E-09	1,11E-16	188	11
			$-(1,1)^\top$	$-(1,1,1)^\top$	-8,77E-01	-9,12E-01	3,89E+00	81	41
		-10,10	$(10,10)^\top$	$(1,1,1)^\top$	-2,27E-08	-1,64E-09	5,17E-16	216	11
			$(10,10)^\top$	$-(1,1,1)^\top$	-6,69E-01	7,19E-01	4,33E+00	265	28
			$-(10,10)^\top$	$(1,1,1)^\top$	-2,10E-08	-1,69E-08	7,30E-16	270	15
			$-(10,10)^\top$	$-(1,1,1)^\top$	-1,29E-01	-1,32E+00	1,89E+00	81	24

Table 3: Results for Example 4.

inner problem for any initial point y^0. This is the reason why BOBYQA finally returned the values of $\vartheta(\tilde{x})$ far from the true solution.

Generally, when the number of failures was large in comparison with the total number of outer iterations (NFV), no solution was obtained. On the other hand, if the total number of failures was small, the fact that the inner problem was not solved successfully in some iterations did not influence the computation of a correct (δ, ε)-pessimistic solution of the problem.

Figure 4 depicts function $\vartheta(x)$ in logarithmic scale for different values of RHOEND, various starting points and bounds $a_1 = a_2 = a,\ b_1 = b_2 = b$:

Line 1: $\texttt{RHOBEG} = 10^{-1},\ \texttt{RHOEND} = 10^{-6},\ a, b = -2, 2,\ x^0 = (1,1)^\top,\ y^0 = (1,1,1)^\top$;
Line 2: $\texttt{RHOBEG} = 10^{-1},\ \texttt{RHOEND} = 10^{-6},\ a, b = -10, 10,\ x^0 = (10,10)^\top,\ y^0 = (1,1,1)^\top$;
Line 3: $\texttt{RHOBEG} = 10^{-1},\ \texttt{RHOEND} = 10^{-10},\ a, b = -2, 2,\ x^0 = -(1,1)^\top,\ y^0 = (1,1,1)^\top$;
Line 4: $\texttt{RHOBEG} = 10^{-1},\ \texttt{RHOEND} = 10^{-10},\ a, b = -10, 10,\ x^0 = (10,10)^\top,\ y^0 = (1,1,1)^\top$.

4 Conclusion

Our aim was to develop a new numerical procedure by merging two existing codes that would compute a pessimistic solution of MPEC with non-unique lower-level solution map, or a relaxation of a pessimistic solution in cases when the pessimistic solution does not exist. Numerical experiments on small-dimensional programs with complementarity constraints are very promising despite frequent early termination of the algorithm.

In our future research, we will test the proposed method on medium-dimensional programs with complementarity constraints and, using different solvers for the inner problem, also on test problems from other classes of MPECs. Also, by using test problems from a

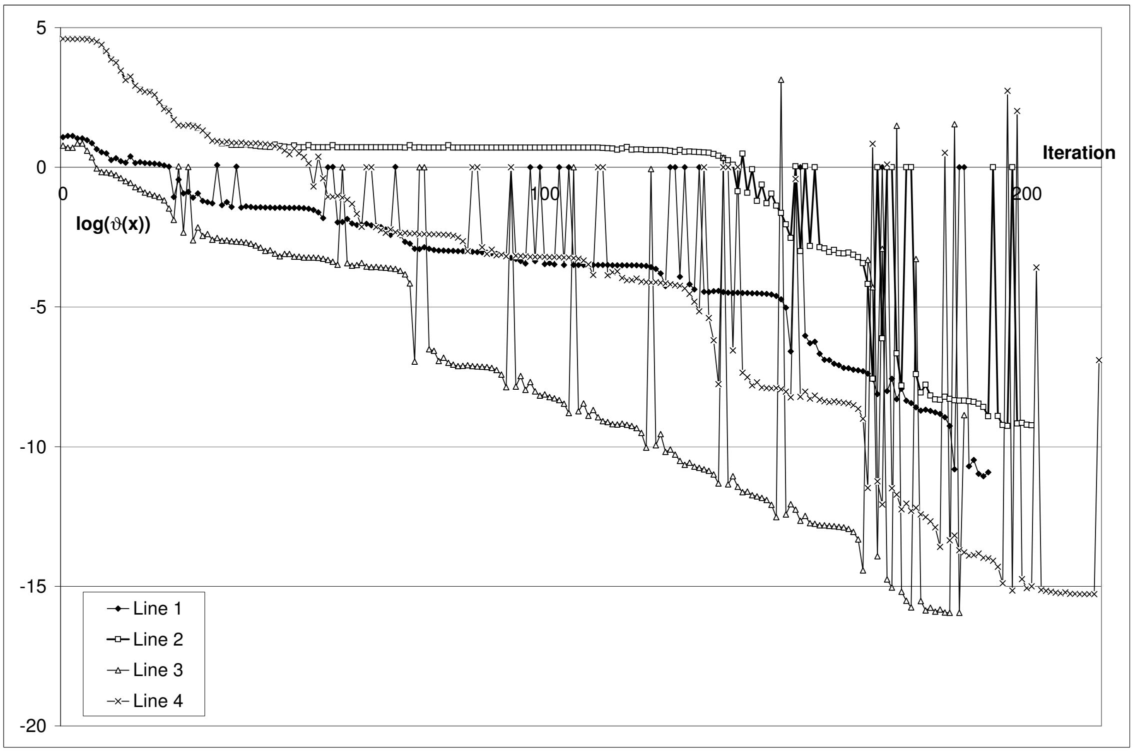

Figure 4: Function $\log(\vartheta(x))$ for Example 4.

class of programs with continuous value function ϑ, we plan to compare the performance of our method composed of BOBYQA and UFO with a method composed of standard bundle method and UFO, the latter combination leveraging from the available first order information of the minimized value function.

Acknowledgment

We are grateful to prof. Powell for his kind help in providing the Fortran version of BOBYQA and we thank prof. Kočvara for drawing our attention towards BOBYQA.

References

[1] A. Antoniou and W.S. Lu: *Practical Optimization*. Springer, 2007.

[2] J.-P. Aubin and H. Frankowska: *Set-valued Analysis*, Birkhäuser, Boston, 1990.

[3] M.S. Bazaraa, H.D. Sherali and C.M. Shetty: *Nonlinear Programming, Theory and Algorithms*. Wiley, 2006.

[4] A.R. Conn, N.I.M. Gould and P.L. Toint: *Trust-region methods*. SIAM, 2000.

[5] A.R. Conn, K. Scheinberg and L.N. Vicente: *Introduction to Derivative-Free Optimization*. SIAM, 2009.

[6] S. Dempe: A bundle algorithm applied to bilevel programming problems with non-unique lower level solutions, *Computational Optimization and Applications* 15, 145–166, 2000.

[7] S. Dempe: *Foundations of bi-level programming*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[8] P. Loridan and J. Morgan: New results on Approximate Solutions in Two-Level Optimization, *Optimization* 20 (6), 819–836, 1989.

[9] P. Loridan and J. Morgan: ε-regularized two-level optimization problems: approximation and existence results, *Optimization - Fifth French-German Conference Castel Novel 1988, Lecture Notes in Mathematics 1405*, Springer Verlag, 99–113, 1989.

[10] P. Loridan and J. Morgan: Weak Via Strong Stackelberg Problem: New Results, *J. of Global Optimization* 8, 263–287, 1996.

[11] L. Lukšan, C. Matonoha and J. Vlček: Interior-Point Method for Nonlinear Programming with Complementarity Constraints. Technical Report V-1039, ICS AS CR, Prague, 2008.

[12] L. Lukšan, M. Tůma, J. Vlček, N. Ramešová, M. Šiška, J. Hartman and C. Matonoha: UFO 2008 - Interactive system for universal functional optimization. Technical Report V-1040, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague 2008.

[13] Z.-Q. Luo, J.-S. Pang and D. Ralph: *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge, 1996.

[14] J. V. Outrata, M. Kočvara and J. Zowe: *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints*, Kluwer Academic Publisher, Dordrecht, The Netherlands, 1998.

[15] M.J.D. Powell: The BOBYQA algorithm for bound constrained optimization without derivatives. Technical report DAMTP 2009/NA06, Cambridge, 2009.

[16] R.T. Rockafellar and R.J.B. Wets: *Variational Analysis*, Springer, 1998.

[17] H. von Stackelberg: *Marktform und Gleichgewicht*, Springer, Berlin, 1934.