



národní
úložiště
šedé
literatury

Sparse Test Problems for Unconstrained Optimization

Lukšan, Ladislav
2010

Dostupný z <http://www.nusl.cz/ntk/nusl-41177>

Dílo je chráněno podle autorského zákona č. 121/2000 Sb.

Tento dokument byl stažen z Národního úložiště šedé literatury (NUŠL).

Datum stažení: 27.09.2021

Další dokumenty můžete najít prostřednictvím vyhledávacího rozhraní [nusl.cz](http://www.nusl.cz) .



Institute of Computer Science
Academy of Sciences of the Czech Republic

Sparse Test Problems for Unconstrained Optimization

Ladislav Lukšan, Ctirad Matonoha, Jan Vlček

Technical report No. 1064

January 2010



Sparse Test Problems for Unconstrained Optimization¹

Ladislav Lukšan, Ctirad Matonoha, Jan Vlček

Technical report No. 1064

January 2010

Abstract:

This report contains a description of subroutines which can be used for testing unconstrained optimization codes. These subroutines can easily be obtained either by using the anonymous ftp address <ftp://ftp.cs.cas.cz/pub/msdos/opt> (file TEST25.FOR) or from the web homepage <http://www.cs.cas.cz/luksan/test.html>. Furthermore, all test problems contained in these subroutines are presented in the analytic form.

Keywords:

unconstrained optimization, test problems

¹This work was supported by the Grant Agency of the Czech Republic, project No. 201/09/1957, and the institutional research plan No. AVOZ10300504

1 Introduction

This report describes subroutines TIUD25, TIUS25, TFFU25, TFGU25, TFBU25, which contain 82 sparse problems for testing unconstrained optimization codes. Some of these problems were chosen from [19], but additional problems with sparse Hessian matrices are included. All subroutines are written in the standard Fortran 77 language. Their names are derived from the following rule:

- The first letter is T - test subroutines.
- The second letter is either I - initiation, or F - objective function.
- The third letter is either U - initiation of unconstrained problem, or F - computation of the function value, or G - computation of the gradient vector, or B - computation of both the function value and the gradient vector simultaneously.
- The fourth letter is either U - universal subroutine, or D - subroutine for problems with unknown sparsity patterns, or S - subroutine for problems with given sparsity patterns.

The last two digits determine a given collection (the numbering corresponds to the UFO system [20], which contains similar collections).

Initiation subroutines use the following parameters (array dimensions are given in parentheses):

N	input	number of variables,
M	output	number of elements of the sparse Hessian matrix,
X(N)	output	vector of variables,
IH(N+1)	output	pointers of diagonal elements of the sparse Hessian matrix,
JH(M)	output	column indices of nonzero elements of the sparse Hessian matrix,
FMIN	output	lower bound of the objective function value,
XMAX	output	maximum stepsize,
NEXT	input	number of the problem selected,
IERR	output	error indicator (0 - correct data, 1 - N is too small).

Although N is an input parameter, it can be changed by the initiation subroutine when its value does not satisfy the required conditions. For example, most of the problems require N to be even or a multiple of a positive integer.

Evaluation subroutines use the following parameters (array dimensions are given in parentheses):

N	input	number of variables,
X(N)	input	vector of variables,
F	output	value of the objective function,
G(N)	output	gradient of the objective function,
NEXT	input	number of the problem selected.

2 Test problems for general unconstrained optimization

Calling statements have the form

```
CALL TIUD25(N,X,FMIN,XMAX,NEXT,IERR)
CALL TIUS25(N,M,X,IH,JH,FMIN,XMAX,NEXT,IERR)
CALL TFFU25(N,X,F,NEXT)
CALL TFGU25(N,X,G,NEXT)
CALL TFBU25(N,X,F,G,NEXT)
```

with the following significance:

- TIUD25 - initiation of vector of variables X , which has dimension N .
- TIUS25 - initiation of vector of variables X and the pattern of the sparse Hessian matrix IH , JH .
- TFFU25 - evaluation of the general objective function value F at the point X .
- TFGU25 - evaluation of the general objective function gradient G at the point X .
- TFBU25 - evaluation of the general objective function value F and gradient G at the point X .

We seek a minimum of a general objective function $F(x)$ from the starting point \bar{x} . For positive integers k and l , we use the notation $\text{div}(k, l)$ for integer division, i.e., maximum integer not greater than k/l , and $\text{mod}(k, l)$ for the remainder after integer division, i.e., $\text{mod}(k, l) = l(k/l - \text{div}(k, l))$. The description of individual problems follows.

Problem 1. Chained Rosenbrock function [7].

$$F(x) = \sum_{i=2}^n [100(x_{i-1}^2 - x_i)^2 + (x_{i-1} - 1)^2],$$

$$\bar{x}_i = -1.2, \text{ mod}(i, 2) = 1, \quad \bar{x}_i = 1.0, \text{ mod}(i, 2) = 0.$$

Problem 2. Chained Wood function [7].

$$F(x) = \sum_{j=1}^k [100(x_{i-1}^2 - x_i)^2 + (x_{i-1} - 1)^2 + 90(x_{i+1}^2 - x_{i+2})^2$$

$$+ (x_{i+1} - 1)^2 + 10(x_i + x_{i+2} - 2)^2 + (x_i - x_{i+2})^2/10],$$

$$i = 2j, \quad k = (n - 2)/2,$$

$$\bar{x}_i = -3, \text{ mod}(i, 2) = 1, \quad i \leq 4, \quad \bar{x}_i = -2, \text{ mod}(i, 2) = 1, \quad i > 4,$$

$$\bar{x}_i = -1, \text{ mod}(i, 2) = 0, \quad i \leq 4, \quad \bar{x}_i = 0, \text{ mod}(i, 2) = 0, \quad i > 4.$$

Problem 3. Chained Powell singular function [7].

$$F(x) = \sum_{j=1}^k [(x_{i-1} + 10x_i)^2 + 5(x_{i+1} - x_{i+2})^2 + (x_i - 2x_{i+1})^4 + 10(x_{i-1} - x_{i+2})^4],$$

$$\begin{aligned}
i &= 2j, \quad k = (n-2)/2, \\
\bar{x}_i &= 3, \quad \text{mod}(i, 4) = 1, \quad \bar{x}_i = -1, \quad \text{mod}(i, 4) = 2, \\
\bar{x}_i &= 0, \quad \text{mod}(i, 4) = 3, \quad \bar{x}_i = 1, \quad \text{mod}(i, 4) = 0.
\end{aligned}$$

Problem 4. Chained Cragg and Levy function [7].

$$\begin{aligned}
F(x) &= \sum_{j=1}^k \left[(\exp(x_{i-1}) - x_i)^4 + 100(x_i - x_{i+1})^6 + \tan^4(x_{i+1} - x_{i+2}) + x_{i-1}^8 + (x_{i+2} - 1)^2 \right], \\
i &= 2j, \quad k = (n-2)/2, \\
\bar{x}_i &= 1, \quad i = 1, \quad \bar{x}_i = 2, \quad i > 1.
\end{aligned}$$

Problem 5. Generalized Broyden tridiagonal function [22].

$$\begin{aligned}
F(x) &= \sum_{i=1}^n |(3 - 2x_i)x_i - x_{i-1} - x_{i+1} + 1|^p, \\
p &= 7/3, \quad x_0 = x_{n+1} = 0, \\
\bar{x}_i &= -1, \quad i \geq 1.
\end{aligned}$$

Problem 6. Generalized Broyden banded function [22].

$$\begin{aligned}
F(x) &= \sum_{i=1}^n \left| (2 + 5x_i^2)x_i + 1 + \sum_{j \in J_i} x_j(1 + x_j) \right|^p, \\
p &= 7/3, \quad J_i = \{j : j \neq i, \max(1, i-5) \leq j \leq \min(n, i+1)\}, \\
\bar{x}_i &= -1, \quad i \geq 1.
\end{aligned}$$

Problem 7. Seven-diagonal generalization of the Broyden tridiagonal function [7].

$$\begin{aligned}
F(x) &= \sum_{i=1}^n |(3 - 2x_i)x_i - x_{i-1} - x_{i+1} + 1|^p + \sum_{i=1}^{n/2} |x_i + x_{i+n/2}|^p, \\
p &= 7/3, \quad x_0 = x_{n+1} = 0, \\
\bar{x}_i &= -1, \quad i \geq 1.
\end{aligned}$$

Problem 8. Sparse modification of the Nazareth trigonometric function.

$$\begin{aligned}
F(x) &= \frac{1}{n} \sum_{i=1}^n \left(n + i - \sum_{j \in J_i} (a_{ij} \sin x_j + b_{ij} \cos x_j) \right)^2, \\
a_{ij} &= 5[1 + \text{mod}(i, 5) + \text{mod}(j, 5)], \quad b_{ij} = (i + j)/10, \\
J_i &= \{j : \max(1, i-2) \leq j \leq \min(n, i+2)\} \cup \{j : |j - i| = n/2\}, \\
\bar{x}_i &= 1/n, \quad i \geq 1.
\end{aligned}$$

Problem 9. Another trigonometric function.

$$\begin{aligned}
F(x) &= \frac{1}{n} \sum_{i=1}^n \left(i(1 - \cos x_i) + \sum_{j \in J_i} (a_{ij} \sin x_j + b_{ij} \cos x_j) \right), \\
a_{ij} &= 5[1 + \text{mod}(i, 5) + \text{mod}(j, 5)], \quad b_{ij} = (i + j)/10, \\
J_i &= \{j : \max(1, i-2) \leq j \leq \min(n, i+2)\} \cup \{j : |j - i| = n/2\}, \\
\bar{x}_i &= 1/n, \quad i \geq 1.
\end{aligned}$$

Problem 10. Toint trigonometric function [26].

$$\begin{aligned}
F(x) &= \frac{1}{n} \sum_{i=1}^n \sum_{j \in J_i} a_{ij} \sin(b_{ij} + c_i x_i + c_j x_j), \\
a_{ij} &= 5[1 + \text{mod}(i, 5) + \text{mod}(j, 5)], \quad b_{ij} = (i + j)/10, \\
c_i &= 1 + i/10, \quad c_j = 1 + j/10, \\
J_i &= \{j : \max(1, i - 2) \leq j \leq \min(n, i + 2)\} \cup \{j : |j - i| = n/2\}, \\
\bar{x}_i &= 1, \quad i \geq 1.
\end{aligned}$$

Problem 11. Augmented Lagrangian function [7].

$$\begin{aligned}
F(x) &= \sum_{i \in J} \left\{ \exp \left(\prod_{j=1}^5 x_{i+1-j} \right) + 10 \left[\left(\sum_{j=1}^5 x_{i+1-j}^2 - 10 - \lambda_1 \right)^2 \right. \right. \\
&\quad \left. \left. + (x_{i-3} x_{i-2} - 5x_{i-1} x_i - \lambda_2)^2 + (x_{i-4}^3 + x_{i-3}^3 + 1 - \lambda_3)^2 \right] \right\}, \\
\lambda_1 &= -0.002008, \quad \lambda_2 = -0.001900, \quad \lambda_3 = -0.000261, \\
J &= \{i, \text{mod}(i, 5) = 0\}, \\
\bar{x}_i &= -2, \quad \text{mod}(i, 5) = 1, \quad i \leq 2, \quad \bar{x}_i = -1, \quad \text{mod}(i, 5) = 1, \quad i > 2, \\
\bar{x}_i &= 2, \quad \text{mod}(i, 5) = 2, \quad i \leq 2, \quad \bar{x}_i = -1, \quad \text{mod}(i, 5) = 2, \quad i > 2, \\
\bar{x}_i &= 2, \quad \text{mod}(i, 5) = 3, \quad \bar{x}_i = -1, \quad \text{mod}(i, 5) = 4, \\
\bar{x}_i &= -1, \quad \text{mod}(i, 5) = 0.
\end{aligned}$$

Problem 12. Generalization of the Brown function 1 [7].

$$\begin{aligned}
F(x) &= \sum_{i=2}^n \left[(x_{i-1} - 3)^2 + (x_{i-1} - x_i)^2 + \exp(20(x_{i-1} - x_i)) \right] \\
\bar{x}_i &= 0, \quad \text{mod}(i, 2) = 1, \quad \bar{x}_i = -1, \quad \text{mod}(i, 2) = 0.
\end{aligned}$$

Problem 13. Generalization of the Brown function 2 [7].

$$\begin{aligned}
F(x) &= \sum_{i=2}^n \left[(x_{i-1}^2)^{(x_i^2+1)} + (x_i^2)^{(x_{i-1}^2+1)} \right], \\
\bar{x}_i &= -1, \quad \text{mod}(i, 2) = 1, \quad \bar{x}_i = 1, \quad \text{mod}(i, 2) = 0.
\end{aligned}$$

Problem 14. Discrete boundary value problem [22].

$$\begin{aligned}
F(x) &= \sum_{i=1}^n \left[2x_i - x_{i-1} - x_{i+1} + h^2(x_i + ih + 1)^3/2 \right]^2, \\
h &= 1/(n + 1), \quad x_0 = x_{n+1} = 0, \\
\bar{x}_i &= ih(1 - ih), \quad i \geq 1.
\end{aligned}$$

Problem 15. Discretization of a variational problem [26].

$$\begin{aligned}
F(x) &= 2 \sum_{i=1}^n \left[(x_i(x_i - x_{i+1}))/h + 2h \sum_{i=0}^n [(\exp(x_{i+1}) - \exp(x_i))/(x_{i+1} - x_i)] \right], \\
h &= 1/(n + 1), \quad x_0 = x_{n+1} = 0, \\
\bar{x}_i &= ih(1 - ih), \quad i \geq 1.
\end{aligned}$$

Problem 16. Banded trigonometric problem.

$$\begin{aligned} F(x) &= \sum_{i=1}^n i [(1 - \cos x_i) + \sin x_{i-1} - \sin x_{i+1}], \\ x_0 &= x_{n+1} = 0, \\ \bar{x}_i &= 1, \quad i \geq 1. \end{aligned}$$

Problem 17. Variational problem 1 [9].

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x) = \int_0^1 \left[\frac{1}{2} \dot{x}^2(t) + \exp(x(t)) - 1 \right] dt,$$

where $x(0) = 0$ and $x(1) = 0$. We use the trapezoidal rule together with 3-point finite differences on a uniform grid having $n + 1$ internal nodes. The starting point is given by the formula $\bar{x}_i = x(t_i) = ih(1 - ih)$, where $h = 1/(n + 1)$.

Problem 18. Variational problem 2 [9].

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x) = \int_0^1 [\dot{x}^2(t) - x^2(t) - 2tx(t)] dt,$$

where $x(0) = 0$ and $x(1) = 0$. We use the trapezoidal rule together with 3-point finite differences on a uniform grid having $n + 1$ internal nodes. The starting point is given by the formula $\bar{x}_i = x(t_i) = ih(1 - ih)$, where $h = 1/(n + 1)$.

Problem 19. Variational problem 3 [9].

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x) = \int_0^1 [\dot{x}^2(t) + x^2(t) + 2x(t) \exp(2t)] dt,$$

where $x(0) = 1/3$ and $x(1) = \exp(2/3)$. We use the trapezoidal rule together with 3-point finite differences on a uniform grid having $n + 1$ internal nodes. The starting point is given by the formula $\bar{x}_i = x(t_i) = (ih \exp(2) + 1)/3$, where $h = 1/(n + 1)$.

Problem 20. Variational problem 4 [9] (Calvar 3 [10]).

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x) = \int_0^1 [\exp(-2x^2(t))(\dot{x}^2(t) - 1)] dt,$$

where $x(0) = 1$ and $x(1) = 0$. We use the trapezoidal rule together with 3-point finite differences on a uniform grid having $n + 1$ internal nodes. The starting point is given by the formula $\bar{x}_i = x(t_i) = 1 - ih$, where $h = 1/(n + 1)$.

Problem 21. Variational problem 5 [9] (Calvar 1 [10]).

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x) = \int_0^1 \left[x^2(t) + \dot{x}(t) \arctan \dot{x}(t) - \log \sqrt{1 + \dot{x}^2(t)} \right] dt,$$

where $x(0) = 1$ and $x(1) = 2$. We use the trapeziodal rule together with 3-point finite differences on a uniform grid having $n + 1$ internal nodes. The starting point is given by the formula $\bar{x}_i = x(t_i) = ih + 1$, where $h = 1/(n + 1)$.

Problem 22. Variational problem Calvar 2 [10].

This problem is a finite difference analogue of a variational problem defined as a minimization of the functional

$$F(x) = \int_0^1 \left[100(x(t) - \dot{x}^2(t))^2 + (1 - \dot{x}(t))^2 \right] dt,$$

where $x(0) = 0$ and $x(1) = 0$. We use the trapeziodal rule together with 3-point finite differences on a uniform grid having $n + 1$ internal nodes. The starting point is given by the formula $\bar{x}_i = x(t_i) = ih(1 - ih)$, where $h = 1/(n + 1)$.

Problem 23. Extended Rosenbrock function [22].

$$\begin{aligned} F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\ f_k(x) &= 10(x_k^2 - x_{k+1}) \quad , \quad \text{mod}(k, 2) = 1 \\ f_k(x) &= x_{k-1} - 1 \quad , \quad \text{mod}(k, 2) = 0 \\ \bar{x}_l &= -1.2, \quad \text{mod}(l, 2) = 1, \quad \bar{x}_l = 1.0, \quad \text{mod}(l, 2) = 0 \end{aligned}$$

Problem 24. Extended Powell singular function [22].

$$\begin{aligned} F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\ f_k(x) &= x_k + 10x_{k+1} \quad , \quad \text{mod}(k, 4) = 1, \\ f_k(x) &= \sqrt{5} (x_{k+2} - x_{k+3}) \quad , \quad \text{mod}(k, 4) = 2, \\ f_k(x) &= (x_{k+1} - 2x_{k+2})^2 \quad , \quad \text{mod}(k, 4) = 3, \\ f_k(x) &= \sqrt{10} (x_k - x_{k+3})^2 \quad , \quad \text{mod}(k, 4) = 0, \\ \bar{x}_l &= 3, \quad \text{mod}(l, 4) = 1, \quad \bar{x}_l = -1, \quad \text{mod}(l, 4) = 2, \\ \bar{x}_l &= 0, \quad \text{mod}(l, 4) = 3, \quad \bar{x}_l = 1, \quad \text{mod}(l, 4) = 0. \end{aligned}$$

Problem 25. Broyden tridiagonal function [22].

$$\begin{aligned} F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\ f_k(x) &= (3 - 2x_k)x_k - x_{k-1} - 2x_{k+1} + 1, \\ m &= n, \quad x_0 = x_{n+1} = 0, \\ \bar{x}_l &= -1, \quad l \geq 1. \end{aligned}$$

Problem 26. Problem 201 in [25].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= x_k - 1, \quad k = 1, \\
f_k(x) &= 10(k-1)(x_k - x_{k-1})^2, \quad 1 < k \leq n, \\
\bar{x}_l &= -1.2, \quad 1 \leq l < n, \quad x_l = -1, \quad l = n.
\end{aligned}$$

Problem 27. Generalized Broyden tridiagonal function [16].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= (3 - 2x_k) x_k + 1 - x_{k-1} - x_{k+1}, \\
m &= n, \quad x_0 = x_{n+1} = 0, \\
\bar{x}_l &= -1, \quad l \geq 1.
\end{aligned}$$

Problem 28. Generalized Broyden banded function [16].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= (2 + 5x_k^2)x_k + 1 + \sum_{j \in J_k} x_j(1 + x_j), \\
m &= n, \quad J_k = \{j : j \neq k, \max(1, k-5) \leq j \leq \min(n, k+1)\}, \\
\bar{x}_l &= -1, \quad l \geq 1.
\end{aligned}$$

Problem 29. Chained Freudenstein and Roth function [28].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= x_i + x_{i+1}((5 - x_{i+1})x_{i+1} - 2) - 13, \quad \text{mod}(k, 2) = 1, \\
f_k(x) &= x_i + x_{i+1}((1 + x_{i+1})x_{i+1} - 14) - 29, \quad \text{mod}(k, 2) = 0, \\
m &= 2(n-1), \quad i = \text{div}(k+1, 2), \\
\bar{x}_l &= 0.5, \quad l < n, \quad \bar{x}_l = -2, \quad l = n.
\end{aligned}$$

Problem 30. Wright and Holt zero residual problem [29].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= (x_i^a - x_j^b)^c, \\
a &= 1, \quad k \leq m/2, \quad a = 2, \quad k > m/2, \\
b &= 5 - \text{div}(k, m/4), \quad c = \text{mod}(k, 5) + 1, \\
m &= 5n, \quad i = \text{mod}(k, n/2) + 1, \quad j = i + n/2, \\
\bar{x}_l &= \sin^2(l), \quad l \geq 1.
\end{aligned}$$

Problem 31. Toint quadratic merging problem [28].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= x_i + 3x_{i+1}(x_{i+2} - 1) + x_{i+3}^2 - 1 & , \quad \text{mod}(k, 6) = 1, \\
f_k(x) &= (x_i + x_{i+1})^2 + (x_{i+2} - 1)^2 - x_{i+3} - 3 & , \quad \text{mod}(k, 6) = 2, \\
f_k(x) &= x_i x_{i+1} - x_{i+2} x_{i+3} & , \quad \text{mod}(k, 6) = 3, \\
f_k(x) &= 2x_i x_{i+2} + x_{i+1} x_{i+3} - 3 & , \quad \text{mod}(k, 6) = 4, \\
f_k(x) &= (x_i + x_{i+1} + x_{i+2} + x_{i+3})^2 + (x_i - 1)^2 & , \quad \text{mod}(k, 6) = 5, \\
f_k(x) &= x_i x_{i+1} x_{i+2} x_{i+3} + (x_{i+3} - 1)^2 - 1 & , \quad \text{mod}(k, 6) = 0, \\
m &= 3(n - 2), \quad i = 2 \operatorname{div}(k + 5, 6) - 1, \\
\bar{x}_l &= 5, \quad l \geq 1.
\end{aligned}$$

Problem 32. Chained exponential problem [16].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= 4 - \exp(x_i) - \exp(x_{i+1}) & , \quad \text{mod}(k, 2) = 1, \quad i = 1, \\
f_k(x) &= 8 - \exp(3x_{i-1}) - \exp(3x_i) \\
&\quad + 4 - \exp(x_i) - \exp(x_{i+1}) & , \quad \text{mod}(k, 2) = 1, \quad 1 < i < n, \\
f_k(x) &= 8 - \exp(3x_{i-1}) - \exp(3x_i) & , \quad \text{mod}(k, 2) = 1, \quad i = n, \\
f_k(x) &= 6 - \exp(2x_i) - \exp(2x_{i+1}) & , \quad \text{mod}(k, 2) = 0, \\
m &= 2n - 1, \quad i = \operatorname{div}(k + 1, 2), \\
\bar{x}_l &= 0.2, \quad l \geq 1.
\end{aligned}$$

Problem 33. Chained serpentine function [17].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= 10(2x_i / (1 + x_i^2) - x_{i+1}) & , \quad \text{mod}(k, 2) = 1, \\
f_k(x) &= x_i - 1 & , \quad \text{mod}(k, 2) = 0, \\
m &= 2(n - 1), \quad i = \operatorname{div}(k + 1, 2), \\
\bar{x}_l &= -0.8, \quad l \geq 1.
\end{aligned}$$

Problem 34. Chained and modified problem HS47 [17].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= 10(x_i^2 - x_{i+1}) & , \quad \text{mod}(k, 6) = 1, \\
f_k(x) &= x_{i+2} - 1 & , \quad \text{mod}(k, 6) = 2, \\
f_k(x) &= (x_{i+3} - 1)^2 & , \quad \text{mod}(k, 6) = 3, \\
f_k(x) &= (x_{i+4} - 1)^3 & , \quad \text{mod}(k, 6) = 4,
\end{aligned}$$

$$\begin{aligned}
f_k(x) &= x_i^2 x_{i+3} + \sin(x_{i+3} - x_{i+4}) - 10, & \text{mod}(k, 6) = 5, \\
f_k(x) &= x_{i+1} + x_{i+2}^4 x_{i+3}^2 - 20, & \text{mod}(k, 6) = 0, \\
m &= 6(\text{div}(n - 5, 3) + 1), & i = 3 \text{div}(k + 5, 6) - 2, \\
\bar{x}_l &= -1, & l \geq 1.
\end{aligned}$$

Problem 35. Chained and modified problem HS48 [17].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= 10(x_i^2 - x_{i+1}) & , & \text{mod}(k, 7) = 1, \\
f_k(x) &= 10(x_{i+1}^2 - x_{i+2}) & , & \text{mod}(k, 7) = 2, \\
f_k(x) &= (x_{i+2} - x_{i+3})^2 & , & \text{mod}(k, 7) = 3, \\
f_k(x) &= (x_{i+3} - x_{i+4})^2 & , & \text{mod}(k, 7) = 4, \\
f_k(x) &= x_i + x_{i+1}^2 + x_{i+2} - 30 & , & \text{mod}(k, 7) = 5, \\
f_k(x) &= x_{i+1} - x_{i+2}^2 + x_{i+3} - 10 & , & \text{mod}(k, 7) = 6, \\
f_k(x) &= x_i x_{i+4} - 10 & , & \text{mod}(k, 7) = 0, \\
m &= 7(\text{div}(n - 5, 3) + 1), & i = & 3 \text{div}(k + 6, 7) - 2, \\
\bar{x}_l &= -1, & l \geq & 1.
\end{aligned}$$

Problem 36. Sparse signomial function [17].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= y_j - \sum_{p=1}^3 (p^2/j) \prod_{q=1}^4 \text{sign}(x_{i+q}) |x_{i+q}|^{q/(pj)}, \\
m &= 4(\text{div}(n - 4, 2) + 1), & i = & 2 \text{div}(k + 3, 4) - 2, & j = & \text{mod}(k - 1, 4) + 1, \\
\bar{x}_l &= -0.8 & , & \text{mod}(l, 4) = 1, & y_1 &= 14.4, \\
\bar{x}_l &= 1.2 & , & \text{mod}(l, 4) = 2, & y_2 &= 6.8, \\
\bar{x}_l &= -1.2 & , & \text{mod}(l, 4) = 3, & y_3 &= 4.2, \\
\bar{x}_l &= 0.8 & , & \text{mod}(l, 4) = 0, & y_4 &= 3.2.
\end{aligned}$$

Problem 37. Sparse exponential function [17].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= y_j - \sum_{p=1}^3 (p^2/j) \exp \left(\sum_{q=1}^4 x_{i+q} q / (pj) \right), \\
m &= 4(\text{div}(n - 4, 2) + 1), & i = & 2 \text{div}(k + 3, 4) - 2, & j = & \text{mod}(k - 1, 4) + 1, \\
\bar{x}_l &= -0.8 & , & \text{mod}(l, 4) = 1, & y_1 &= 35.8, \\
\bar{x}_l &= 1.2 & , & \text{mod}(l, 4) = 2, & y_2 &= 11.2, \\
\bar{x}_l &= -1.2 & , & \text{mod}(l, 4) = 3, & y_3 &= 6.2, \\
\bar{x}_l &= 0.8 & , & \text{mod}(l, 4) = 0, & y_4 &= 4.4.
\end{aligned}$$

Problem 38. Sparse trigonometric function [17].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= y_j - \sum_{q=1}^4 [(-1)^q j q^2 \sin(x_{i+q}) + j^2 q \cos(x_{i+q})], \\
m &= 4(\operatorname{div}(n-4, 2) + 1), \quad i = 2 \operatorname{div}(k+3, 4) - 2, \quad j = \operatorname{mod}(k-1, 4) + 1, \\
\bar{x}_i &= -0.8 \quad , \quad \operatorname{mod}(l, 4) = 1, \quad y_1 = 30.6, \\
\bar{x}_i &= 1.2 \quad , \quad \operatorname{mod}(l, 4) = 2, \quad y_2 = 72.2, \\
\bar{x}_i &= -1.2 \quad , \quad \operatorname{mod}(l, 4) = 3, \quad y_3 = 124.4, \\
\bar{x}_i &= 0.8 \quad , \quad \operatorname{mod}(l, 4) = 0, \quad y_4 = 187.4.
\end{aligned}$$

Problem 39. Countercurrent reactors problem 1 [6] (modified).

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= \alpha - (1 - \alpha)x_{k+2} - x_k(1 + 4x_{k+1}) \quad , \quad k = 1, \\
f_k(x) &= -(2 - \alpha)x_{k+2} - x_k(1 + 4x_{k-1}) \quad , \quad k = 2, \\
f_k(x) &= \alpha x_{k-2} - (1 - \alpha)x_{k+2} - x_k(1 + 4x_{k+1}) \quad , \quad \operatorname{mod}(k, 2) = 1, \quad 2 < k < n-1, \\
f_k(x) &= \alpha x_{k-2} - (2 - \alpha)x_{k+2} - x_k(1 + 4x_{k-1}) \quad , \quad \operatorname{mod}(k, 2) = 0, \quad 2 < k < n-1, \\
f_k(x) &= \alpha x_{k-2} - x_k(1 + 4x_{k+1}) \quad , \quad k = n-1, \\
f_k(x) &= \alpha x_{k-2} - (2 - \alpha) - x_k(1 + 4x_{k-1}) \quad , \quad k = n, \\
\alpha &= 1/2, \\
\bar{x}_l &= 0.1 \quad , \quad \operatorname{mod}(l, 8) = 1, \quad \bar{x}_l = 0.2 \quad , \quad \operatorname{mod}(l, 8) = 2, \\
\bar{x}_l &= 0.3 \quad , \quad \operatorname{mod}(l, 8) = 3, \quad \bar{x}_l = 0.4 \quad , \quad \operatorname{mod}(l, 8) = 4, \\
\bar{x}_l &= 0.5 \quad , \quad \operatorname{mod}(l, 8) = 5, \quad \bar{x}_l = 0.4 \quad , \quad \operatorname{mod}(l, 8) = 6, \\
\bar{x}_l &= 0.3 \quad , \quad \operatorname{mod}(l, 8) = 7, \quad \bar{x}_l = 0.2 \quad , \quad \operatorname{mod}(l, 8) = 0.
\end{aligned}$$

Problem 40. Tridiagonal system [15].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= 4(x_k - x_{k+1}^2) \quad , \quad k = 1, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) + 4(x_k - x_{k+1}^2) \quad , \quad 1 < k < n, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) \quad , \quad k = n, \\
\bar{x}_l &= 12, \quad l \geq 1.
\end{aligned}$$

Problem 41. Structured Jacobian problem [11].

$$F(x) = \frac{1}{2} \sum_{k=1}^n f_k^2(x),$$

$$\begin{aligned}
f_k(x) &= -2x_k^2 + 3x_k - 2x_{k+1} + 3x_{n-4} - x_{n-3} \\
&\quad - x_{n-2} + 0.5x_{n-1} - x_n + 1, \quad k = 1, \\
f_k(x) &= -2x_k^2 + 3x_k - x_{k-1} - 2x_{k+1} + 3x_{n-4} - x_{n-3} \\
&\quad - x_{n-2} + 0.5x_{n-1} - x_n + 1, \quad 1 < k < n, \\
f_k(x) &= -2x_k^2 + 3x_k - x_{k-1} + 3x_{n-4} - x_{n-3} \\
&\quad - x_{n-2} + 0.5x_{n-1} - x_n + 1, \quad k = n, \\
\bar{x}_l &= -1, \quad l \geq 1.
\end{aligned}$$

Problem 42. Modified discrete boundary value problem [17].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= 2x_k + (1/2)h^2(x_k + hk + 1)^3 - x_{k-1} - x_{k+1} + 1, \\
h &= 1/(n+1), \quad x_0 = x_{n+1} = 0, \\
\bar{x}_l &= lh(lh - 1), \quad l \geq 1.
\end{aligned}$$

Problem 43. Chained and modified problem HS48 [17].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= 10(x_i^2 - x_{i+1}) \quad , \quad \text{mod}(k, 7) = 1, \\
f_k(x) &= x_{i+1} + x_{i+2} - 2 \quad , \quad \text{mod}(k, 7) = 2, \\
f_k(x) &= x_{i+3} - 1 \quad , \quad \text{mod}(k, 7) = 3, \\
f_k(x) &= x_{i+4} - 1 \quad , \quad \text{mod}(k, 7) = 4, \\
f_k(x) &= x_i + 3x_{i+1} \quad , \quad \text{mod}(k, 7) = 5, \\
f_k(x) &= x_{i+2} + x_{i+3} - 2x_{i+4} \quad , \quad \text{mod}(k, 7) = 6, \\
f_k(x) &= 10(x_{i+1}^2 - x_{i+4}) \quad , \quad \text{mod}(k, 7) = 0, \\
m &= 7(\text{div}(n - 5, 3) + 1) \quad , \quad i = 3 \text{div}(k + 6, 7) - 2, \\
\bar{x}_l &= -1, \quad l \geq 1.
\end{aligned}$$

Problem 44. Attracting-Repelling problem [17].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^m f_k^2(x), \\
f_k(x) &= x_1 - 1 \quad , \quad k = 1, \\
f_k(x) &= 10(x_i^2 - x_{i+1}), \quad , \quad k > 1, \quad \text{mod}(k, 2) = 0, \\
f_k(x) &= 2 \exp(-(x_i - x_{i+1})^2) + \exp(-2(x_{i+1} - x_{i+2})^2) \quad , \quad k > 1, \quad \text{mod}(k, 2) = 1, \\
m &= 2(n - 1), \quad i = \text{div}(k, 2), \\
\bar{x}_l &= -1.2 \quad , \quad \text{mod}(l, 2) = 1, \quad \bar{x}_l = 1.0 \quad , \quad \text{mod}(l, 2) = 0.
\end{aligned}$$

Problem 45. Countercurrent reactors problem 2 [6] (modified).

$$F(x) = \frac{1}{2} \sum_{k=1}^n f_k^2(x),$$

$$\begin{aligned}
f_k(x) &= x_1 - (1 - x_1)x_{k+2} - \alpha(1 + 4x_{k+1}) & , k = 1, \\
f_k(x) &= -(1 - x_1)x_{k+2} - \alpha(1 + 4x_k) & , k = 2, \\
f_k(x) &= \alpha x_1 - (1 - x_1)x_{k+2} - x_k(1 + 4x_{k-1}) & , k = 3, \\
f_k(x) &= x_1x_{k-2} - (1 - x_1)x_{k+2} - x_k(1 + 4x_{k-1}) & , 3 < k < n - 1, \\
f_k(x) &= x_1x_{k-2} - x_k(1 + 4x_{k-1}) & , k = n - 1, \\
f_k(x) &= x_1x_{k-2} - (1 - x_1) - x_k(1 + 4x_{k-1}) & , k = n, \\
\alpha &= 0.414214, \\
\bar{x}_i &= 0.1 \quad , \text{mod}(i, 8) = 1, & \bar{x}_i &= 0.2 \quad , \text{mod}(i, 8) = 2, \\
\bar{x}_i &= 0.3 \quad , \text{mod}(i, 8) = 3, & \bar{x}_i &= 0.4 \quad , \text{mod}(i, 8) = 4, \\
\bar{x}_i &= 0.5 \quad , \text{mod}(i, 8) = 5, & \bar{x}_i &= 0.4 \quad , \text{mod}(i, 8) = 6, \\
\bar{x}_i &= 0.3 \quad , \text{mod}(i, 8) = 7, & \bar{x}_i &= 0.2 \quad , \text{mod}(i, 8) = 0.
\end{aligned}$$

Problem 46.

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= 5 - (i + 1)(1 - \cos x_k) - \sin x_k - \sum_{j=5i+1}^{5i+5} \cos x_j, \\
i &= \text{div}(k - 1, 5), \quad \bar{x}_l = 1/n, \quad l \geq 1.
\end{aligned}$$

Problem 47. Trigonometric - exponential system (trigexp 1) [27].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= 3x_k^3 + 2x_{k+1} - 5 + \sin(x_k - x_{k+1}) \sin(x_k + x_{k+1}), \quad k = 1, \\
f_k(x) &= 3x_k^3 + 2x_{k+1} - 5 + \sin(x_k - x_{k+1}) \sin(x_k + x_{k+1}) \\
&\quad + 4x_k - x_{k-1} \exp(x_{k-1} - x_k) - 3, \quad 1 < k < n, \\
f_k(x) &= 4x_k - x_{k-1} \exp(x_{k-1} - x_k) - 3, \quad k = n, \\
\bar{x}_i &= 0, \quad i \geq 1.
\end{aligned}$$

Problem 48. Trigonometric - exponential system (trigexp 2) [27].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= 3(x_k - x_{k+2})^3 - 5 + 2x_{k+1}, \\
&\quad + \sin(x_k - x_{k+1} - x_{k+2}) \sin(x_k + x_{k+1} - x_{k+2}) \quad , \text{mod}(k, 2) = 1, k = 1, \\
f_k(x) &= -6(x_{k-2} - x_k)^3 + 10 - 4x_{k-1} \\
&\quad - 2 \sin(x_{k-2} - x_{k-1} - x_k) \sin(x_{k-2} + x_{k-1} - x_k) \\
&\quad + 3(x_k - x_{k+2})^3 - 5 + 2x_{k+1} \\
&\quad + \sin(x_k - x_{k+1} - x_{k+2}) \sin(x_k + x_{k+1} - x_{k+2}) \quad , \text{mod}(k, 2) = 1, 1 < k < n, \\
f_k(x) &= -6(x_{k-2} - x_k)^3 + 10 - 4x_{k-1}
\end{aligned}$$

$$\begin{aligned}
& - 2 \sin(x_{k-2} - x_{k-1} - x_k) \sin(x_{k-2} + x_{k-1} - x_k) \quad , \text{ mod } (k, 2) = 1, \quad k = n, \\
f_k(x) & = 4x_k - (x_{k-1} - x_{k+1}) \exp(x_{k-1} - x_k - x_{k+1}) - 3 \quad , \text{ mod } (k, 2) = 0, \\
\bar{x}_i & = 1, \quad i \geq 1.
\end{aligned}$$

Problem 49. Singular Broyden problem [11].

$$\begin{aligned}
F(x) & = \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) & = ((3 - 2x_k)x_k - 2x_{k+1} + 1)^2 \quad , \quad k = 1, \\
f_k(x) & = ((3 - 2x_k)x_k - x_{k-1} - 2x_{k+1} + 1)^2 \quad , \quad 1 < k < n, \\
f_k(x) & = ((3 - 2x_k)x_k - x_{k-1} + 1)^2 \quad , \quad k = n, \\
\bar{x}_i & = -1, \quad i \geq 1.
\end{aligned}$$

Problem 50. Five-diagonal system [15].

$$\begin{aligned}
F(x) & = \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) & = 4(x_k - x_{k+1}^2) + x_{k+1} - x_{k+2}^2 \quad , \quad k = 1, \\
f_k(x) & = 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) \\
& + 4(x_k - x_{k+1}^2) + x_{k+1} - x_{k+2}^2 \quad , \quad k = 2, \\
f_k(x) & = 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) \\
& + 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} - x_{k+2}^2 \quad , \quad 2 < k < n - 1, \\
f_k(x) & = 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) \\
& + 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} \quad , \quad k = n - 1, \\
f_k(x) & = 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) + x_{k-1}^2 - x_{k-2} \quad , \quad k = n, \\
\bar{x}_i & = -2, \quad i \geq 1.
\end{aligned}$$

Problem 51. Seven-diagonal system [15].

$$\begin{aligned}
F(x) & = \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) & = 4(x_k - x_{k+1}^2) + x_{k+1} - x_{k+2}^2 + x_{k+2} - x_{k+3}^2 \quad , \quad k = 1, \\
f_k(x) & = 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) \\
& + 4(x_k - x_{k+1}^2) + x_{k-1}^2 + x_{k+1} - x_{k+2}^2 + x_{k+2} - x_{k+3}^2 \quad , \quad k = 2, \\
f_k(x) & = 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) \\
& + 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} - x_{k+2}^2 \\
& + x_{k-2}^2 + x_{k+2} - x_{k+3}^2 \quad , \quad k = 3, \\
f_k(x) & = 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) \\
& + 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} - x_{k+2}^2 \\
& + x_{k-2}^2 + x_{k+2} - x_{k-3} - x_{k+3}^2 \quad , \quad 3 < k < n - 2, \\
f_k(x) & = 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k)
\end{aligned}$$

$$\begin{aligned}
& + 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} - x_{k+2}^2 \\
& + x_{k-2}^2 + x_{k+2} - x_{k-3} \qquad \qquad \qquad , \quad k = n - 2, \\
f_k(x) & = 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) \\
& + 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} \\
& + x_{k-2}^2 - x_{k-3} \qquad \qquad \qquad , \quad k = n - 1, \\
f_k(x) & = 8x_k(x_k^2 - x_{k-1}) - 2(1 - x_k) + x_{k-1}^2 - x_{k-2} \\
& + x_{k-2}^2 - x_{k-3} \qquad \qquad \qquad , \quad k = n, \\
\bar{x}_i & = -3, \quad i \geq 1.
\end{aligned}$$

Problem 52. Extended Freudenstein and Roth function [5].

$$\begin{aligned}
F(x) & = \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k & = x_k + ((5 - x_{k+1})x_{k+1} - 2)x_{k+1} - 13 \quad , \quad \text{mod}(k, 2) = 1, \\
f_k & = x_{k-1} + ((x_k + 1)x_k - 14)x_k - 29 \quad , \quad \text{mod}(k, 2) = 0, \\
\bar{x}_i & = 90 \quad , \quad \text{mod}(i, 2) = 1, \quad \bar{x}_i = 60 \quad , \quad \text{mod}(i, 2) = 0.
\end{aligned}$$

Problem 53. Extended Cragg and Levy problem [22].

$$\begin{aligned}
F(x) & = \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) & = (\exp(x_k) - x_{k+1})^2 \quad , \quad \text{mod}(k, 4) = 1, \\
f_k(x) & = 10(x_k - x_{k+1})^3 \quad , \quad \text{mod}(k, 4) = 2, \\
f_k(x) & = \tan^2(x_k - x_{k+1}) \quad , \quad \text{mod}(k, 4) = 3, \\
f_k(x) & = x_k - 1 \quad , \quad \text{mod}(k, 4) = 0, \\
\bar{x}_i & = 1 \quad , \quad \text{mod}(i, 4) = 1, \quad \bar{x}_i = 2 \quad , \quad \text{mod}(i, 4) \neq 1.
\end{aligned}$$

Problem 54. Broyden tridiagonal problem [22].

$$\begin{aligned}
F(x) & = \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) & = x_k(0.5x_k - 3) + 2x_{k+1} - 1 \quad , \quad k = 1, \\
f_k(x) & = x_k(0.5x_k - 3) + x_{k-1} + 2x_{k+1} - 1 \quad , \quad 1 < k < n, \\
f_k(x) & = x_k(0.5x_k - 3) - 1 + x_{k-1} \quad , \quad k = n, \\
\bar{x}_i & = -1, \quad i \geq 1.
\end{aligned}$$

Problem 55. Extended Powell badly scaled function [22].

$$\begin{aligned}
F(x) & = \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) & = 10000 x_k x_{k+1} - 1 \quad , \quad \text{mod}(k, 2) = 1, \\
f_k(x) & = \exp(-x_{k-1}) + \exp(-x_k) - 1.0001 \quad , \quad \text{mod}(k, 2) = 0, \\
\bar{x}_i & = 0 \quad , \quad \text{mod}(i, 2) = 1, \quad \bar{x}_i = 1 \quad , \quad \text{mod}(i, 2) = 0.
\end{aligned}$$

Problem 56. Extended Wood problem [12].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= -200x_k(x_{k+1} - x_k^2) - (1 - x_k) & , \quad \text{mod}(k, 4) = 1, \\
f_k(x) &= 200(x_k - x_{k-1}^2) + 20.2(x_k - 1) + 19.8(x_{k+2} - 1) & , \quad \text{mod}(k, 4) = 2, \\
f_k(x) &= -180x_k(x_{k+1} - x_k^2) - (1 - x_k) & , \quad \text{mod}(k, 4) = 3, \\
f_k(x) &= 180(x_k - x_{k-1}^2) + 20.2(x_k - 1) + 19.8(x_{k-2} - 1) & , \quad \text{mod}(k, 4) = 0, \\
\bar{x}_i &= -3, \quad \text{mod}(i, 2) = 1, \quad \bar{x}_i = -1, \quad \text{mod}(i, 2) = 0.
\end{aligned}$$

Problem 57. Tridiagonal exponential problem [5].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= x_k - \exp(\cos(k(x_k + x_{k+1}))) & , \quad k = 1, \\
f_k(x) &= x_k - \exp(\cos(k(x_{k-1} + x_k + x_{k+1}))) & , \quad 1 < k < n, \\
f_k(x) &= x_k - \exp(\cos(k(x_{k-1} + x_k))) & , \quad k = n, \\
\bar{x}_i &= 1.5, \quad i \geq 1.
\end{aligned}$$

Problem 58. Brent problem [4].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= 3x_k(x_{k+1} - 2x_k) + x_{k+1}^2/4 & , \quad k = 1, \\
f_k(x) &= 3x_k(x_{k+1} - 2x_k + x_{k-1}) + (x_{k+1} - x_{k-1})^2/4 & , \quad 1 < k < n, \\
f_k(x) &= 3x_k(20 - 2x_k + x_{k-1}) + (20 - x_{k-1})^2/4 & , \quad k = n, \\
\bar{x}_i &= 10, \quad i \geq 1.
\end{aligned}$$

Problem 59. Troesch problem [24].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= 2x_k + \rho h^2 \sinh(\rho x_k) - x_{k+1} & , \quad k = 1, \\
f_k(x) &= 2x_k + \rho h^2 \sinh(\rho x_k) - x_{k-1} - x_{k+1} & , \quad 1 < k < n, \\
f_k(x) &= 2x_k + \rho h^2 \sinh(\rho x_k) - x_{k-1} - 1 & , \quad k = n, \\
\rho &= 10, \quad h = 1/(n + 1), \\
\bar{x}_i &= 1, \quad i \geq 1.
\end{aligned}$$

Problem 60. Flow in a channel [3].

$$F(x) = \frac{1}{2} f^T(x) f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear ordinary differential equation

$$u'''' = R(u'u'' - uu'''), \quad R = 500$$

over unit interval Ω with boundary conditions $u(0) = 0$, $u'(0) = 0$, $u(1) = 1$, $u'(1) = 0$. We use standard 5-point finite differences on a uniform grid having 5000 internal nodes. The initial approximate solution is a discretization of $u_0(x) = (x - 1/2)^2$.

Problem 61. Swirling flow [3].

$$F(x) = \frac{1}{2}f^T(x)f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following system of two nonlinear ordinary differential equations

$$\begin{aligned} u'''' + R(uu'' + vv') &= 0 \\ v'' + R(uv' + u'v) &= 0, \quad R = 500 \end{aligned}$$

over unit interval Ω with boundary conditions $u(0) = u'(0) = u(1) = u'(1) = 0$, $v(0) = -1$, $v(1) = 1$. We use standard 5-point finite differences on a uniform grid having 2500 internal nodes. The initial approximate solution is a discretization of $u_0(x) = (x - 1/2)^2$ and $v_0(x) = x - 1/2$.

Problem 62. Bratu problem [13].

$$F(x) = \frac{1}{2}f^T(x)f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u + R \exp(u) = 0, \quad R = 6.8$$

over unit square Ω with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 63. Poisson problem 1 [11].

$$F(x) = \frac{1}{2}f^T(x)f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u = \frac{u^3}{1 + x^2 + y^2}$$

over unit square Ω with Dirichlet boundary conditions $u(0, y) = 1$, $u(1, y) = 2 - \exp(y)$, $u(x, 0) = 1$, $u(x, 1) = 2 - \exp(x)$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = -1$.

Problem 64. Poisson problem 2 [21].

$$F(x) = \frac{1}{2}f^T(x)f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u + \sin(2\pi u) + \sin\left(2\pi \frac{\partial u}{\partial x}\right) + \sin\left(2\pi \frac{\partial u}{\partial y}\right) + f(x, y) = 0,$$

where $f(x, y) = 1000((x-1/4)^2 + (y-3/4)^2)$, over unit square Ω with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 65. Porous medium problem [8].

$$F(x) = \frac{1}{2}f^T(x)f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u^2 + R\left(\frac{\partial u^3}{\partial x} + f(x, y)\right) = 0, \quad R = 50,$$

where $f(1/71, 1/71) = 1$ and $f(x, y) = 0$ for $(x, y) \neq (1/71, 1/71)$, over unit square Ω with Dirichlet boundary conditions $u(0, y) = 1$, $u(1, y) = 0$, $u(x, 0) = 1$, $u(x, 1) = 0$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 1 - xy$.

Problem 66. Convection-diffusion problem [14].

$$F(x) = \frac{1}{2}f^T(x)f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u - Ru\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + f(x, y) = 0, \quad R = 20,$$

where $f(x, y) = 2000x(1-x)y(1-y)$, over unit square Ω with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 67. Nonlinear biharmonic problem [18].

$$F(x) = \frac{1}{2}f^T(x)f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta\Delta u + R(\max(0, u) + \text{sign}(x - 1/2)) = 0, \quad R = 500$$

over unit square Ω with the boundary conditions $u = 0$ on $\partial\Omega$ and $\partial u(0, y)/\partial x = 0$, $\partial u(1, y)/\partial x = 0$, $\partial u(x, 0)/\partial y = 0$, $\partial u(x, 1)/\partial y = 0$. We use standard 13-point finite differences on a shifted uniform grid having 50×50 internal nodes [13]. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 68. Driven cavity problem [13].

$$F(x) = \frac{1}{2}f^T(x)f(x),$$

where equation $f(x) = 0$ is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta\Delta u + R\left(\frac{\partial u}{\partial y}\frac{\partial\Delta u}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial\Delta u}{\partial y}\right) = 0, \quad R = 500$$

over unit square Ω with the boundary conditions $u = 0$ on $\partial\Omega$ and $\partial u(0, y)/\partial x = 0$, $\partial u(1, y)/\partial x = 0$, $\partial u(x, 0)/\partial y = 0$, $\partial u(x, 1)/\partial y = 1$. We use standard 13-point finite differences on a shifted uniform grid having 50×50 internal nodes [13]. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 69.

$$\begin{aligned} F(x) &= \frac{1}{2}\sum_{k=1}^n f_k^2(x), \\ f_k(x) &= 2x_k - x_{k+1} - x_{k-1} \\ &\quad + h^2\left(x_k^3 + 2.10^{-4}(2.10^{-4}a_2 - 1)x_k - 10^9 \exp(-3.10^4 a_2)\right), \\ h &= 1/(n+1), \quad a_1 = hk, \quad a_2 = (a_1 - 1/2)^2, \\ x_0 &= x_{n+1} = 0, \quad \bar{x}_l = 5 \min(lh, 1 - lh), \quad l \geq 1. \end{aligned}$$

Problem 70.

$$\begin{aligned} F(x) &= \frac{1}{2}\sum_{k=1}^n f_k^2(x), \\ f_k(x) &= 2x_k - x_{k+1} - x_{k-1} \\ &\quad + h^2\left(x_k^3 \exp(x_k) + 5.10^8 \exp(-10^4 a_2)\sqrt{|a_1 - 1/2|}(x_{k+1} - x_{k-1}) + a_3\right), \\ h &= 1/(n+1), \quad a_1 = hk, \quad a_2 = (a_1 - 1/2)^2, \quad a_3 = 10^6 \text{sign}(a_1 - 1/2), \\ x_0 &= x_{n+1} = 0, \quad \bar{x}_l = 5 \min(lh, 1 - lh), \quad l \geq 1. \end{aligned}$$

Problem 71. Problem 202 in [25].

$$F(x) = \frac{1}{2}\sum_{k=1}^n f_k^2(x),$$

$$\begin{aligned}
f_k(x) &= x_k - \frac{x_{k+1}^2}{10}, \quad 1 \leq k < n, \\
f_k(x) &= x_k - \frac{x_1^2}{10}, \quad k = n, \\
\bar{x}_l &= 2, \quad l \geq 1.
\end{aligned}$$

Problem 72. Problem 206 in [25].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= x_{k-1} - 2x_k + x_{k+1} - h^2 \exp(x_k), \quad 1 \leq k \leq n, \\
h &= 1/(n+1), \quad x_0 = x_{n+1} = 0, \\
\bar{x}_l &= 1, \quad l \geq 1.
\end{aligned}$$

Problem 73. Problem 207 in [25].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= (3 - x_k/10)x_k + 1 - x_{k-1} - 2x_{k+1}, \quad 1 \leq k \leq n, \\
x_0 &= x_{n+1} = 0, \\
\bar{x}_l &= -1, \quad l \geq 1.
\end{aligned}$$

Problem 74. Problem 208 in [25].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= (1 + x_k^2)x_k + 1 - \sum_{i \in I_k} (x_i + x_i^2), \quad 1 \leq k \leq n, \\
I_k &= \{i : i \neq k, \max(1, k-3) \leq i \leq \min(n, k+3)\}, \\
\bar{x}_l &= -1, \quad l \geq 1.
\end{aligned}$$

Problem 75. Problem 212 in [25].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= x_k, \quad k = 1, \\
f_k(x) &= \cos(x_{k-1}) + x_k - 1, \quad 1 < k \leq n, \\
\bar{x}_l &= 1/2, \quad l \geq 1.
\end{aligned}$$

Problem 76. Problem 213 in [25].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k^2(x), \\
f_k(x) &= 2x_k + h^2(x_k + \sin(x_k)) - x_{k-1} - x_{k+1}, \quad 1 \leq k \leq n, \\
h &= 1/(n+1), \quad x_0 = 0, \quad x_{n+1} = 1, \\
\bar{x}_l &= 1, \quad l \geq 1.
\end{aligned}$$

Problem 77. Problem 214 in [25].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k(x), \\
f_k(x) &= x_k(2 + 5x_k^2) + 1 - \sum_{i \in I_k} x_i(1 + x_i), \quad 1 \leq k \leq n, \\
I_k &= \{i : i \neq k, \max(1, k-5) \leq i \leq \min(n, k+1)\}, \\
\bar{x}_l &= -1, \quad l \geq 0.
\end{aligned}$$

Problem 78. Ascher and Russel boundary value problem [2].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k(x), \\
f_k(x) &= 2x_k - 2h^2 \left(x_k^2 + \frac{x_{k+1} - x_{k-1}}{2h} \right) - x_{k-1} - x_{k+1}, \quad 1 \leq k \leq n, \\
h &= 1/(n+1), \quad x_0 = 0, \quad x_{n+1} = 1/2, \\
\bar{x}_l &= 1, \quad l \geq 1.
\end{aligned}$$

Problem 79. Allgower and Georg boundary value problem [1].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k(x), \\
f_k(x) &= 2x_k + 0.3h^2 [\exp(20(x_k + 25(kh - 1))) - \exp(-20(x_k + 25kh)) - t_k] \\
&\quad - x_{k-1} - x_{k+1}, \\
t_k &= \text{sign}(kh - 0.009), \quad k \geq 1, \\
h &= 0.01/(n+1), \quad x_0 = 0, \quad x_{n+1} = 25, \\
\bar{x}_l &= 1, \quad l \geq 1.
\end{aligned}$$

Problem 80. Potra and Rheinboldt boundary value problem [23].

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k(x), \\
f_k(x) &= 2x_k - x_{k-1} - x_{k+1} + h^2(x_k^2 + x_k + 0.1x_{k+n/2} - 1.2), \quad 1 \leq k \leq n/2, \\
f_k(x) &= 2x_k - x_{k-1} - x_{k+1} + h^2(0.2x_{k-n/2}^2 + x_k^2 + 2x_k - 0.6), \quad n/2 < k \leq n, \\
h &= 1/(n/2 + 1), \quad x_0 = x_{n+1} = 0, \\
\bar{x}_l &= lh(1 - lh), \quad \bar{x}_{l+n/2} = \bar{x}_l, \quad 1 \leq l \leq n/2.
\end{aligned}$$

Problem 81.

$$\begin{aligned}
F(x) &= \frac{1}{2} \sum_{k=1}^n f_k(x), \\
f_k(x) &= 4x_k - x_{k-1} - x_{k+1} - x_{k-\sqrt{n}} - x_{k+\sqrt{n}} + h^2 \exp(x_k), \quad 1 \leq k \leq n, \\
h &= 1/(\sqrt{n} + 1), \quad x_l = 0 \quad \text{for } l < 1 \quad \text{or } l > n, \\
\bar{x}_l &= 1, \quad 1 \leq l \leq n.
\end{aligned}$$

Problem 82.

$$\begin{aligned} F(x) &= \frac{1}{2} \sum_{k=1}^n f_k(x), \\ f_k(x) &= 4x_k - x_{k-1} - x_{k+1} - x_{k-\sqrt{n}} - x_{k+\sqrt{n}} + h^2 x_k^2 - y_k, \quad 1 \leq k \leq n, \\ h &= 1/(\sqrt{n} + 1), \quad x_l = 0 \quad \text{for } l < 1 \quad \text{or } l > n, \\ \bar{x}_l &= 1, \quad 1 \leq l \leq n. \end{aligned}$$

References

- [1] Allgover, E.L., and Georg, K., *Computational Solution of Nonlinear Systems of Equations*, Lectures in Applied Mathematics 26, American Mathematical Society, 1990.
- [2] Ascher, U.M., and Russel, R.D., *Numerical Boundary Value ODEs*, Birkhauser, Boston 1985.
- [3] Averick, B.M., and Carter, R.G., and Moré, J.J., *The Minpack-2 Test Problem Collection*, Research Report No. ANL/MCS-TM-150, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne 1991.
- [4] Alefeld, G., and Gienger, A., and Potra, F., *Efficient Validation of Solutions of Nonlinear Systems*, SIAM Journal on Numerical Analysis, Vol 31, pp. 252-260, 1994.
- [5] Bing, Y., Lin, G., *An Efficient Implementation of Merrill's Method for Sparse or Partially Separable Systems of Nonlinear Equations*, SIAM Journal on Optimization, Vol. 2, pp. 206-221, 1991.
- [6] Bogle, I.D.L., Perkins, J.D., *A New Sparsity Preserving Quasi-Newton Update for Solving Nonlinear Equations*, SIAM Journal on Scientific and Statistical Computations, Vol. 11, pp. 621-630, 1990.
- [7] Conn, A.R., Gould, N.I.M, Toint, P., *Testing a Class of Methods for Solving Minimization Problems with Simple Bounds on the Variables*, Mathematics of Computation, Vol. 50, pp. 399-430, 1988.
- [8] Eisenstat, S.C., Walker, H.F., *Choosing the Forcing Terms in an Inexact Newton Method*, SIAM Journal on Scientific Computation, Vol. 17, pp. 16-32, 1996.
- [9] Gill, P.E., Murray, W., *The Numerical Solution of a Problem in the Calculus of Variations*, Technical Report NAC 28, National Physical Laboratory, Teddington, Middlesex, England 1972.
- [10] Gill, P.E., Murray, W., *Conjugate-Gradient Methods for Large-Scale Nonlinear Optimization*, Technical Report SOL 79-15, System Optimization Laboratory, Department of Operations Research, Stanford University, Stanford, California, 1979.

- [11] Gomez-Ruggiero, M.A., Martinez, J.M., and Moretti, A.C., *Comparing Algorithms for Solving Sparse Nonlinear Systems of Equations*, SIAM Journal on Scientific and Statistical Computations, Vol. 13, pp. 459-483, 1992.
- [12] Incerti, S., Zirilli, F., and Parisi, V., *Algorithm 111. A Fortran Subroutine for Solving Systems of Nonlinear Simultaneous Equations*, Computer Journal, Vol 24, pp. 87-91, 1981.
- [13] Kaporin, I.E., Axelsson, O., *On a Class of Nonlinear Equation Solvers Based on the Residual Norm Reduction Over a Sequence of Affine Subspaces*, SIAM Journal on Scientific and Statistical Computations, Vol 16, pp. 228-249, 1995.
- [14] Kelley, C.T., *Iterative Methods for Linear and Nonlinear Equations*, SIAM, Philadelphia, Pennsylvania, 1995.
- [15] Li, G., *Successive Column Correction Algorithms for Solving Sparse Nonlinear Systems of Equations*, Mathematical Programming, Vol. 43, pp. 187-207, 1989.
- [16] Lukšan, L., *Inexact Trust Region Method for Large Sparse Nonlinear Least Squares*, Kybernetika, Vol. 29, pp. 305-324, 1993.
- [17] Lukšan L., *Hybrid Methods for Large Sparse Nonlinear Least Squares*, Journal of Optimization Theory and Applications, Vol. 89, pp. 575-595, 1996.
- [18] Lukšan, L., Vlček, J., *Computational Experience with Globally Convergent Descent Methods for Large Sparse Systems of Nonlinear Equations*, Optimization Methods and Software, Vol. 8, pp. 201-223, 1998.
- [19] Lukšan, L., Vlček, J., *Sparse and Partially Separable Test Problems for Unconstrained and Equality Constrained Optimization*, Research Report No. V-767, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, Czech Republic, 1998.
- [20] Lukšan, L., Šiška, M., Tůma, M., Vlček, J., Ramešová, N., *Interactive System for Universal Functional Optimization (UFO), Version 2002*, Research Report No. V-883, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, Czech Republic, 2002.
- [21] Martinez, J.M., *A Quasi-Newton Method with Modification of One Column per Iteration*, Computing, Vol. 33, pp. 353-362, 1984.
- [22] Moré, J.J., Garbow, B.S., Hillström, K.E., *Testing Unconstrained Optimization Software*, ACM Transactions on Mathematical Software, Vol. 7, pp. 17-41, 1981.
- [23] Potra, F.A., and Rheinboldt, W.C., *On the Monotone Convergence of Newton's Method*, Computing, Vol 36, pp. 81-90, 1986.
- [24] Roberts, S.M., and Shipman, J.S., *On the Closed Form Solution of Troesch's Problem*, Journal of Computational Physics, Vol. 21, pp. 291-304, 1976.

- [25] Roose, A., Kulla, V., Lomp, M., Meressoo, T., *Test Examples for systems of nonlinear equations*, Estonian Software and Computer Service Company, Tallin 1990.
- [26] Toint, P.L., *Some Numerical Results Using a Sparse Matrix Updating Formula in Unconstrained Optimization*, Mathematics of Computation, Vol. 143, pp. 839-851, 1978.
- [27] Toint, P.L., *Numerical Solution of Large Sets of Algebraic Equations*, Mathematics of Computation, Vol. 46, pp. 175-189, 1986.
- [28] Toint, P.L., *On Large Scale Nonlinear Least Squares Calculations*, SIAM Journal on Scientific and Statistical Computations, Vol. 8, pp. 416-435, 1987.
- [29] Wright, S.J., Holt, J.N., *An Inexact Levenberg-Marquardt Method for Large Sparse Nonlinear Least Squares*, Journal of the Australian Mathematical Society, Vol. B26, pp. 387-403, 1985.