Institute of Computer Science
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Logics with Truth Constants for Delimiting Idempotents

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Technical report No. 1055

November 2009

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Abstract:

The propositional logics in the language of Hájek's Basic Fuzzy Logic enriched with truth constants for idempotent elements delimiting the components of a continuous t-norm are investigated. An axiomatization is proposed for each of the logics, together with some completeness results. Computational complexity of the sets of tautologies is investigated.

Keywords:
Propositional fuzzy logic, truth constants, axiomatization, computational complexity

[1]Supported by grant ICC/08/E018 of the Grant Agency of the Czech Republic (a part of ESF Eurocores-LogICCC project FP006).

Logics with Truth Constants for Delimiting Idempotents

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1 Introduction

In this paper we investigate the propositional logic of standard algebras (for Hájek's Basic Fuzzy Logic, see [1]) in a language expanded by truth constants for the idempotent elements delimiting the Ł-, G-, and Π-components. We start from a given standard algebra and try to present a suitable axiomatization of its tautologies in the expanded language under the given semantics. Naturally, the logic depends on the algebra as well as on a chosen enumeration of truth constants.

A particular case of this general setting was already discussed in [4] and in [5], where only one delimiting constant is considered.

Moreover, Hájek's paper [2] on logic of truth hedges is in some points similar to the present material.

This paper is organized as follows. In Section 2 we describe the set of constants introduced and their semantics. We give axioms which describe the set of constants (in particular, its ordering) and show its standard completeness. In section 3 we additionally consider the types of components inbetween the constants. Section 4 gives complexity results for the logics.

2 Logics with truth constants for endpoints

2.1 Language and semantics

According to the Mostert-Shields representation theorem, each continuous t-norm $*$ imposes the following kind of structure to the real unit interval $[0, 1]$: the latter consists of a closed set I of elements of $[0, 1]$ which are idempotent w. r. t. $*$, while on closures of the open intervals which constitute the complement

*supported by grant ICC/08/E018 of the Grant Agency of the Czech Republic (a part of ESF Eurocores-LogICCC project FP006)

of I, $*$ is isomorphic either to the Łukasiewicz t-norm on $[0,1]$ or to the product t-norm on $[0,1]$.

It follows that with each $*$, one can distinguish three types of intervals on $[0,1]$: intervals on which $*$ is isomorphic to the Łukasiewicz t-norm, intervals on which $*$ is isomorphic to the product t-norm, and intervals of idempotent elements. In the last case, we consider only the *maximal* intervals of idempotents w. t. t. inclusion; it is obvious that on each such interval, $*$ is isomorphic to the Gödel t-norm. Each interval of one of the three above types is delimited by two idempotent elements, its *endpoints*.

For a given standard algebra $[0,1]_*$ let $\mathrm{EP}(*)$ be the set of endpoints of its Ł-, G-, and Π-intervals (we write EP if $*$ is clear from context). Note that for each $*$ the set $\mathrm{EP}(*)$ is countable. Moreover, it is a consequence of the representation theorem that if two standard algebras $[0,1]_{*_1}$ and $[0,1]_{*_2}$ have the same set of endpoints and for $x, y \in EP(*_1)$ we have $[x,y]$ is an Ł-component, (G-component, Π-component) in $[0,1]_{*_1}$ iff $[x,y]$ is an Ł-component (G-component, Π-component respectively) in $[0,1]_{*_2}$, then $[0,1]_{*_1}$ and $[0,1]_{*_2}$ are isomorphic. If moreover, the isomorphism function is the same for $[x,y]$ in $[0,1]_{*_1}$ and $[0,1]_{*_2}$, then they coincide.

Definition 2.1. *(i) Fix $*$ and let* EP *be its set of endpoints. Assume $a : \mathbb{N} \longrightarrow \mathrm{EP}$ is a given enumeration of* EP*, i. e., a maps (some initial segment of) $\mathbb{N}$ bijectively onto* EP*. Denote $\mathbb{N}_0 = \mathrm{Dom}(a)$.*[1] *So $a_i = a(i)$ is the i-th endpoint in our enumeration of $\mathrm{EP}(*)$. Assume for convenience $a_0 = 0$.*
(ii)Furthemore, let $+ : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ be the function assigning to each $i \in \mathbb{N}_0$ an index j s. t. $a_j = \min\{x : x \in \mathrm{EP}\,\mathrm{and}\, a_i < x\}$, $+(i) = i$ if no such j exists. We write i^+ for $+(i)$.

Given $*$, introduce a set of truth constants $\mathcal{C}_* = \{c_i\}_{i=0}^{\mathrm{card(EP)}-1}$. The semantics for these new elements of the language is the following: $e(c_i) = a_i$ for any evaluation e in $[0,1]_*$ (hence c_0 denotes 0). We can define the $+$ function on $\mathcal{C}_*$: For each $i \in \mathbb{N}_0$ we define $c_i^+ = c_{(i^+)}$.

The semantics for the propositional BL-language expanded with the set $\mathcal{C}_*$ is given by a continuous t-norm *and* the mapping a enumerating the endpoints of $*$; two algebras given by isomorphic t-norms, with a different enumeration of endpoints in each case, will have different sets of tautologies in the expanded language.

2.2 Axioms for constants and completeness results

For each $*$, we define the propositional logic $BL_{EP(*)}$. We have already defined the set $EP(*)$ and the corresponding set $\mathcal{C}$ of new propositional constants. Recall that $\mathbb{N}_0$ is the set of natural numbers enumerating both $EP(*)$ and $\mathcal{C}$.

We introduce a set of formulas which are tautologies of $[0,1]_*$ in the expanded propositional language and axiom candidates. To indicate idempotence of the

[1] We assume $0 \in \mathbb{N}$ and thus the endpoints are indexed from zero.

elements denoted by the constants we add, for each $i \in \mathbb{N}_0$, an axiom

$$c_i \,\&\, c_i \equiv c_i.$$

To capture the strict linear ordering of the cutpoints, add for $i, j \in \mathbb{N}_0$ such that $a_i < a_j$, the formulas

$$c_i \to c_j$$

and

$$(c_j \to c_i) \to c_i.$$

Note that, assuming c_i, c_j are evaluated by idempotents, $(c_j \to c_i) \to c_i$ is valid iff either $e(c_i) < e(c_j) < 1$, or $e(c_i) = e(c_j) = 1$. Note also that $c_i \to c_i^+$ is an instance of the second type of formula.

We now define the logic $BL_{EP(*)}$ and demonstrate some completeness results. Note that each logic $BL_{EP(*)}$ is tailored to a particular continuous t-norm $*$.

Definition 2.2. *Let $*$ be a continuous t-norm. The axioms of the logic $BL_{EP(*)}$ are the axioms of BL plus the following formulas:*

(EP_1^i)	$c_i \,\&\, c_i \equiv c_i$ for each $i \in \mathbb{N}_0$
$(EP_2^{i,j})$	$c_i \to c_j$ for each $i, j \in \mathbb{N}_0$ s.t. $a_i < a_j$
$(EP_3^{i,j})$	$(c_j \to c_i) \to c_i$ for each $i, j \in \mathbb{N}_0$ s.t. $a_i < a_j$

The deduction rule is modus ponens.

Definition 2.3. *Let $*$ be a continuous t-norm and $EP(*)$ the set of its endpoints. A $BL_{EP(*)}$-algebra is a structure for the language of BL-algebras expanded with a set $\mathcal{S}_*$ of constants that makes valid all the axioms of $BL_{EP(*)}$, evaluating $e(c_i) = s_i, i \in \mathbb{N}_0, s_i \in \mathcal{A}$ for all evaluations e.*

$BL_{EP(*)}$-algebras are defined by a set of propositional formulas and therefore form a variety in the given language.

By a *standard* $BL_{EP(*)}$-algebra we mean an algebra which is standard and belongs to the variety generated by $[0,1]_*$.

Denoting $s_i = e(c_i)$ for all $i \in \mathbb{N}_0$ and S^{ep} be the set of all s_i, which are idempotent elements but not necessarily endpoints of Ł-, G- or Π-components in $\mathbf{A}$, the ordering of s_i is as follows:

Observation 2.4. *Let $*$ be a continuous t-norm, EP the set of its endpoints, $\mathbf{A}$ a $BL_{EP(*)}$-chain and $s_i = e(c_i)$ in $\mathbf{A}$. Assume $a_i, a_j, a_k \in EP$. Then*

(i) if $a_i < a_j$ in $[0,1]_*$, then $s_i \leq s_j$ in $\mathbf{A}$;

(ii) if $s_i, s_j < 1$ in $\mathbf{A}$ and $a_i < a_j$ in $[0,1]_*$, then $s_i < s_j$ in $\mathbf{A}$.

Proof. (i) Follows from the fact that $EP_3^{i,j}$ is valid in $\mathbf{A}$.
(ii) We have $s_i \leq s_j < 1$ by assumptions. Moreover, both s_i and s_j are idempotents in $\mathbf{A}$. Then the axiom $EP_4^{i,j}$ yields the truth value 1 iff $s_i < s_j$ (in that case, $s_j \Rightarrow s_i = s_i$; if $s_i = s_j < 1$, we have $s_j \Rightarrow s_i = 1$ and $1 \Rightarrow s_i = s_i$). $\mathcal{QED}$

Theorem 2.5. (Completeness) *Let $*$ be a continuous t-norm and EP($*$) the set of its endpoints. Let φ be a formula in the language of $BL_{EP(*)}$. Then the following are equivalent:*

- *(i)* $\vdash_{BL_{EP}} \varphi$
- *(ii) φ holds in any $BL_{EP(*)}$-algebra A*
- *(iii) φ holds in any $BL_{EP(*)}$-chain A.*

Proof. By inspection of the completeness proof for BL-algebras. $\mathcal{QED}$

Theorem 2.6. (Standard completeness) *Let $*$ be a continuous t-norm and $EP(*)$ be the set of its endpoints. Let φ be a formula in the language of $BL_{EP(*)}$. Then $BL_{EP(*)} \vdash \varphi$ iff φ holds in all standard $BL_{EP(*)}$-algebras.*

Proof. Assume φ is not provable in $BL_{EP(*)}$; then by Theorem 2.5 there is a $BL_{EP(*)}$-chain $\mathbf{A}$ in which φ does not hold under some evaluation e. We may assume $\mathbf{A}$ saturated. Let $\{v_1, \ldots, v_m\}$ be the values of all subformulas of φ under e. Each v_i, $i = 1, dots, m$, either is an idempotent of $\mathbf{A}$, or belongs to some L- or Π-component of A. Let V be a subset of the domain of $\mathbf{A}$ which contains, for each $i = 1, \ldots, m$ v_i whenever v_i is idempotent of $\mathbf{A}$, and the delimiting idempotent endpoints of v_i if it is not idempotent. Then V is a finite subset of the domain of $\mathbf{A}$. Denote $S = \{s_i = e(c_i), i \in \mathrm{N}_0\}$. Consider $S \cup V$ as a set ordered with the ordering of $\mathbf{A}$. Embed this ordered set into $[0,1]$ (with 1-1 embedding). Then it is obvious that one can define on $[0,1]$ L- and Π-components corresponding by their types to those components of $\mathbf{A}$ which are delimited by elements of V; denote this algebra B. (Propositional constants are evaluated by the s_i-images in B.) This shows that the counterexample evaluation of φ can be embedded into $[0,1]$, where it yields value less than 1.

Finally, B is a standard BL_{EP}-algebra. $\mathcal{QED}$

3 Axioms for components

We suggest a way of describing the isomorphism type of the intervals inbetween endpoints (Ł, G, Π) by means of a suitable translation of formulas. For each particular continuous t-norm, the ultimate goal of the endeavour is to find a complete axiomatics for the BL_{EP}-algebra given by it. We retain the terminology and notation from the previous section, i.e., each continuous t-norm $*$ determines the set EP of its endpoints, as well as their enumeration $\mathbb{N}_0$ and the corresponding set of truth constants $\mathcal{C}$.

Assume $*$ is given. For each $i \in \mathbb{N}_0$, we define a translation function, operating on formulas of the language of BL. The result of the translation of a formula φ will be denoted $\varphi^{[c_i, {c_i}^+]}$. The translation function is defined by induction on the formula structure as follows:

$$\overline{0}^{[c_i,c_i^+]} = c_i$$
$$\overline{1}^{[c_i,c_i^+]} = c_i^+$$
$$p^{[c_i,c_i^+]} = (p \vee c_i) \wedge c_i^+$$
$$(\varphi \& \psi)^{[c_i,c_i^+]} = \varphi^{[c_i,c_i^+]} \& \psi^{[c_i,c_i^+]}$$
$$(\varphi \to \psi)^{[c_i,c_i^+]} = (\varphi^{[c_i,c_i^+]} \to \psi^{[c_i,c_i^+]}) \wedge c_i^+$$

Observation 3.1. *Let $*$ be a continuous t-norm, EP($*$) the set of its endpoints. For any $i \in \mathbb{N}_0$ and any φ, the following holds:*

(i) for any evaluation e we have $e(\varphi^{[c_i,c_i^+]}) \in [c_i, c_i^+]$

(ii) $\varphi^{[c_i,c_i^+]} \equiv ((\varphi^{[c_i,c_i^+]} \vee c_i) \wedge c_i^+)$

(iii) $\varphi^{[c_i,c_i^+]} \equiv (\varphi^{[c_i,c_i^+]} \,\&\, c_i^+)$

Proof. (i) by induction on formula structure, using the above definition of the $[c_i, c_i^+]$-translation function; (ii) follows from (i) by completeness; (iii) follows from (i) by virtue of basic facts on behaviour of idempotent elements of $*$. $\mathcal{QED}$

Observation 3.2. *Let $*$ be a continuous t-norm, EP($*$) the set of its endpoints. Then for any $i \in \mathbb{N}_0$:*

$$(\neg\varphi)^{[c_i,c_i^+]} \text{ is } (\varphi^{[c_i,c_i^+]} \to c_i) \wedge c_i^+$$
$$(\varphi \wedge \psi)^{[c_i,c_i^+]} \text{ is } \varphi^{[c_i,c_i^+]} \wedge \psi^{[c_i,c_i^+]}$$
$$(\varphi \vee \psi)^{[c_i,c_i^+]} \text{ is } (((\varphi^{[c_i,c_i^+]} \to \psi^{[c_i,c_i^+]}) \wedge c_i^+) \to \psi^{[c_i,c_i^+]}) \wedge (((\psi^{[c_i,c_i^+]} \to \varphi^{[c_i,c_i^+]}) \wedge c_i^+) \to \varphi^{[c_i,c_i^+]}) \wedge c_i^+$$

Proof. (and) $(\varphi \wedge \psi)^{[c_i,c_i^+]}$ is
$(\varphi \,\&\, (\varphi \to \psi))^{[c_i,c_i^+]}$, which is by definition
$\varphi^{[c_i,c_i^+]} \,\&\, ((\varphi^{[c_i,c_i^+]} \to \psi^{[c_i,c_i^+]}) \wedge c_i^+)$, which distributes to
$(\varphi^{[c_i,c_i^+]} \,\&\, (\varphi^{[c_i,c_i^+]} \to \psi^{[c_i,c_i^+]})) \wedge (\varphi^{[c_i,c_i^+]} \,\&\, c_i^+)$, which, using the above lemma, is equivalent to
$\varphi^{[c_i,c_i^+]} \wedge \psi^{[c_i,c_i^+]}$. $\mathcal{QED}$

Theorem 3.3. *Let $*$ be a continuous t-norm, EP($*$) the set of its endpoints, and A the BL_{EP}-algebra given by $*$ on $[0,1]$. Let $i \in \mathbb{N}_0$ be such that $*$ on $[c_i, c_i^+]$ is isomorphic to the Łukasiewicz t-norm (the Gödel t-norm, the product t-norm respectively). Then $\varphi \equiv \psi$ is a tautology of $[0,1]_{\text{Ł}}$ ($[0,1]_{\text{G}}$, $[0,1]_{\Pi}$ respectively) iff $\varphi^{[c_i,c_i^+]} \equiv \psi^{[c_i,c_i^+]}$ is a tautology of A.*

Proof. Let us assume that $*$ on $[c_i, c_i^+]$ is isomorphic to Łukasiewicz t-norm; the proofs for Gödel and product are analogous.

On $[c_i, c_i^+]$ define $x \Rightarrow^{[c_i,c_i^+]} y = \min\{x \Rightarrow y, c_i^+\}$. Then it is obvious that $S^i = ([c_i, c_i^+], c_i, c_i^+, *, \Rightarrow^{[c_i,c_i^+]})$ is an MV-algebra isomorphic to $[0,1]_{\text{Ł}}$. Suppose φ, ψ are two formulas of at most n free variables $p_1, \dots, p_n$. Assume $v_1, \dots, v_n \in [0,1]$. Observe that $\varphi^{[c_i,c_i^+]}(p_1/v_1, \dots, p_n/v_n)$ yields the same value in $[0,1]_*$ as $\varphi(p_1/(v_1 \vee c_i) \wedge c_i^+, \dots, p_n/(v_n \vee c_i) \wedge c_i^+)$ evaluated in S^i, by definition of the $[c_i, c_i^+]$-translation function. Thus, if two formulas φ and ψ are equal in $[0,1]_{\text{Ł}}$ in all evaluations, so will their translations $\varphi^{[c_i,c_i^+]}$ and $\psi^{[c_i,c_i^+]}$ be in $[0,1]_*$. Vice versa, if $\varphi^{[c_i,c_i^+]}$ and $\psi^{[c_i,c_i^+]}$ are equal under all evaluations in $[0,1]_*$, they are equal under all evaluations in $[c_i, c_i^+]$, thus φ and ψ are equal under all evaluations in $[0,1]_{\text{Ł}}$. $\mathcal{QED}$

In particular, if φ is a tautology of $[0,1]_{\text{Ł}}$ ($[0,1]_{\text{G}}$, $[0,1]_\Pi$ respectively), and the interval $[a_i, a_i^+]$ in $*$ is an Ł-component (G-component, Π-component respectively), then $\varphi^{[c_i,c_i^+]} \equiv c_i^+$ is a tautology of the $BL_{EP(*)}$-algebra given by $*$.

Let (Ł) denote the additional axiom $\neg\neg\varphi \to \varphi$ of Łukasiewicz logic, (G) denote the axiom $\varphi \to \varphi \& \varphi$ of Gödel logic, and (Π) denote the axiom $(\varphi \to \chi) \vee ((\varphi \to (\varphi \,\&\, \psi)) \to \psi)$ of product logic. For $i \in N$, denote

Ł^i the formula $\text{Ł}^{[c_i,c_i^+]} \equiv c_i^+$

G^i the formula $\text{G}^{[c_i,c_i^+]} \equiv c_i^+$

Π^i the formula $\Pi^{[c_i,c_i^+]} \equiv c_i^+$

We refine the calculus BL_{EP} with a specification of the isomorphism type of each of the components of $*$.

Definition 3.4. *Let $*$ be a continuous t-norm, EP the set of its endpoints. The logic $BL_{COMP(*)}$ has as axioms the axioms of $BL_{EP(*)}$ plus the following formulas, for all $i \in \mathbb{N}_0$:*

$(COMP_{\text{Ł}}^i)$	L^i whenever $[a_i, a_i^+]$ in $[0,1]_*$ is a copy of $[0,1]_{\text{Ł}}$
$(COMP_G^i)$	G^i whenever $[a_i, a_i^+]$ in $[0,1]_*$ is a copy of $[0,1]_G$
$(COMP_\Pi^i)$	Π^i whenever $[a_i, a_i^+]$ in $[0,1]_*$is a copy of $[0,1]_\Pi$

The deduction rule is modus ponens.

As in the case of the logic BL_{EP}, one can state a completeness theorem w. r. t. (linearly ordered) BL_{COMP}-algebras, and also a standard completeness theorem w. r. t. all standard BL_{COMP}-algebras. It remains open whether the logic BL_{COMP} is complete with respect to the single standard BL_{COMP}-algebra given by $*$.

4 Complexity issues

We analyze the computational complexity of the set of propositional 1-tautologies of each of the BL_{EP}-algebras given by $*$.

If $*$ is a finite ordinal sum, we show that the set of propositional 1-tautologies of $[0,1]_*$ in the language enriched with the constants $\mathcal{C}$ is in coNP (in fact, it is coNP-complete).

Next we address infinite sums. Although there exist infinite sums whose sets of tautologies (in the language of BL_{EP}-algebras are in coNP, it is also true that some others are undecidable. There are (classes of) standard algebras which are infinite sums with a less favourable ordering/numbering of delimiting idempotents and whose sets of 1-tautologies in the enriched language are non-arithmetical.

Let $*$ be a continuous t-norm and $\mathbf{A}$ be the standard $BL_{EP(*)}$-algebra given by $*$. One may distinguish the following sets of formulas (in all cases φ stands for a propositional formula in the BL-language with constants ($\mathcal{C}$) and $e_{\mathbf{A}}$ runs over evaluations in $\mathbf{A}$).

$$\begin{aligned}
\mathrm{TAUT}_1^{\mathbf{A}} &= \{\varphi : \forall e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) = 1)\} \\
\mathrm{TAUT}_{\mathrm{pos}}^{\mathbf{A}} &= \{\varphi : \forall e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) > 0)\} \\
\mathrm{SAT}_1^{\mathbf{A}} &= \{\varphi : \exists e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) = 1)\} \\
\mathrm{SAT}_{\mathrm{pos}}^{\mathbf{A}} &= \{\varphi : \exists e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) > 0)\}
\end{aligned}$$

These sets are referred to as 1-tautologies, positive tautologies, 1-satisfiable formulas and positively satisfiable formulas of $\mathbf{A}$.

For a class K of algebras of the same type, one may generalize:

$$\begin{aligned}
\mathrm{TAUT}_1^{K} &= \{\varphi : \forall \mathbf{A} \in K\, \forall e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) = 1)\} \\
\mathrm{TAUT}_{\mathrm{pos}}^{K} &= \{\varphi : \forall \mathbf{A} \in K\, \forall e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) > 0)\} \\
\mathrm{SAT}_1^{K} &= \{\varphi : \exists \mathbf{A} \in K\, \exists e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) = 1)\} \\
\mathrm{SAT}_{\mathrm{pos}}^{K} &= \{\varphi : \exists \mathbf{A} \in K\, \exists e_{\mathbf{A}}(e_{\mathbf{A}}(\varphi) > 0)\}
\end{aligned}$$

4.1 Finite ordinal sums

If the number of Ł-, G-, and Π-components in a continuous t-norm is finite, then so is the number of its endpoints and the computational situation is straightforward, regardless of the enumeration a of endpoints.

Let $*$ be a continuous t-norm which is a finite ordinal sum of Ł-, G-, and Π-components, A the BL_{EP}-algebra given by $*$, and n the number of components in A. W. l. o. g., we may assume that the endpoints in A are enumerated in increasing order (w. r. t. their real ordering), a_0 being 0 and a_n being 1.

For any formula φ, let $|\varphi|$ denote the number of occurrences of propositional variables, and denote $m = 2|\varphi| - 1$ (so m is the number of subformulas in φ). Fix an enumeration of all subformulas of φ; assume φ gets the index 1.

Theorem 4.1. TAUT_1^A *is a co-NP-complete set.*

Proof. TAUT_1^A is trivially coNP-hard as the tautologies of $[0,1]_*$ (the standard algebra in a language without constants), which is a coNP-hard set, can be reduced to it (using identity).

We use a modification of the algorithm in [3], which is a nondeterministic acceptor of non-tautologies of A, running in polynomial time w. r. t. m (the size of the input φ). This will entail that the 1-tautologies of A are in coNP.

`nameSubformulas()` Introduce variables $x_1 \ldots, x_m$, and assign the variable x_i to the subformula φ_i of φ (x_1 is assigned to φ).

Set $V = \{a_0, \ldots, a_n\} \cup \{x_1, \ldots, x_m\}$.

`guessOrder()` Guess a linear ordering $\preceq$ of elements of V, such that $x_1 \prec a_n = 1$.

`checkOrder()` Check that $\preceq$ satisfies basic natural conditions: first, that it preserves the strict ordering of the endpoints a_i on the real unit interval,
second, any variable assigned to the constant 0 must be $\approx$-equal to a_0, the variable denoting the least endpoint.

We say that variables x_j s. t. $a_i \preceq x_j \preceq a_{i+1}$ *belong to* i.

`checkExternal()` Check external soundness of $\preceq$: for φ_i, φ_j subformulas of φ ($1 \leq i, j \leq m$),

– if $\varphi_i \,\&\, \varphi_j$ is a subformula φ_k of φ for some $k \in \{1, \ldots, m\}$ and, for some $l \in \{0, \ldots, n\}$, we have $x_i \preceq a_l \preceq x_j$, then $x_k \approx x_i$;

– if $\varphi_i \to \varphi_j$ is a subformula φ_k of φ for some $k \in \{1, \ldots, m\}$ and $x_i \preceq x_j$, then $x_k \approx a_n$;

– if $\varphi_i \to \varphi_j$ is a subformula φ_k of φ for some $k \in \{1, \ldots, m\}$ and for some $l \in \{0, \ldots, n\}$, we have $x_j \prec a_l \preceq x_i$, then $x_k \approx x_j$.

`checkInternal()` Check internal soundness of $\preceq$ for each interval $[a_i, a_{i+1}]$, $i = 0, \ldots, n-1$ in $\preceq$. Consider variables in i. Construct a system $\mathcal{S}_i$ of equations and inequalities; $\mathcal{S}_i$ is initially empty. For each subformula φ_l which is $\varphi_j \,\&\, \varphi_k$, if x_j and x_k are in i, check x_l is also in i and put equation $x_j * x_k = x_l$ into $\mathcal{S}_i$. For each subformula φ_l which is $\varphi_j \to \varphi_k$, such that $x_k \prec x_j$, if x_j and x_k are in i, check x_l is also in i and put equation $x_j \Rightarrow x_k = x_l$ into $\mathcal{S}_i$.

Further, put all equations and inequalities defined by $\preceq$ for the variables in i into $\mathcal{S}_i$. Check whether the system $\mathcal{S}_i$ has a solution in the i-th component of $\mathbf{A}$.

`end`

It is shown in [3] that the last check can be performed (nondeterministically) in polynomial time w.r.t. the size of $\mathcal{S}$, for all three types of basic components. This concludes the proof.

$\mathcal{QED}$

Now we examine continuous t-norms with infinitely many endpoints.

First, an example of a continuous t-norm which is an infinite ordinal sum for which the set of 1-tautologies of the corresponding BL_{EP}-algebra is coNP-complete.

Lemma 4.2. *Let $*$ be a continuous t-norm which is an infinite sum of Ł-components, with endpoints enumerated by ω. Then the set of 1-tautologies of the resulting BL_{EP}-algebra is coNP-complete.*

Proof. Obvious. $\mathcal{QED}$

However, the following statement holds for ordinal sums whose endpoints are ordered and enumerated by ω, but with both Ł- and Π-components.

Observation 4.3. *Let S be any subset of N. Let $*$ be a continuous t-norm with endpoints ordered and enumerated by ω. Assume $*$ has two types of components Ł and Π, and the distribution of these copies the characteristic function of S in such a way that Łstands for 1 whereas Π stands for 0. . Then char(S) is reducible to TAUT(A).*

Proof. Take a formula λ which is valid in $[0,1]_{\text{Ł}}$ but not in $[0,1]_{\Pi}$. Then one can reduce membership is S to tautologousness in $[0,1]_*$ by asking, for a given $i \in S$, about the validity of of $\lambda^{[c_{i-1},c_i]}$ in the BL_{EP}-algebra given by $*$. $\mathcal{QED}$

The latter statement not only shows that tautologies of BP_{EP}-algebras can be placed arbitrarily high in the arithmetical hierarchy, but also that they can be non-arithmetical.

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