Properties of Fuzzy Logical Operations

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Abstract

We deal with geometrical and differential properties of triangular norms (t-norms for short), i.e. binary operations which implement logical conjunctions in fuzzy logic. The first part discusses the problem of a visual characterization of the associativity of t-norms. The results given by web geometry are adopted, mainly the concept of the Reidemeister closure condition, in order to characterize the shape of level sets of t-norms. This way, a visual characterization of the associativity is provided for general, continuous, and continuous Archimedean t-norms. The second part deals with differential properties of continuous Archimedean t-norms. It is shown that partial derivatives of such a t-norm on a particular subset of its domain correspond directly to the generator (or to the derivative of the generator) of the t-norm. As the result, several methods which reconstruct multiplicative and additive generators of continuous Archimedean t-norms are introduced. The presented results contribute to a partial solution of an open problem whether a non-trivial convex combination of two t-norms can be a triangular norm again.

1. Introduction

The fuzzy logic has been proposed as an alternative to the classical Boolean logic. The notion “fuzzy” was firstly introduced in 1965 by Zadeh in his paper [40] where he defined fuzzy logic and fuzzy sets.

The main idea of the fuzzy logic is to enlarge the set of truth values, i.e. 0 and 1 (false and true), to the real unit interval $[0,1]$. In comparison to the classical logic where a statement can be either true or false, the generalization to the fuzzy logic allows to express also a partial truth of a statement as it admits degrees of truth.

Generalization of the set of truth values hangs together with a generalization of the logical operations. The logical conjunction is usually implemented by a *triangular norm* (shortly, a *t-norm*). Although the notion of a t-norm was originally introduced within the framework of probabilistic metric spaces [37], it has found a successful application in fuzzy logic. The currently studied fuzzy logics, as will be described in the sequel, are primarily based on t-norms. Another important logical connective, the implication, is usually implemented by a *residuum* (also *residuated implication*) which is derived from a t-norm in order to form an adjoint pair and work correctly in the generalized Modus Ponens rule.

The logical calculus which is able to cope with partially true statements is called a fuzzy or many-valued logic. The beginning of many-valued reasoning dates back to 1920 when Łukasiewicz proposed his three-valued logic [23] and to the work of Post [36] in 1921. Now, one of the most successful fuzzy logics is the *Basic Fuzzy Logic* (BL for short) which has been introduced by Hájek [15] and fully described in his monograph [16]. We remark that BL includes the fuzzy logics, known so far at the time of its introduction, as its special cases. The semantical counterpart of BL is represented by BL-algebras which play an analogous role as Boolean algebras for the classical Boolean logic. An example of a BL-algebra is the real unit interval $[0,1]$ endowed with a continuous t-norm which represents a conjunction and the corresponding residuum which represents an implication. Such a BL-algebra is called a standard BL-algebra. Hájek proved that BL is sound and complete with respect to the class of BL-algebras. This means that a formula is provable in BL if and only if it is a tautology in all BL-algebras. BL is complete even with respect to standard BL-algebras. This fact is known as the *Standard Completeness Theorem* of BL [11].

2. Preliminaries

We present here some basic facts about triangular norms. The proofs and more details can be found e.g. in the monographs on triangular norms [7, 20]. Another good introduction to triangular norms can also be given by monographs on fuzzy sets and fuzzy logic [22, 30].

Definition 2.1 *A* triangular norm *(a* t-norm *for short) is a binary operation $T\colon [0,1]\times[0,1]\to[0,1]$ such that for all $x,y,z\in[0,1]$ the following axioms are satisfied:*

(T1) $T(x,y)=T(y,x)\,,$ *(commutativity)*

(T2) $T(x,T(y,z))=T(T(x,y),z)\,,$ *(associativity)*

(T3) $x\le y\Rightarrow T(x,z)\le T(y,z)\,,$ *(monotonicity)*

(T4) $T(x,1)=x\,.$ *(neutral element)*

The three most common t-norms are the *minimum t-norm*, $T_{\mathbf{M}}(x,y)=\min\{x,y\}$, the *Łukasiewicz t-norm*, $T_{\mathbf{L}}(x,y)=\max\{x+y-1,0\}$, and the *product t-norm*, $T_{\mathbf{P}}(x,y)=x\cdot y$.

A continuous t-norm T is called *Archimedean* if $T(x,x)<x$ for all $x\in\,]0,1[$. A t-norm which is continuous and strictly increasing on the half-open square $]0,1]^2$ is said to be *strict*; such a t-norm is always Archimedean. A continuous Archimedean t-norm is called *nilpotent* if it is not strict. Thus every continuous Archimedean t-norm is either strict or nilpotent. For example, the product t-norm is strict, the Łukasiewicz t-norm is nilpotent, and the minimum t-norm is an example of a continuous t-norm which is not Archimedean.

Every continuous Archimedean t-norm can be represented by a one-dimensional real function called *generator*. This result is formalized by the *Representation Theorem* [1, 14, 21, 27]:

Theorem 2.2 (Representation Theorem) *For a function $T\colon [0,1]^2\to[0,1]$ the following statements are equivalent:*

1. *T is a continuous Archimedean t-norm.*
2. *T has a continuous* additive generator, *i.e., there exists a continuous strictly decreasing function $t\colon[0,1]\to[0,\infty]$ with $t(1)=0$ such that $T(x,y)=t^{(-1)}\left(t(x)+t(y)\right)$ holds for all $(x,y)\in[0,1]^2$. Here, $t^{(-1)}$ denotes the pseudo-inverse of t which is (in this case) defined as:*

$$t^{(-1)}(y)=\begin{cases}0 & \text{if}\quad y>t(0)\,,\\ t^{-1}(y) & \text{if}\quad y\le t(0)\,.\end{cases}$$

3. *T has a continuous* multiplicative generator, *i.e., there exists a continuous strictly increasing function $\theta\colon[0,1]\to[0,1]$ with $\theta(1)=1$ such that $T(x,y)=\theta^{(-1)}\left(\theta(x)\cdot\theta(y)\right)$ holds for all $(x,y)\in[0,1]^2$. Here, $\theta^{(-1)}$ denotes the pseudo-inverse of θ which is (in this case) defined as:*

$$\theta^{(-1)}(y)=\begin{cases}0 & \text{if}\quad y<\theta(0)\,,\\ \theta^{-1}(y) & \text{if}\quad y\ge\theta(0)\,.\end{cases}$$

The *support* of a binary operation $T\colon [0,1]^2\to[0,1]$, denoted by $\operatorname{supp}T$, is the closure of the set

$$\left\{(x,y)\in[0,1]^2 \mid T(x,y)>0\right\}\,.$$

3. Current situation of the studied problem

3.1. Convex combinations of t-norms

This work has been primarily inspired by the long standing open problem of convex combinations of triangular norms and summarizes the results which have been achieved while solving this problem. This problem has been formulated, for example, in the list of open problems by Alsina, Frank, and Schweizer [6]:

Problem 3.1 *Is the arithmetic mean, or for that matter any convex combination, of two distinct t-norms ever a t-norm?*

We recall that a convex combination of two t-norms T_1, T_2 is a function $F=\alpha T_1+(1-\alpha)T_2$ where $\alpha\in[0,1]$. It is immediate that for trivial convex combinations, i.e. for $\alpha\in\{0,1\}$ or for $T_1=T_2$, the answer is positive. A positive example can be given even for non-trivial convex combinations of non-continuous t-norms [17, 34, 39]. For example, let T_1 be an ordinal sum of the product t-norm $T_{\mathbf{P}}$ on the carrier $[0,\frac{1}{2}]$. Let T_2 be a binary operation on $[0,1]$ such that $T_2(x,y)=0$ for $x,y\in[0,\frac{1}{2}]$ and $T_2(x,y)=\min\{x,y\}$ otherwise. It is easy to check that T_2 is a left-continuous t-norm. Observe now that any convex combination of T_1 and T_2 is a left-continuous t-norm. However, for continuous t-norms the problem still has not been answered completely although it is conjectured that for the continuous t-norms the answer to the question posed in Problem 3.1 is “never” [6].

Thus, in order to exclude the trivial cases mentioned above, whenever we write “convex combination” we mean a function $\alpha T_1 + (1-\alpha) T_2$ where $\alpha \in \,]0,1[$, $T_1 \neq T_2$, and both t-norms are continuous.

In the rest of this section we briefly outline the results related to the convex combinations of t-norms which have been done so far. In the historically first paper dealing with this problem, Tomás [38] has given a result on strict t-norms under additional (and rather restrictive) constraints. In the papers by Ouyang, Fang and Li [31, 32], the whole class of continuous t-norms is treated under no additional assumptions. For example, they prove [31] that a convex combination of a continuous Archimedean t-norm and a continuous non-Archimedean t-norm is never a t-norm. In other words, if a convex combination of two continuous t-norms is a t-norm again, then both combined t-norms are ordinal sums with the same structure of summand carriers. By this result, in order to clarify the convex structure of the class of continuous t-norms it is sufficient to clarify the convex structure of the class of continuous Archimedean t-norms. By another result of theirs [31], a convex combination of a strict and a nilpotent t-norm is never a t-norm. Thus even the latter task can be subdivided into solving the convex structure of the nilpotent class and of the strict class separately. Another result is due to Jenei [17] and applies to all pairs of left-continuous t-norms with an additional property that both t-norms share an involutive level set. An immediate consequence of this result is that a convex combination of two nilpotent t-norms, T_1 and T_2, such that $\operatorname{supp} T_1 = \operatorname{supp} T_2$, is never a t-norm. Let us mention also the recent result by Mesiar and Mesiarová-Zemánková [26] where it is stated that a convex combination of two continuous t-norms with the same diagonal is never a t-norm. (We recall that a *diagonal* of a t-norm T is the function $x \mapsto T(x,x)$.)

Two new, recently published [33, 34], results on this topic are presented here. Using a web-geometrical approach to describe associativity of t-norms, it is proven that any convex combination of two nilpotent t-norms is never a t-norm. Furthermore, using an idea of reconstruction of generators according to partial derivatives of t-norms, several new results on the problem of convex combinations of strict t-norms are presented.

3.2. Associativity of t-norms

The commutativity, the non-decreasingness and the existence of a neutral element have an easy graphical interpretation. However, the question how to visually interpret the associativity is a long-standing open problem within the community of people dealing with t-norms. Some results have been done, mainly thanks to the effort of Jenei [18], and Maes and De Baets [24, 25], yet a satisfactory answer to the question still has not been given.

The theory of *web geometry* [9, 2, 3, 4] has come with results which answer such, and similar, kinds of questions in a rather intuitive way. In particular, associative loops are characterized by the *Reidemeister closure condition*. These results were, however, done to characterize algebraic properties of loops. Although t-norms do not form loops, there are, fortunately, some similarities between t-norms and loops (monotonicity, neutral element, ...). We will show that some modifications of the Reidemeister closure condition can still be applied to t-norms in order to characterize their associativity.

Motivation 3.2 *Consider the Łukasiewicz t-norm,* $T_{\mathbf{L}}(x,y) = \max\{x+y-1, 0\}$. *The structure of its level sets is extremely simple as they are formed by parallel lines.*

Notice the following easy property of these sets: draw a rectangle (by vertical and horizontal lines) anywhere in the support of the operation and denote the level sets passing through the vertices of the rectangle. Now draw another rectangle such that three of its vertices match the three distinct denoted level sets. The fourth vertex of the rectangle shall, naturally, match the fourth denoted level set.

The property described in Motivation 3.2 characterizes associativity and corresponds to the Reidemeister closure condition introduced by web geometry [9, 2, 3, 4].

3.3. Reconstruction of generators

When a continuous (multiplicative or additive) generator is defined, it is easy to construct the corresponding (continuous Archimedean) t-norm. The reverse task, however, is not so trivial. One way how to obtain a generator of a continuous Archimedean t-norm is to use the proof of the Representation Theorem. This proof is constructive, however, it does not need to result in an explicit formula of the generator. This significantly reduces the usability of this method. Another possibility is to use the results given by Pi-Calleja [5, 35] and by Craigen and Páles [12]. Both these results give explicit formulas for additive generators of strict t-norms. However, the computations of formulas are rather non-intuitive and non-straightforward which disallows an easy usage. The formulas also show no direct relation between t-norms and their generators.

In this work, an alternative [28, 29] is presented. It is shown that partial derivatives of t-norms admit to obtain formulas for generators in a closed form. As the partial derivatives need not exist, this approach cannot be applied to all continuous Archimedean t-norms, but it seems general enough for all practical applications. It is even shown that every continuous t-norm can be approximated (with an arbitrary precision) by a t-norm from the class of strict t-norms on which one of the introduced methods is applicable. An advantage of this approach is that it relates (the shape of) the generator directly to (the shape of) the t-norm and that it is based on the basic differential calculus which makes the computational procedure straightforward. Benefiting from the fact that computation with the first derivatives is well described and can be well algorithmized, these methods can be easily applicable both by a manual computation and by computational systems. Furthermore, a simplified proof of the Representation Theorem for a subclass of strict t-norms is given as one of the results based on this approach.

4. Results

4.1. Associativity of t-norms

Let $F\colon [0,1]^2 \to [0,1]$ be a commutative and non-decreasing binary operation satisfying $F(x,1) = x$ for all $x \in [0,1]$.

By a *rectangle* we mean a set of four points $\mathbf{R} = \{x_1^{\mathbf{R}}, x_2^{\mathbf{R}}\} \times \{y_1^{\mathbf{R}}, y_2^{\mathbf{R}}\} \subset [0,1]^2$ where $x_1^{\mathbf{R}} \le x_2^{\mathbf{R}}$ and $y_1^{\mathbf{R}} \le y_2^{\mathbf{R}}$. Let $\mathbf{P}, \mathbf{R} \subset [0,1]^2$ be two rectangles. We say that $\mathbf{P} \approx_F \mathbf{R}$ if and only if $F\left(x_i^{\mathbf{P}}, y_j^{\mathbf{P}}\right) = F\left(x_i^{\mathbf{R}}, y_j^{\mathbf{R}}\right)$ for all $i, j \in \{1,2\}$; $\mathbf{P} \sim_F^{k,l} \mathbf{R}$ if and only if the equality $F\left(x_i^{\mathbf{P}}, y_j^{\mathbf{P}}\right) = F\left(x_i^{\mathbf{R}}, y_j^{\mathbf{R}}\right)$ is violated for at most $i = k$ and $j = l$; $\mathbf{P} \sim_F \mathbf{R}$ if and only if the equality $F\left(x_i^{\mathbf{P}}, y_j^{\mathbf{P}}\right) = F\left(x_i^{\mathbf{R}}, y_j^{\mathbf{R}}\right)$ is violated for at most one combination of i and j. Clearly, $\approx_F$, $\sim_F^{k,l}$, and $\sim_F$ are equivalences, $\approx_F$ is a subrelation of $\sim_F^{k,l}$, and $\sim_F^{k,l}$ is a subrelation of $\sim_F$ for any $k, l \in \{1,2\}$.

Theorem 4.1 *Let $T\colon [0,1]^2 \to [0,1]$ be a non-decreasing, commutative binary operation which satisfies $T(x,1) = x$ for every $x \in [0,1]$.*

- *T is associative if and only if $\mathbf{P} \sim_T^{1,1} \mathbf{R}$ implies $\mathbf{P} \approx_T \mathbf{R}$ for every pair of rectangles, $\mathbf{P}$ and $\mathbf{R}$, such that $\mathbf{P} = \{x_1^{\mathbf{P}}, x_2^{\mathbf{P}}\} \times \{y_1^{\mathbf{P}}, 1\} \subset [0,1]^2$ and $\mathbf{R} = \{x_1^{\mathbf{R}}, 1\} \times \{y_1^{\mathbf{R}}, y_2^{\mathbf{R}}\} \subset [0,1]^2$.*
- *If T is continuous then it is associative if and only if $\mathbf{P} \sim_T^{1,1} \mathbf{R}$ implies $\mathbf{P} \approx_T \mathbf{R}$ for every pair of rectangles, $\mathbf{P}, \mathbf{R} \subset [0,1]^2$.*
- *If T is continuous and Archimedean then it is associative if and only if $\mathbf{P} \sim_T \mathbf{R}$ implies $\mathbf{P} \approx_T \mathbf{R}$ for every pair of rectangles, $\mathbf{P}, \mathbf{R} \subset \operatorname{supp} T \cap \left]0,1\right]^2$.*

4.2. Reconstruction of generators

We denote by t', θ' the derivatives of generators t, θ, respectively. We denote by

$$\begin{aligned} \mathrm{D}T(x,y) &= \lim_{h \to 0} \frac{T(x+h,y) - T(x,y)}{h} \\ &= \lim_{z \to x} \frac{T(z,y) - T(x,y)}{z - x} . \end{aligned}$$

the partial derivative of a t-norm T with respect to the first variable.

Assumption 4.2 *The partial derivative $\mathrm{D}T$ will be considered only in the support $\operatorname{supp} T$. In particular,*

$$\mathrm{D}T(1,y) = \lim_{x \to 1_-} \frac{y - T(x,y)}{1 - x}$$

is the left *partial derivative with respect to the first variable. If T is strict, then*

$$\mathrm{D}T(0,y) = \lim_{x \to 0_+} \frac{T(x,y)}{x}$$

is the right *partial derivative. For T nilpotent, we require the second argument $y > 0$; then $\mathrm{D}T(x,y)$ is defined for all $x \in [N_T(y), 1]$, in particular,*

$$\mathrm{D}T(N_T(y), y) = \lim_{z \to N_T(y)_+} \frac{T(z,y)}{z - N_T(y)} \tag{1}$$

is the right *partial derivative. Since T is nilpotent, the negation N_T is involutive. Therefore, substituting $x = N_T(y)$, we can write (1) as*

$$\mathrm{D}T(x, N_T(x)) = \lim_{z \to x_+} \frac{T(z, N_T(x))}{z - x} .$$

For T nilpotent and $y = 0$, the line $\{(x,0) \mid x \in \mathbb{R}\}$ intersects $\operatorname{supp} T$ only at a single point $(1,0)$ and $\mathrm{D}T(x,0)$ is undefined for any $x \in [0,1]$.

We say that a strict t-norm T is *annihilator-differentiable* if the function $\mathrm{D}T(0,y)$ is defined for all $y \in [0,1]$.

Theorem 4.3 (Reconstruction along annihilator) *Let T be a strict annihilator-differentiable t-norm and let $\xi\colon [0,1] \to [0,1]\colon y \mapsto \mathrm{D}T(0,y)$. Then $\xi(0) = 0$, $\xi(1) = 1$, and the restriction of ξ to $]0,1[$ is either (1) the constant 0, (2) the constant 1, or (3) a bijection on $]0,1[$. Moreover, in case (3) the function ξ is a multiplicative generator of T.*

Theorem 4.4 (Reconstruction along level set) *Let T be a continuous Archimedean t-norm. Suppose that T has an absolutely continuous additive generator with a non-zero finite derivative at some point $a \in]0,1]$. (We take the left derivative at 1.) Let $\mathrm{D}T$ be the partial derivative of T with respect to the first variable in the support $\operatorname{supp} T$. Suppose that $\mathrm{D}T(z, I_T(z,a))$ exists for almost all $z \in [a,1]$. Suppose further that $\mathrm{D}T(a, I_T(a,z))$ exists and is in $]0,\infty[$ for almost all $z \in [0,a[$. Then T has an additive generator*

$$t^*(x) = \int_x^1 v(z)\,\mathrm{d}z\,,$$

where

$$v(z) = \begin{cases} \mathrm{D}T(z, I_T(z,a)) & \text{if} \quad z \geq a\,, \\ \frac{1}{\mathrm{D}T(a, I_T(a,z))} & \text{if} \quad z < a \end{cases}$$

for almost all $z \in [0,1]$. Explicitly, if $x \geq a$ then

$$t^*(x) = \int_x^1 \mathrm{D}T(z, I_T(z,a))\,\mathrm{d}z$$

and if $x < a$ then

$$t^*(x) = \int_x^a \frac{1}{\mathrm{D}T(a, I_T(a,z))}\,\mathrm{d}z + \int_a^1 \mathrm{D}T(z, I_T(z,a))\,\mathrm{d}z\,.$$

Remark 4.5 *We admit that the function v may attain zero or infinite value at some points. Then we obtain an infinite value of t'. However, this may happen only in countably many points and this does not influence the integral defining t. The assumption of absolute continuity includes also the convergence of the integral.*

As a special case of Theorem 4.4, we obtain:

Theorem 4.6 (Reconstruction along unit) *Let T be a continuous Archimedean t-norm and let t be an additive generator of T such that t is absolutely continuous at $]0,1]$ and $t'(1) = b_{t,1} \in]-\infty,0[$. Suppose that $\mathrm{D}T(1,y) \in]0,\infty[$ for almost all $y \in]0,1]$. Then*

$$t'(y) = \frac{b_{t,1}}{\mathrm{D}T(1,y)} \qquad \text{(almost everywhere in }]0,1])$$

and

$$t(y) = \int_y^1 \frac{-b_{t,1}}{\mathrm{D}T(1,u)}\,\mathrm{d}u$$

for all $y \in]0,1]$.

Theorem 4.4 allows us to reconstruct an additive generator when a non-negative constant $a \in]0,1]$ is given. The following theorem shows that even $a = 0$ can be used. However, this works for nilpotent t-norms only.

Theorem 4.7 *Let T be a nilpotent t-norm. Suppose that T has an absolutely continuous additive generator with a non-zero finite (right) derivative at the point 0. Let $\mathrm{D}T$ be the right partial derivative of T with respect to the first variable in the support $\operatorname{supp} T$. Suppose that $\mathrm{D}T\,(z, N_T(z))$ exists for almost all $z \in [0,1]$. Then T has an additive generator*

$$t^*(x) = \int_x^1 \mathrm{D}T\,(z, N_T(z))\,\mathrm{d}z\,.$$

4.3. Convex combinations of t-norms

With the help of web geometry, the following result can be achieved:

Theorem 4.8 *Let T_1 and T_2 be two continuous Archimedean t-norms such that $\operatorname{supp} T_1 \neq \operatorname{supp} T_2$. Then no non-trivial convex combination of T_1 and T_2 is a t-norm.*

According to the result by Jenei [17], a convex combination of two nilpotent t-norms with the same support is never a t-norm. Therefore Theorem 4.8 brings the following result:

Corollary 4.9 *A non-trivial convex combination of two distinct nilpotent t-norms is never a t-norm.*

Theorem 4.8 also gives an alternative proof of the result by Ouyang and Fang [31]:

Corollary 4.10 *A non-trivial convex combination of a strict and a nilpotent t-norm is never a t-norm.*

Now, we present some results on convex combinations of strict t-norms based on the reconstruction methods. Let T be a strict annihilator-differentiable t-norm and let $\xi\colon [0,1] \to [0,1]\colon y \mapsto \mathrm{D}T(0,y)$. Then T is said to be

- *annihilator-weak* (and we write $T \in \mathcal{T}_{\mathrm{AW}}$) if $\xi(x) = 0$ for all $x \in]0,1[$,
- *annihilator-strong* (and we write $T \in \mathcal{T}_{\mathrm{AS}}$) if $\xi(x) = 1$ for all $x \in]0,1[$,
- *annihilator-reconstructible* (and we write $T \in \mathcal{T}_{\mathrm{AR}}$) if ξ is a bijection.

The set of all strict t-norms which are not annihilator-differentiable will be denoted by $\mathcal{T}_{\mathrm{N}}$.

Let T be a continuous Archimedean t-norm with a multiplicative generator θ such that θ' is continuous at 1 and $\theta'(1) \in]0,\infty[$. Then we say that T belongs to the class $\mathcal{T}_{\mathrm{UR}}$.

Proposition 4.11 *Let T_1 and T_2 belong to two distinct classes from $\mathcal{T}_{\mathrm{AR}}$, $\mathcal{T}_{\mathrm{AW}}$, $\mathcal{T}_{\mathrm{AS}}$. Then no non-trivial convex combination of T_1 and T_2 is a t-norm.*

Proposition 4.12 *Let $T_1, T_2 \in \mathcal{T}_{\mathrm{AR}} \cap \mathcal{T}_{\mathrm{UR}}$ be strict t-norms. Let $\theta_1\colon y \mapsto \mathrm{D}T\,(0,y)$ and $\theta_2\colon y \mapsto \mathrm{D}T\,(0,y)$ be multiplicative generators of T_1 and T_2, respectively.*

If a non-trivial convex combination of T_1 and T_2 is a t-norm then for each $y \in [0,1]$ at least one of the following conditions is satisfied:

$$\theta_2'(y) = \frac{\theta_2'(1)}{\theta_1'(1)}\,\theta_1'(y)\,,$$

$$\frac{\theta_1'(y)}{\theta_1(y)} = \frac{\theta_2'(y)}{\theta_2(y)}\,.$$

Corollary 4.13 *Let $T_1, T_2 \in \mathcal{T}_{\mathrm{AR}} \cap \mathcal{T}_{\mathrm{UR}}$ be two distinct strict t-norms such that their multiplicative generators, $\theta_1\colon y \mapsto \mathrm{D}T\,(0,y)$ and $\theta_2\colon y \mapsto \mathrm{D}T\,(0,y)$, are absolutely continuous. If there exists $a \in]0,1[$ such that $\theta_1(a) = \theta_2(a)$ then no non-trivial convex combination of T_1 and T_2 is a t-norm.*

5. Summary

We summarize here briefly the contributions of the thesis:

- Some results of web geometry, namely the Reidemeister closure condition, have been generalized also for algebras which do not form loops. (T-norms can be considered as commutative integral monoids on $[0,1]$.)
- A tool which visually characterizes associativity of general t-norms has been given.
- It has been shown that the generators or their derivatives correspond in many cases directly to the partial derivatives of continuous Archimedean t-norms. These results contribute to both practical applications (they allow a straightforward computation) and theoretical research (they give a new insight into the subject). A theoretical contribution has been, furhermore, illustrated by the results on convex combinations of strict t-norms and by the alternative proof of the Representation Theorem.
- The question of convex combinations of t-norms has been answered negatively for all nilpotent t-norms. In the case of strict t-norms, the problem has been divided into several subclasses and a possible further research has been outlined.
- We remark that the thesis also contributes to the question: “Which subsets of its domain uniquely determine an Archimedean t-norm?” Several results [8, 10, 13, 19] (and a summarization [20]) have been published giving concrete types of subsets of the unit square. Knowing functional values on the points of such a subset, an Archimedean t-norm is determined uniquely. Here a similar result is given yet the first partial derivatives are considered instead of functional values.

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