Syntactic Approach to Fuzzy Modal Logic in MTL

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Abstract

We study provability in Hilbert-style calculi obtained by adding standard modal logic axioms to the Monoidal T-norm based Logic (MTL) by automated theorem proving methods. The aim of this paper is to present some basic properties of systems K, D, T, S4 and S5 over MTL. These systems are defined in the same way as are in classical propositional logic. It is shown that many classically valid formulae become unprovable.

1. Introduction

In logic it is quite common to enrich the expressive power of given system by new logical connectives or operators. The most prominent such systems over classical propositional calculus (CPC) are modal logics, which introduce new operators formalising a necessity and a possibility. The practical importance of these logics constantly grows and are studied not only over classical logic but also over non-classical logics. Interesting candidates for such generalisations are mathematical fuzzy logics.

The basic generally studied modal logic is the minimal normal modal logic K. A similar role in mathematical fuzzy logic has, from some point of view, Esteva and Godo's Monoidal T-norm based Logic (MTL) [5], which is the logic of left-continuous t-norms and their residua.

Fuzzy (or more precisely many-valued) modal logics have already been studied in the literature, e.g., [9, 8, 6]. However, in most cases only very strong modal logics like S4 and S5 have been considered. The systematic study of modal logics starting from the minimal normal modal logic K is relatively recent, see, e.g., [3].

In [3], a semantic approach is used to build a minimal normal modal logic over finite residuated lattices. The syntactic problems which this brings are discussed in [2]. Our starting point is completely different, we are interested solely in these syntactic notions. We enrich the Hilbert-style calculus for MTL by standard modal axioms and by the methods of automated theorem proving we study provability and unprovability in obtained systems.

Similar problems were quite extensively studied in intuitionistic modal logics, for some discussions see, e.g., [12]. In [11], automated theorem proving methods, which are very similar to ours, were used to study dependencies in modal logics over CPC.

We emphasise that in this paper we only touch some basic properties. However, all of them can be proved by automated or semi-automated theorem proving methods. The work on this approach is currently in progress and a much more comprehensive paper is being planned. From these reasons and to make the paper shorter some proofs are omitted.

The paper is organised as follows. In Section 2 we set up terminology and in Section 3 we discuss the provability and unprovability of some formulae in K, D, T, S4 and S5 over MTL. The choice of studied systems and formulae is mainly influenced by [10].

2. Preliminaries

2.1. Monoidal T-norm based Logic MTL

We define standard Hilbert style calculus for the Monoidal T-norm based Logic (MTL), which consists of axioms and modus ponens as the only deduction rule. The language of MTL consists of implication ($\rightarrow$), multiplicative ($\&$) and additive ($\wedge$) conjunctions and a constant for falsity ($\overline{0}$).

Definition 2.1 *We define the monoidal t-norm based logic MTL as a Hilbert style calculus with following formulae as axioms*

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$,

(A2) $(\varphi \,\&\, \psi) \rightarrow \varphi$,

(A3) $(\varphi \,\&\, \psi) \rightarrow (\psi \,\&\, \varphi)$,

(A4a) $(\varphi \,\&\, (\varphi \rightarrow \psi)) \rightarrow (\varphi \wedge \psi)$,

(A4b) $(\varphi \wedge \psi) \rightarrow \varphi$,

(A4c) $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$,

(A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \,\&\, \psi) \rightarrow \chi)$,

(A5b) $((\varphi \,\&\, \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$,

(A6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$,

(A7) $\overline{0} \rightarrow \varphi$.

The only deduction rule of MTL is modus ponens

(MP) If φ is derivable and $\varphi \rightarrow \psi$ is derivable then ψ is derivable.

Let us note properties stated by each axiom, following [8, 5]. Axiom (A1) is the transitivity of implication. Axiom (A2) states that multiplicative conjunction implies its first member. Axiom (A3) is the commutativity of multiplicative conjunction. Axioms (A4c), (A4b) and (A4a) state that additive conjunction is commutative, implies its first member and one implication of the divisibility property. Axioms (A5a) and (A5b) represent residuation. Axiom (A6) is a variant of proof by cases, and states that if both $\varphi \rightarrow \psi$ and $\psi \rightarrow \varphi$ implies χ, then χ. Axiom (A7) states that false implies everything.

Further logical connectives—pseudo-complement negation ($\neg$), disjunction ($\vee$) and equivalence ($\equiv$)—are definable in MTL. Therefore, we read them as following abbreviations

$$\neg\varphi =_{df} \varphi \rightarrow \overline{0},$$
$$\varphi \vee \psi =_{df} ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi),$$
$$\varphi \equiv \psi =_{df} (\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi).$$

For some purposes can be suitable to have an involutive negation which we obtain by adding axiom $\neg\neg\varphi \rightarrow \varphi$ to MTL. The system so obtained is called Involutive Monoidal T-norm based Logic (IMTL). If we add the contraction axiom $\varphi \rightarrow \varphi \,\&\, \varphi$ to MTL we obtain Gödel logic (G). The last two axiomatic extensions of MTL mentioned in the paper are Hájek's Basic Logic (BL) and Łukasiewicz logic (Ł). These logics are obtained by adding the divisibility axiom $\varphi \wedge \psi \rightarrow \varphi \,\&\, (\varphi \rightarrow \psi)$ to MTL and IMTL, respectively.

The following theorems of MTL are very useful for our purposes. An interested reader can find proofs in [8].

Lemma 2.2 *The following formulae are provable in MTL:*

(F1) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$,

(F2) $\varphi \rightarrow \varphi$,

(F3) $(\varphi \,\&\, (\varphi \rightarrow \psi)) \rightarrow \psi$,

(F4) $(\varphi \rightarrow (\psi \rightarrow (\varphi \,\&\, \psi))$,

(F5) $(\varphi \wedge \psi) \rightarrow \varphi, (\varphi \wedge \psi) \rightarrow \psi, (\varphi \,\&\, \psi) \rightarrow (\varphi \wedge \psi)$,

(F6) $((\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)) \rightarrow (\varphi \rightarrow (\psi \wedge \psi))$,

(F7) $\varphi \rightarrow (\varphi \vee \psi), \psi \rightarrow (\varphi \vee \psi)$,

(F8) $((\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi)) \rightarrow ((\varphi \vee \psi) \rightarrow \chi)$,

(F9) $\varphi \rightarrow \neg\neg\varphi$,

(F10) $(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$,

(F11) $(\varphi \equiv \psi) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \chi))$,

(F12) $(\varphi \equiv \psi) \rightarrow ((\chi \rightarrow \varphi) \equiv (\chi \rightarrow \psi))$,

(F13) $(\neg\varphi \vee \neg\psi) \equiv \neg(\varphi \wedge \psi)$.

It is worth pointing out that we will restrict our attention to the syntactic aspects of MTL. To emphasise this approach we completely ignore the semantics of MTL. An interested reader can consult [5].

2.2. Modal logics

For our purposes only a very limited introduction to modal logics is needed, for a detail treatment we refer the reader to, e.g., [10, 4, 1]. We obtain modal logics by adding a unary modal necessity operator box ($\Box$) to our language. Another standard modal operator is a possibility operator diamond ($\Diamond$) which is usually defined as an abbreviation for $\neg\Box\neg$. Although in logics with a non-involutive negation ($\neg\neg\varphi\rightarrow\varphi$ is not true) this definition evidently leads to some problems, we use this approach for simplicity and to stress these problems,

$$\Diamond\varphi =_{df} \neg\Box\neg\varphi,$$

The properties of a modal operator box depends on chosen axioms. Some of the most widely studied are these:

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$,

(4) $\Box\varphi \rightarrow \Box\Box\varphi$,

(T) $\Box\varphi \rightarrow \varphi$,

(D) $\Box\varphi \rightarrow \Diamond\varphi$,

(B) $\varphi \rightarrow \Box\Diamond\varphi$,

(E) $\Diamond\varphi \rightarrow \Box\Diamond\varphi$.

We also need some derivational rules dealing with modalities. The most common is the necessitation rule

(Nec) $\varphi/\Box\varphi$.

In Table 1 are presented some of the most prominent modal logics over CPC. All of them are so called normal modal logics, which means that contains the minimal normal modal logic K.

Logic	Additional axioms and rules
K	(K) and (Nec)
D	(K), (Nec) and (D)
T	(K), (Nec) and (T)
S4	(K), (Nec), (T) and (4)
S5	(K), (Nec), (T) and (E)

Table 1: Modal logics in CPC.

We construct our systems of modal logics over MTL in the very same way as are in CPC. It, not so surprisingly, turns out that this leeds to some problems.

Let us remark that when proving that some formulae are equivalent in some modal logics over CPC, we can use the interdefinability of logical connectives, which is mostly impossible in MTL.

2.3. Automated theorem proving methods

All given results can be obtained automatically or semi-automatically by automated theorem proving. There is a well known technique for encoding a propositional Hilbert-style calculus into classical first-order logic through terms. The key idea is that formula variables are encoded as first-order variables and propositional connectives as first-order function symbols. For details, see, e.g., [13].

We used freely available software—E prover version 1.0-004 Temi[1] and finite-domain model finder Paradox 3.0[2]. No special prover setting is needed for our purposes, but can lead to great speed improvements. However, these aspects are too complex to be discussed here. Moreover, all presented proofs are easy to find for anyone familiar with the Hilbert-style calculus for MTL and counterexamples can be find completely automatically even with default setting. More complex problems are not included in this paper.

2.4. Models

The standard way to prove that some formula φ is not provable from the given set of formulae Γ is to present a model M in which all formulae from Γ are true, but formula φ is false. In our case, we will present tables with finitely many elements which are labelled by integers starting from 0. We always interpret $\overline{0}$ as 0 in a model and truth in a given model M is the maximal value in this model M, e.g., in a four element model true formulae are these with value 3. A function from atoms or formula variables to elements of model is called a valuation. The definition of a valuation can be easily extended to all formulae in a standard way. To show that Γ is true in M we must show that all formulae from Γ are true in M under all valuations. To show that φ is not true in M it is enough to find a valuation for which φ is not true in M.

We present tables for every connective separately and for better readability even for some defined connectives, but never for negation which corresponds to the first column of implication. Let us note that some formulae

[1] http://www.eprover.org/
[2] http://www.cs.chalmers.se/ koen/folkung/

have smaller counterexamples than these presented, but we tried to make the paper more compact.

3. Modal logics in MTL

3.1. K_{MTL}

The basic generally studied modal logic is K. This system is obtained by adding an axiom which distribute box over implication and the necessitation rule. In the same way we define K_{MTL} over MTL.

Definition 3.1 *Logic K_{MTL} is obtained by adding axiom* (K) *and the derivational rule* (Nec) *to MTL.*

By an easy application of (Nec) and (K) we immediately obtain that the derivation rule

(DR1) $\varphi \to \psi / \Box\varphi \to \Box\psi$

is valid in K_{MTL}.

The fundamental property of the classical modal logic K is the distributivity of box over conjunction. However, in K_{MTL} this is not true. Moreover, we cannot interchange box with diamond, because we don't have an involutive negation. From this follows that the same problem is with the distribution of diamond over disjunction, which is a part of popular diamond based definition of K over CPC. Nevertheless, at least some implications can still be proved.

Lemma 3.2 *The following formulae are provable in K_{MTL}:*

(a) $(\Box\varphi \,\&\, \Box\psi) \to \Box(\varphi \,\&\, \psi)$,

(b) $\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$,

(c) $(\Diamond\varphi \vee \Diamond\psi) \to \Diamond(\varphi \vee \psi)$,

(d) $\Box\varphi \to \neg\Diamond\neg\varphi$.

Proof:

(a)

1: $\Box\varphi \to \Box(\psi \to (\varphi \,\&\, \psi))$ (F4), (DR1)

2: $\Box(\psi \to (\varphi \,\&\, \psi)) \to (\Box\psi \to \Box(\varphi \,\&\, \psi))$ (K)

3: $(\Box\varphi \,\&\, \Box\psi) \to \Box(\varphi \,\&\, \psi)$ 1, 2, (A1), (A5a)

(b)

4: $\Box(\varphi \wedge \psi) \to \Box\varphi$ (F5), (DR1)

5: $\Box(\varphi \wedge \psi) \to \Box\psi$ (F5), (DR1)

6: $\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$ (A1), (F4), (F5), (F6)

(c) Let us remark that $\Diamond\varphi$ is an abbreviation for $\neg\Box\neg\varphi$.

7: $\Box(\neg(\varphi \vee \psi)) \to \Box\neg\varphi$ (F7), (F10), (DR1)

8: $\Box(\neg(\varphi \vee \psi)) \to \Box\neg\psi$ (F7), (F10), (DR1)

9: $\Box(\neg(\varphi \vee \psi)) \to (\Box\neg\varphi \wedge \Box\neg\psi)$ (A1), (F4), (F5), (F6)

10: $(\Diamond\varphi \vee \Diamond\psi) \to \Diamond(\varphi \vee \psi)$ (F10), (F13), (A1)

An alternative slightly shorter proof uses the derivational rule (DR2), which we show later on.

(d)

11: $\Box\varphi \to \Box\neg\neg\varphi$ (F9), (DR1)

12: $\Box\neg\neg\varphi \to \neg\neg\Box\neg\neg\varphi$ (F9)

13: $\Box\varphi \to \neg\Diamond\neg\varphi$ (A1)

■

Lemma 3.3 *The following formulae are not provable in K_{MTL}:*

(a) $\Box(\varphi \,\&\, \psi) \to (\Box\varphi \,\&\, \Box\psi)$,

(b) $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$,

(c) $\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$,

(d) $\neg\Diamond\neg\varphi \to \Box\varphi$.

Proof: For (a) use Table 2 and $\varphi = 0$ and $\psi = 0$.

$\&$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

$\to$	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

$\Box$	
0	1
1	1
2	2

Table 2: Truth tables over K_{MTL}.

For (b) and (c) use Table 3 and $\varphi = 1, \psi = 2$ and $\varphi = 2, \psi = 3$, respectively.

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	1	1	1
2	0	0	2	0	2	2
3	0	1	0	3	1	3
4	0	1	2	1	4	4
5	0	1	2	3	4	5

$\&$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	1	0	1
2	0	0	2	0	2	2
3	0	1	0	3	1	3
4	0	0	2	1	2	4
5	0	1	2	3	4	5

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	4	5	5	5
2	3	3	5	3	5	5
3	2	4	2	5	4	5
4	1	3	4	3	5	5
5	0	1	2	3	4	5

$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	4	3	4	5
2	2	4	2	5	4	5
3	3	3	5	3	5	5
4	4	4	4	5	4	5
5	5	5	5	5	5	5

$\Box$	
0	2
1	4
2	4
3	4
4	4
5	5

$\Diamond$	
0	0
1	1
2	1
3	1
4	1
5	3

Table 3: Truth tables over K_{MTL}.

Table 4 and $\varphi = 1$ is a counterexample for (d).

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

$\Box$	
0	0
1	0
2	2

$\Diamond$	
0	0
1	2
2	2

Table 4: Truth tables over K_{MTL}.

■

It is evident that the models in Table 2 and 3 have an involutive negation and satisfy the divisibility axiom and so are counterexamples to (a), (b) and (c) also in K_{IMTL}, K_{BL} and even $K_{Ł}$. All these systems are obtained in the very same way as K_{MTL} from MTL. The completely different situation is in K_G where $\varphi \,\&\, \psi \equiv \varphi \wedge \psi$ is true and we can prove formulae (a) and (b) similarly to Lemma 3.2. Formula (c) then easily follows from (b).

A different situation is with (d) which is easily provable if we have an involutive negation, but is false in K_G as follows from Table 4.

If we take into account (F10) we can prove similarly to (DR1) that the derivational rule

(DR2) $\varphi \to \psi / \Diamond\varphi \to \Diamond\psi$

is valid in K_{MTL}.

In K, we can also prove the partial distribution of box over disjunction and diamond over conjunction which holds even in K_{MTL}.

Lemma 3.4 *The following formulae are provable in K_{MTL}*

(a) $\Diamond(\varphi \wedge \psi) \to (\Diamond\varphi \wedge \Diamond\psi)$,

(b) $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$.

Proof: Both proofs are very similar to Lemma 3.2b. In (a), we only use (DR2) instead of (DR1) and for (b) the proof reads as follows:

14:	$\Box\varphi \to \Box(\varphi \vee \psi)$	(F7), (DR1)
15:	$\Box\psi \to \Box(\varphi \vee \psi)$	(F7), (DR1)
16:	$(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$	(F4), (F5), (F8)

■

The following distributivity of diamond over implication remains true in K_{MTL} only partially.

Lemma 3.5 *The following formula is provable in K_{MTL}*

$$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi).$$

Proof:

17:	$\varphi \to ((\varphi \to \psi) \to \psi)$	(F3), (A5b)
18:	$\Box(\varphi \to (\neg\psi \to \neg(\varphi \to \psi)))$	(F10), (Nec)
19:	$\Box\varphi \to (\Box\neg\psi \to \Box\neg(\varphi \to \psi))$	(K), (K)
20:	$\neg\Box\neg(\varphi\to\psi)\to(\Box\varphi\to\neg\Box\neg\psi)$	(F10), (F1)

■

The opposite implication in the previous lemma, which is true in classical logic, is not true in K_{MTL} and has a three element counterexample.

We have shown that some important modal formulae of K are not provable in K_{MTL}. A stronger system can be thus easily obtained by adding these formulae to K_{MTL}. On the other hand, some axiomatics of K are same even over MTL. For example, if we take (DR1) and $\Box(\varphi\to\varphi)$ instead of the necessitation rule (Nec) we obtain again K_{MTL}. Moreover, we obtain K_{MTL} even if we replace in this system axiom (K) with $(\Box\varphi \,\&\, \Box\psi) \to \Box(\varphi \,\&\, \psi)$.

3.2. D_{MTL}

Logic D, which has deontic interpretations, is the least standard system we are going to study, but both its standard axiomatics remain equivalent.

Definition 3.6 *Logic D_{MTL} is an axiomatic extension of K_{MTL} by axiom* (D).

Lemma 3.7 *The following formula is provable in D_{MTL}*

$$\Diamond(\varphi \to \varphi).$$

Proof: We obtain $\Diamond(\varphi \to \varphi)$ immediately from (F2) by the necessitation and (D). ■

The previous formula form an alternative axiomatic system of $\mathrm{D_{MTL}}$ as we have already noted. If we add $\Diamond(\varphi \to \varphi)$ to $\mathrm{K_{MTL}}$ then (D) is provable by Lemma 3.5.

3.3. T_{MTL}

The rest of the paper deals with logics containing axiom (T). This axiom is sometimes called the axiom of necessity.

Definition 3.8 *Logic T_{MTL} is an axiomatic extension of K_{MTL} by axiom* (T).

The following formula well illustrates problems we are facing with our diamond definition over MTL.

Lemma 3.9 *The following formula is not provable in T_{MTL}*

$$\Diamond(\varphi \to \Box\varphi).$$

Proof: Use Table 5 and $\varphi = 1$.

&, ∧	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

□	
0	0
1	0
2	1
3	3

→	0	1	2	3
0	3	3	3	3
1	0	3	3	3
2	0	1	3	3
3	0	1	2	3

◇	
0	0
1	3
2	3
3	3

Table 5: Truth tables over $\mathrm{T_{MTL}}$.

■

However, some diamond based formulae are still provable.

Lemma 3.10 *The following formula is provable in T_{MTL}*

$$\varphi \to \Diamond\varphi.$$

Proof: Follows immediately from $\Box\neg\varphi \to \neg\varphi$ by (F10) and (F9). ■

The previous lemma with the transitivity of implication gives that axiom (D) is provable in $\mathrm{T_{MTL}}$ and thus $\mathrm{T_{MTL}}$ is an extension of $\mathrm{D_{MTL}}$.

In CPC, the axiomatic extension of K by the previous formula proves axiom (T), but in $\mathrm{K_{MTL}}$ it is not the case. There is a three element counterexample. It is also well known that if we take rule (DR1), axiom (T) and formula $\Box(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi))$ in CPC, we obtain T. It turns out that over MTL we obtain exactly $\mathrm{T_{MTL}}$. On the other hand, if we take another classicaly equivalent axiomatics which has $\varphi \to \Diamond\varphi$ instead of (T), we obtain a weaker system.

Corollary 3.11 *The following formulae are provable in T_{MTL}:*

(a) $\Box\Diamond\varphi \to \Diamond\varphi$,

(b) $\Box\varphi \to \Diamond\Box\varphi$,

(c) $\Diamond\varphi \to \Diamond\Diamond\varphi$,

(d) $\Box\Box\varphi \to \Box\varphi$.

Together with the opposite implications these formulae form so called reduction laws. These opposite implications

(R1) $\Diamond\varphi \to \Box\Diamond\varphi$,

(R2) $\Diamond\Box\varphi \to \Box\varphi$,

(R3) $\Diamond\Diamond\varphi \to \Diamond\varphi$,

(R4) $\Box\varphi \to \Box\Box\varphi$

lead in classical logic to the well known axiomatic extensions of T. If we add (R3) or (R4) to T we obtain S4 and if we add (R1) or (R2) to T we obtain S5 which is a proper extension of S4.

It turns out that over $\mathrm{T_{MTL}}$ the situation slightly changes.

Lemma 3.12 *The following provability conditions hold*

(a) T_{MTL}, *R2*$\vdash$*R1*,

(b) T_{MTL}, *R2*$\vdash$*R4*,

(c) T_{MTL}, *R1*$\vdash$*R3*,

(d) T_{MTL}, *R4*$\vdash$*R3*,

(e) T_{MTL}, *R1*$\nvdash$*R2*,

(f) T_{MTL}, *R3*$\nvdash$*R4*.

Proof: For (e) and (f) use Table 5 and $\varphi = 2$. ∎

Thus, we have two non-equivalent axiomatics of S4 and two non-equivalent axiomatics of S5 over MTL. We will briefly study the three of them.

3.4. $\mathbf{S4_{MTL}}$

The first system is obtained by adding (R4), called axiom (4), to T_{MTL}. This is the most common definition of axiomatics for S4.

Definition 3.13 *Logic* $S4_{MTL}$ *is an axiomatic extension of* T_{MTL} *by axiom* (4).

The following formulae are the direct consequences of (R4) and thus also (R3) over T_{MTL}.

Lemma 3.14 *The following formulae are provable in* $S4_{MTL}$*:*

(a) $\Diamond\varphi \equiv \Diamond\Diamond\varphi$,

(b) $\Box\varphi \equiv \Box\Box\varphi$,

(c) $\Diamond\Box\Diamond\varphi \to \Diamond\varphi$,

(d) $\Box\Diamond\varphi \equiv \Box\Diamond\Box\Diamond\varphi$,

(e) $\Diamond\Box\varphi \equiv \Diamond\Box\Diamond\Box\varphi$.

Proof: All proofs are the same or very similar as in classical logic. ∎

3.5. $\mathbf{S5_{MTL}}$

The standard definition of S5 uses (R1), called axiom (E), and we define this system over MTL in the same way.

Definition 3.15 *Logic* $S5_{MTL}$ *is an axiomatic extension of* T_{MTL} *by axiom* (E).

However, we already know that this definition leads to the unprovability of axiom (4) in such system. Also the following formulae are not provable.

Lemma 3.16 *The following formulae are not provable in* $S5_{MTL}$*:*

(a) $\Diamond\Box\varphi \to \Box\varphi$,

(b) $\Box\varphi \to \Box\Box\varphi$,

(c) $\Box(\varphi \vee \Box\psi) \to (\Box\varphi \vee \Box\psi)$,

(d) $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \Box\psi)$,

(e) $\Diamond(\varphi \,\&\, \Box\psi) \to (\Diamond\varphi \,\&\, \Box\psi)$,

(f) $(\Diamond\varphi \,\&\, \Box\psi) \to \Diamond(\varphi \,\&\, \Box\psi)$,

(g) $\Diamond(\varphi \wedge \Box\psi) \to (\Diamond\varphi \wedge \Box\psi)$,

(h) $(\Diamond\varphi \wedge \Box\psi) \to \Diamond(\varphi \wedge \Box\psi)$.

Proof: For (a) and (b) use Table 5 and $\varphi = 2$. In all other cases use Table 6. For (c) use $\varphi = 4$ and $\psi = 3$, for (d) use $\varphi = 1$ and $\psi = 3$, for (e) and (g) use $\varphi = 3$ and $\psi = 3$, for (f) and (h) use $\varphi = 4$ and $\psi = 3$.

$\wedge, \&$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	1	0	1
2	0	0	2	2	2	2
3	0	1	2	3	2	3
4	0	0	2	2	4	4
5	0	1	2	3	4	5

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	4	5	4	5
2	1	1	5	5	5	5
3	0	1	4	5	4	5
4	1	1	3	3	5	5
5	0	1	2	3	4	5

$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	3	3	5	5
2	2	3	2	3	4	5
3	3	3	3	3	5	5
4	4	5	4	5	4	5
5	5	5	5	5	5	5

$\Box$	
0	0
1	0
2	0
3	1
4	0
5	5

$\Diamond$	
0	0
1	5
2	5
3	5
4	5
5	5

Table 6: Truth tables over $S5_{MTL}$.

∎

Another very important equivalence is provable in S5 only partially.

Lemma 3.17 *The following formula is provable in $S5_{MTL}$*

$$\varphi \to \Box\Diamond\varphi.$$

Lemma 3.18 *The following formula is not provable in $S5_{MTL}$*

$$\Diamond\Box\varphi \to \varphi.$$

Proof: Use Table 5 and $\varphi = 2$. ■

We can also present some other alternative axiomatics of $S5_{MTL}$. One standard way is to add axiom (B) (the formula from the previous lemma) to $S4_{MTL}$. An alternative way is to add axiom (R2) to T_{MTL}. We already know that (R4) is provable in T_{MTL} with (R2). It is not difficult to show that both axiomatics lead to the same logic.

Definition 3.19 *Logic $S5+_{MTL}$ is an axiomatic extension of T_{MTL} by axiom* (R2).

However, many formulae are not provable even in this stronger system.

Lemma 3.20 *The following formulae are not provable in $S5+_{MTL}$:*

(a) $\Box(\varphi \,\&\, \psi) \to (\Box\varphi \,\&\, \Box\psi)$,

(b) $\neg\Diamond\neg\varphi \to \Box\varphi$,

(c) $(\Box\varphi \to \Diamond\psi) \to \Diamond(\varphi \to \psi)$,

(d) $\Diamond(\varphi \to \Box\varphi)$,

(e) $\Box(\varphi \vee \Box\psi) \to (\Box\varphi \vee \Box\psi)$.

Proof: Use Table (7). For (a) use $\varphi = 3$ and $\psi = 2$, for (b) use $\varphi = 3$, for (c) use $\varphi = 3$ and $\psi = 2$, for (d) use $\varphi = 2$, for (e) use $\varphi = 3$ and $\psi = 4$.

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	2	2	2
3	0	1	2	3	2	3
4	0	1	2	2	4	4
5	0	1	2	3	4	5

&	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	1	0	1
2	0	0	2	2	2	2
3	0	1	2	3	2	3
4	0	0	2	2	4	4
5	0	1	2	3	4	5

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	5	5	5	5
2	1	1	5	5	5	5
3	0	1	4	5	4	5
4	1	1	3	3	5	5
5	0	1	2	3	4	5

$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	3	4	5
2	2	2	2	3	4	5
3	3	3	3	3	5	5
4	4	4	4	5	4	5
5	5	5	5	5	5	5

$\Box$	
0	0
1	1
2	1
3	1
4	4
5	5

$\Diamond$	
0	0
1	1
2	4
3	5
4	4
5	5

Table 7: Truth tables over $S5+_{MTL}$.

■

One more system which is equivalent to $S5+_{MTL}$ is K_{MTL} extended by axioms $\Box\Diamond\Box\varphi \to \varphi$ and $\Diamond\Box\varphi \to \Box\Diamond\Box\Box\varphi$. It is worth pointing out that none of these two formulae is provable in $S5_{MTL}$.

4. Summary and future work

Our paper presents a small introduction to the problems of modal Hilbert-style calculi in mathematical fuzzy logics. We only touch some prominent modal systems and their axiomatics.

We also only slightly touch, in case of modal logic K, problems in axiomatic extensions of MTL, where some formulae unprovable in modal logics over MTL become provable. However, it is not difficult to show that all given counterexamples satisfy the divisibility axiom and some of them even contraction. We also do not discuss the difference between additive and multiplicative conjunctions.

We have shown that some important tautologies are not provable in naively constructed modal systems over MTL. On the other hand, the fact that some formulae are not provable in modal logics over MTL can be seen as an advantage and intended property which enable us to have some formulae, which are over CPC equivalent, true and some false if needed.

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