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2019

Dostupný z http://www.nusl.cz/ntk/nusl-399083

Dílo je chráněno podle autorského zákona č. 121/2000 Sb.

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22.07.2019

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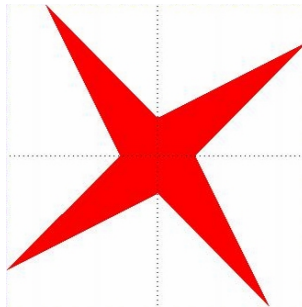
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Technical report No. V-1271

22.07.2019

Abstract:

We prove that the product of a symmetric positive semidefinite matrix and a symmetric matrix has all eigenvalues real. [2]

Keywords:
Symmetric matrix, positive semidefinite matrix, real spectrum.

A real matrix has complex eigenvalues in general. Yet there is a well-known important exception:

Theorem 1. *A symmetric matrix $A \in \mathbb{R}^{n \times n}$ has all eigenvalues real.*

In this note we prove a generalization of this result and we show a way how to construct generally nonsymmetric matrices having real eigenvalues only.

Theorem 2. *If $A, B \in \mathbb{R}^{n \times n}$ are symmetric matrices of which at least one is positive semidefinite, then AB has all eigenvalues real.*

Proof. (a) First, let A be positive semidefinite. Put $C = A^{1/2}$, so that C is the unique symmetric positive semidefinite real matrix satisfying $C^2 = A$ (Horn and Johnson [2, Thm. 7.2.6]). Then

$$AB = C^2 B = CC^T B = C(C^T B)$$

and since $C(C^T B)$ and $(C^T B)C$ have the same spectrum (Horn and Johnson [2, Thm. 1.3.20]), the eigenvalues of AB and $(C^T B)C$ are equal, hence it suffices to prove that $C^T BC$ has all eigenvalues real. Thus let $\lambda \in \mathbb{C}$ be an eigenvalue of $C^T BC$, i.e.,

$$C^T BCx = \lambda x \tag{0.1}$$

holds for some $0 \neq x \in \mathbb{C}^n$ which can be normalized so that $x^* x = 1$, where x^* denotes the conjugate transpose. Premultiplying (0.1) by x^* yields

$$\lambda = x^* C^T BCx = y^* By = \sum_{ij} B_{ij} y_i^* y_j,$$

where $y = Cx \in \mathbb{C}^n$, and consequently

$$\lambda = \sum_i B_{ii} |y_i|^2 + \sum_{i<j} (B_{ij} y_i^* y_j + B_{ji} y_j^* y_i) = \sum_i B_{ii} |y_i|^2 + \sum_{i<j} B_{ij} (y_i^* y_j + y_j^* y_i). \tag{0.2}$$

Because

$$(y_i^* y_j + y_j^* y_i)^* = y_i^* y_j + y_j^* y_i,$$

the number $y_i^* y_j + y_j^* y_i$, being equal to its complex conjugate, is real and this in the light of (0.2) means that λ is also real.

(b) If B is positive semidefinite then by the part (a) above BA has all eigenvalues real and thus the same holds for AB whose spectrum equals to that of BA. $\qquad\square$

To show that Theorem 2 is indeed a generalization of Theorem 1, let us decompose a given symmetric matrix A as $A = AI$ where I is the identity matrix, then the assumption of Theorem 2 are met which implies that all eigenvalues of A are real.

Finally we describe a way how to generate generally nonsymmetric matrices having real eigenvalues only.

Theorem 3. *For any $A, B \in \mathbb{R}^{n \times n}$, the matrices*

$$A^T A B^T B$$

and

$$A^T A (B + B^T)$$

have all eigenvalues real.

Proof. This follows immediately from Theorem 2 since $A^T A$ is symmetric positive semidefinite, and $B^T B$ and $B + B^T$ are symmetric. □

Bibliography

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[2] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.