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2019
Dostupný z http://www.nusl.cz/ntk/nusl-399083

Dílo je chráněno podle autorského zákona č. 121/2000 Sb.

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Datum stažení: 10.04.2024
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# Generalization of a Theorem on Eigenvalues of Symmetric Matrices 

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Technical report No. V-1271
22.07.2019

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## Abstract:

We prove that the product of a symmetric positive semidefinite matrix and a symmetric matrix has all eigenvalues real. ${ }^{2}$


Keywords:
Symmetric matrix, positive semidefinite matrix, real spectrum.

[^0]A real matrix has complex eigenvalues in general. Yet there is a well-known important exception:

Theorem 1. A symmetric matrix $A \in \mathbb{R}^{n \times n}$ has all eigenvalues real.
In this note we prove a generalization of this result and we show a way how to construct generally nonsymmetric matrices having real eigenvalues only.

Theorem 2. If $A, B \in \mathbb{R}^{n \times n}$ are symmetric matrices of which at least one is positive semidefinite, then $A B$ has all eigenvalues real.

Proof. (a) First, let $A$ be positive semidefinite. Put $C=A^{1 / 2}$, so that $C$ is the unique symmetric positive semidefinite real matrix satisfying $C^{2}=A$ (Horn and Johnson [2, Thm. 7.2.6]). Then

$$
A B=C^{2} B=C C^{T} B=C\left(C^{T} B\right)
$$

and since $C\left(C^{T} B\right)$ and $\left(C^{T} B\right) C$ have the same spectrum (Horn and Johnson [2, Thm. 1.3.20]), the eigenvalues of $A B$ and $\left(C^{T} B\right) C$ are equal, hence it suffices to prove that $C^{T} B C$ has all eigenvalues real. Thus let $\lambda \in \mathbb{C}$ be an eigenvalue of $C^{T} B C$, i.e.,

$$
\begin{equation*}
C^{T} B C x=\lambda x \tag{0.1}
\end{equation*}
$$

holds for some $0 \neq x \in \mathbb{C}^{n}$ which can be normalized so that $x^{*} x=1$, where $x^{*}$ denotes the conjugate transpose. Premultiplying (0.1) by $x^{*}$ yields

$$
\lambda=x^{*} C^{T} B C x=y^{*} B y=\sum_{i j} B_{i j} y_{i}^{*} y_{j}
$$

where $y=C x \in \mathbb{C}^{n}$, and consequently

$$
\begin{equation*}
\lambda=\sum_{i} B_{i i}\left|y_{i}\right|^{2}+\sum_{i<j}\left(B_{i j} y_{i}^{*} y_{j}+B_{j i} y_{j}^{*} y_{i}\right)=\sum_{i} B_{i i}\left|y_{i}\right|^{2}+\sum_{i<j} B_{i j}\left(y_{i}^{*} y_{j}+y_{j}^{*} y_{i}\right) . \tag{0.2}
\end{equation*}
$$

Because

$$
\left(y_{i}^{*} y_{j}+y_{j}^{*} y_{i}\right)^{*}=y_{i}^{*} y_{j}+y_{j}^{*} y_{i},
$$

the number $y_{i}^{*} y_{j}+y_{j}^{*} y_{i}$, being equal to its complex conjugate, is real and this in the light of (0.2) means that $\lambda$ is also real.
(b) If $B$ is positive semidefinite then by the part (a) above $B A$ has all eigenvalues real and thus the same holds for $A B$ whose spectrum equals to that of $B A$.

To show that Theorem 2 is indeed a generalization of Theorem 1, let us decompose a given symmetric matrix $A$ as $A=A I$ where $I$ is the identity matrix, then the assumption of Theorem 2 are met which implies that all eigenvalues of $A$ are real.

Finally we describe a way how to generate generally nonsymmetric matrices having real eigenvalues only.

Theorem 3. For any $A, B \in \mathbb{R}^{n \times n}$, the matrices

$$
A^{T} A B^{T} B
$$

and

$$
A^{T} A\left(B+B^{T}\right)
$$

have all eigenvalues real.

Proof. This follows immediately from Theorem 2 since $A^{T} A$ is symmetric positive semidefinite, and $B^{T} B$ and $B+B^{T}$ are symmetric.

## Bibliography

[1] W. Barth and E. Nuding, Optimale Lösung von Intervallgleichungssystemen, Computing, 12 (1974), pp. 117-125.
[2] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.


[^0]:    ${ }^{1}$ This work was supported with institutional support RVO:67985807.
    ${ }^{2}$ Above: Logo of interval computations and related areas (depiction of the solution set of the system $[2,4] x_{1}+[-2,1] x_{2}=[-2,2],[-1,2] x_{1}+[2,4] x_{2}=[-2,2]$ (Barth and Nuding [1])).

