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**Institute of Computer Science**  
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Technical report No. V-1268

14.06.2019



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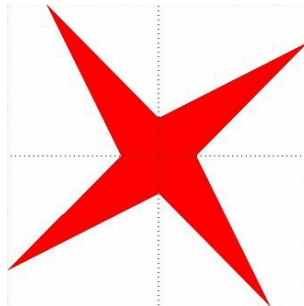
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Abstract:

We consider the conjecture formulated in the title concerning existence of a symmetric singular matrix in a singular symmetric interval matrix. We show by means of a counterexample that it is generally not valid, and we prove that it becomes true under an additional assumption of positive semidefiniteness of the midpoint matrix. The proof is constructive.<sup>2</sup>



Keywords:

Symmetric interval matrix, singularity, positive semidefiniteness.

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<sup>1</sup>This work was supported with institutional support RVO:67985807.

<sup>2</sup>Above: logo of interval computations and related areas (depiction of the solution set of the system  $[2, 4]x_1 + [-2, 1]x_2 = [-2, 2]$ ,  $[-1, 2]x_1 + [2, 4]x_2 = [-2, 2]$  (Barth and Nuding [?])).

# Does a Singular Symmetric Interval Matrix Contain a Symmetric Singular Matrix?

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## Abstract

We consider the conjecture formulated in the title concerning existence of symmetric singular matrix in a singular symmetric interval matrix. We show by means of a counterexample that it is generally not valid, and we prove that it becomes true under an additional assumption of positive semidefiniteness of the midpoint matrix. The proof is constructive.

*Keywords:* symmetric interval matrix, singularity, positive semidefiniteness

*2010 MSC:* 15A09, 65G40

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## 1. Introduction

A square interval matrix  $\mathbf{A} = [A - D, A + D]$  is called singular if it contains a singular matrix, and it is said to be symmetric if both  $A$  and  $D$  are symmetric. Thus unless  $D = 0$ ,  $\mathbf{A}$  contains nonsymmetric matrices as well. This context – namely, presence of both symmetric and nonsymmetric matrices within  $\mathbf{A}$  – leads to a natural question: if a symmetric  $\mathbf{A}$  is singular, does it necessarily contain a symmetric singular matrix?

In Section 2 we show by means of a  $2 \times 2$  counterexample that this conjecture is not true; but then in Section 3 we prove that under an additional assumption of positive semidefiniteness of the midpoint  $A$  it becomes valid. The proof is constructive, and in Section 4 we translate it into the form of an algorithm. It is interesting that it is a two-stage process: first we must find an arbitrary (generally nonsymmetric) singular matrix in  $\mathbf{A}$ , and then we

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\*\*Dedicated to Professor Ilja Černý on the occasion of his 90th birthday.

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exploit the sign structure of its null vector to construct a symmetric singular matrix in  $\mathbf{A}$ .

## 2. Counterexample

The symmetric interval matrix

$$\mathbf{A} = \begin{pmatrix} -1 & [-1, 1] \\ [-1, 1] & 1 \end{pmatrix}$$

is obviously singular since it contains the singular matrix

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},$$

yet each symmetric matrix in  $\mathbf{A}$  is of the form

$$A_t = \begin{pmatrix} -1 & t \\ t & 1 \end{pmatrix}, \quad t \in [-1, 1]$$

and it satisfies  $\det(A_t) = -1 - t^2 < 0$ , i.e., it is nonsingular. Hence, a singular symmetric interval matrix does *not* contain a symmetric singular matrix in the general case.

## 3. Existence of a symmetric singular matrix

We shall show, however, that under an additional assumption the conjecture becomes true.

**Theorem 1.** *A singular symmetric interval matrix  $[A - D, A + D]$  with positive semidefinite  $A$  contains a symmetric singular matrix.*

PROOF. By assumption there exists a singular matrix  $S_0 \in [A - D, A + D]$  and thus also a vector  $x \neq 0$  satisfying  $S_0 x = 0$ . Then we have

$$x^T A x \leq |x^T (A - S_0) x| \leq |x|^T |A - S_0| |x| \leq |x|^T D |x|. \quad (1)$$

Define a diagonal matrix  $T$  by  $T_{ii} = 1$  if  $x_i \geq 0$  and  $T_{ii} = -1$  otherwise, then  $|x| = Tx$  and substituting into (1) we obtain

$$x^T (A - TDT) x \leq 0.$$

Because  $A - TDT$  is symmetric, by the Courant-Fischer theorem [1] we have

$$\lambda_{\min}(A - TDT) = \min_{x' \neq 0} \frac{x'^T (A - TDT) x'}{x'^T x'} \leq \frac{x^T (A - TDT) x}{x^T x} \leq 0.$$

Now define a function  $f$  of one real variable by

$$f(t) = \lambda_{\min}(A - tTDT), \quad t \in [0, 1].$$

Then  $f(0) = \lambda_{\min}(A) \geq 0$  because  $A$  is positive semidefinite by assumption,  $f(1) = \lambda_{\min}(A - TDT) \leq 0$  as proved above, and, moreover,  $f$  is continuous in  $[0, 1]$  since by the Wielandt-Hofman theorem [1] for each  $t_1, t_2 \in [0, 1]$  we have

$$|f(t_1) - f(t_2)| \leq \|(t_1 - t_2)TDT\|_F \leq |t_1 - t_2| \|D\|_F,$$

where  $\|\cdot\|_F$  is the Frobenius norm. In this way the assumptions of the intermediate value theorem are met, hence there exists a  $t^* \in [0, 1]$  such that  $f(t^*) = 0$ . Then

$$S = A - t^*TDT$$

is a symmetric singular matrix in  $[A - D, A + D]$ .

#### 4. Computation of a symmetric singular matrix

We may now sum up the construction given in the proof into the form of an algorithm. Notice that first a singular matrix  $S_0$  must be constructed (by arbitrary means; we recommend the MATLAB file mentioned in the footnote) and then the sign structure of its null vector  $x$  is exploited to construct a real function  $f$  whose zero on the interval  $[0, 1]$  must be found (we recommend to use the classical bisection method which works well despite the lack on any additional information about  $f$ ).

1. Find a singular matrix<sup>1</sup>  $S_0 \in [A - D, A + D]$ .
2. Find an  $x \neq 0$  satisfying  $S_0 x = 0$ .
3.  $T = I$ ; set  $T_{ii} = -1$  whenever  $x_i < 0$ .
4.  $C = TDT$ .
5. Construct a function  $f(t) = \lambda_{\min}(A - tC)$ ,  $t \in [0, 1]$ .
6. Find a zero<sup>2</sup>  $t^*$  of  $f(t)$  in  $[0, 1]$ .

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<sup>1</sup>E.g. by the file available at <http://uivtx.cs.cas.cz/~rohn/other/regising.m>.

<sup>2</sup>E.g. by the interval halving (bisection) method.

7.  $S = A - t^*C$ .

Consider a randomly generated symmetric positive semidefinite integer matrix  $A$  and a symmetric nonnegative integer matrix  $D$ .

A =

|     |      |      |     |     |     |
|-----|------|------|-----|-----|-----|
| 208 | 97   | -8   | 153 | 62  | -89 |
| 97  | 197  | -102 | 71  | 10  | -60 |
| -8  | -102 | 154  | -64 | -2  | -17 |
| 153 | 71   | -64  | 263 | 54  | -32 |
| 62  | 10   | -2   | 54  | 35  | -12 |
| -89 | -60  | -17  | -32 | -12 | 186 |

D =

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 2 | 4 | 7 | 2 | 5 | 2 |
| 4 | 7 | 1 | 6 | 7 | 7 |
| 7 | 1 | 2 | 6 | 8 | 7 |
| 2 | 6 | 6 | 2 | 8 | 6 |
| 5 | 7 | 8 | 8 | 6 | 5 |
| 2 | 7 | 7 | 6 | 5 | 5 |

The computed matrix  $S_0$  is not yet symmetric, but it contains a symmetric integer submatrix  $A(2 : 5, 2 : 5)$ . This nice integer substructure is however destroyed while computing the symmetric singular matrix  $S$  which contains no more integer entry. Finally we compute the rank of  $S$  to demonstrate its singularity.

S0 =

|          |           |           |          |          |          |
|----------|-----------|-----------|----------|----------|----------|
| 208.5947 | 98.1894   | -5.9186   | 153.5947 | 63.4867  | -88.4053 |
| 93.0000  | 190.0000  | -103.0000 | 65.0000  | 3.0000   | -67.0000 |
| -15.0000 | -103.0000 | 152.0000  | -70.0000 | -10.0000 | -24.0000 |
| 151.0000 | 65.0000   | -70.0000  | 261.0000 | 46.0000  | -38.0000 |
| 57.0000  | 3.0000    | -10.0000  | 46.0000  | 29.0000  | -17.0000 |
| -87.0000 | -53.0000  | -10.0000  | -26.0000 | -7.0000  | 191.0000 |

S =

|          |           |           |          |         |          |
|----------|-----------|-----------|----------|---------|----------|
| 207.0808 | 98.8385   | -4.7827   | 153.9192 | 64.2981 | -89.9192 |
| 98.8385  | 193.7827  | -102.4596 | 68.2423  | 6.7827  | -56.7827 |
| -4.7827  | -102.4596 | 153.0808  | -66.7577 | -5.6769 | -13.7827 |
| 153.9192 | 68.2423   | -66.7577  | 262.0808 | 50.3231 | -29.2423 |
| 64.2981  | 6.7827    | -5.6769   | 50.3231  | 32.2423 | -9.7019  |
| -89.9192 | -56.7827  | -13.7827  | -29.2423 | -9.7019 | 183.7019 |

>> rank(S)

ans =

5

## References

- [1] G. H. Golub, C. F. van Loan, Matrix Computations, The Johns Hopkins University Press, Baltimore, 1996.