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# Does a Singular Symmetric Interval Matrix Contain a Symmetric Singular Matrix? 

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http://uivtx.cs.cas.cz/~rohn
Technical report No. V-1268
14.06.2019

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We consider the conjecture formulated in the title concerning existence of a symmetric singular matrix in a singular symmetric interval matrix. We show by means of a counterexample that it is generally not valid, and we prove that it becomes true under an additional assumption of positive semidefiniteness of the midpoint matrix. The proof is constructive. ${ }^{2}$


Keywords:
Symmetric interval matrix, singularity, positive semidefiniteness.

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# Does a Singular Symmetric Interval Matrix Contain a Symmetric Singular Matrix? 

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#### Abstract

We consider the conjecture formulated in the title concerning existence of symmetric singular matrix in a singular symmetric interval matrix. We show by means of a counterexample that it is generally not valid, and we prove that it becomes true under an additional assumption of positive semidefiniteness of the midpoint matrix. The proof is constructive.


Keywords: symmetric interval matrix, singularity, positive semidefiniteness
2010 MSC: 15A09, 65G40

## 1. Introduction

A square interval matrix $\boldsymbol{A}=[A-D, A+D]$ is called singular if it contains a singular matrix, and it is said to be symmetric if both $A$ and $D$ are symmetric. Thus unless $D=0, \boldsymbol{A}$ contains nonsymmetric matrices as well. This context - namely, presence of both symmetric and nonsymmetric matrices within $\boldsymbol{A}$ - leads to a natural question: if a symmetric $\boldsymbol{A}$ is singular, does it necessarily contain a symmetric singular matrix?

In Section 2 we show by means of a $2 \times 2$ counterexample that this conjecture is not true; but then in Section 3 we prove that under an additional assumption of positive semidefiniteness of the midpoint $A$ it becomes valid. The proof is constructive, and in Section 4 we translate it into the form of an algorithm. It is interesting that it is a two-stage process: first we must find an arbitrary (generally nonsymmetric) singular matrix in $\boldsymbol{A}$, and then we

[^1]exploit the sign structure of its null vector to construct a symmetric singular matrix in $\boldsymbol{A}$.

## 2. Counterexample

The symmetric interval matrix

$$
\boldsymbol{A}=\left(\begin{array}{cc}
-1 & {[-1,1]} \\
{[-1,1]} & 1
\end{array}\right)
$$

is obviously singular since it contains the singular matrix

$$
\left(\begin{array}{ll}
-1 & 1 \\
-1 & 1
\end{array}\right)
$$

yet each symmetric matrix in $\boldsymbol{A}$ is of the form

$$
A_{t}=\left(\begin{array}{cc}
-1 & t \\
t & 1
\end{array}\right), \quad t \in[-1,1]
$$

and it satisfies $\operatorname{det}\left(A_{t}\right)=-1-t^{2}<0$, i.e., it is nonsingular. Hence, a singular symmetric interval matrix does not contain a symmetric singular matrix in the general case.

## 3. Existence of a symmetric singular matrix

We shall show, however, that under an additional assumption the conjecture becomes true.

Theorem 1. A singular symmetric interval matrix $[A-D, A+D]$ with positive semidefinite $A$ contains a symmetric singular matrix.

Proof. By assumption there exists a singular matrix $S_{0} \in[A-D, A+D]$ and thus also a vector $x \neq 0$ satisfying $S_{0} x=0$. Then we have

$$
\begin{equation*}
x^{T} A x \leq\left|x^{T}\left(A-S_{0}\right) x\right| \leq|x|^{T}\left|A-S_{0}\right||x| \leq|x|^{T} D|x| . \tag{1}
\end{equation*}
$$

Define a diagonal matrix $T$ by $T_{i i}=1$ if $x_{i} \geq 0$ and $T_{i i}=-1$ otherwise, then $|x|=T x$ and substituting into (1) we obtain

$$
x^{T}(A-T D T) x \leq 0 .
$$

Because $A-T D T$ is symmetric, by the Courant-Fischer theorem [1] we have

$$
\lambda_{\min }(A-T D T)=\min _{x^{\prime} \neq 0} \frac{x^{\prime T}(A-T D T) x^{\prime}}{x^{\prime T} x^{\prime}} \leq \frac{x^{T}(A-T D T) x}{x^{T} x} \leq 0 .
$$

Now define a function $f$ of one real variable by

$$
f(t)=\lambda_{\min }(A-t T D T), \quad t \in[0,1] .
$$

Then $f(0)=\lambda_{\min }(A) \geq 0$ because $A$ is positive semidefinite by assumption, $f(1)=\lambda_{\min }(A-T D T) \leq 0$ as proved above, and, moreover, $f$ is continuous in $[0,1]$ since by the Wielandt-Hofman theorem [1] for each $t_{1}, t_{2} \in[0,1]$ we have

$$
\left|f\left(t_{1}\right)-f\left(t_{2}\right)\right| \leq\left\|\left(t_{1}-t_{2}\right) T D T\right\|_{F} \leq\left|t_{1}-t_{2}\right|\|D\|_{F}
$$

where $\|\cdot\|_{F}$ is the Frobenius norm. In this way the assumptions of the intermediate value theorem are met, hence there exists a $t^{*} \in[0,1]$ such that $f\left(t^{*}\right)=0$. Then

$$
S=A-t^{*} T D T
$$

is a symmetric singular matrix in $[A-D, A+D]$.

## 4. Computation of a symmetric singular matrix

We may now sum up the construction given in the proof into the form of an algorithm. Notice that first a singular matrix $S_{0}$ must be constructed (by arbitrary means; we recommend the MATLAB file mentioned in the footnote) and then the sign structure of its null vector $x$ is exploited to construct a real function $f$ whose zero on the interval $[0,1]$ must be found (we recommend to use the classical bisection method which works well despite the lack on any additional information about $f$ ).

1. Find a singular matrix ${ }^{1} S_{0} \in[A-D, A+D]$.
2. Find an $x \neq 0$ satisfying $S_{0} x=0$.
3. $T=I$; set $T_{i i}=-1$ whenever $x_{i}<0$.
4. $C=T D T$.
5. Construct a function $f(t)=\lambda_{\min }(A-t C), t \in[0,1]$.
6. Find a zero ${ }^{2} t^{*}$ of $f(t)$ in $[0,1]$.
[^2]
## 7. $S=A-t^{*} C$.

Consider a randomly generated symmetric positive semidefinite integer matrix $A$ and a symmetric nonnegative integer matrix $D$.
A =

|  |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: |
| 208 | 97 | -8 | 153 | 62 | -89 |
| 97 | 197 | -102 | 71 | 10 | -60 |
| -8 | -102 | 154 | -64 | -2 | -17 |
| 153 | 71 | -64 | 263 | 54 | -32 |
| 62 | 10 | -2 | 54 | 35 | -12 |
| -89 | -60 | -17 | -32 | -12 | 186 |
|  |  |  |  |  |  |
| D = |  |  |  |  |  |
|  |  |  |  |  |  |
| 2 | 4 | 7 | 2 | 5 | 2 |
| 4 | 7 | 1 | 6 | 7 | 7 |
| 7 | 1 | 2 | 6 | 8 | 7 |
| 2 | 6 | 6 | 2 | 8 | 6 |
| 5 | 7 | 8 | 8 | 6 | 5 |
| 2 | 7 | 7 | 6 | 5 | 5 |

The computed matrix $S_{0}$ is not yet symmetric, but it contains a symmetric integer submatrix $A(2: 5,2: 5)$. This nice integer substructure is however destroyed while computing the symmetric singular matrix $S$ which contains no more integer entry. Finally we compute the rank of $S$ to demonstrate its singularity.
$\mathrm{S} 0=$

| 208.5947 | 98.1894 | -5.9186 | 153.5947 | 63.4867 | -88.4053 |
| ---: | ---: | ---: | ---: | ---: | ---: |
| 93.0000 | 190.0000 | -103.0000 | 65.0000 | 3.0000 | -67.0000 |
| -15.0000 | -103.0000 | 152.0000 | -70.0000 | -10.0000 | -24.0000 |
| 151.0000 | 65.0000 | -70.0000 | 261.0000 | 46.0000 | -38.0000 |
| 57.0000 | 3.0000 | -10.0000 | 46.0000 | 29.0000 | -17.0000 |
| -87.0000 | -53.0000 | -10.0000 | -26.0000 | -7.0000 | 191.0000 |

```
S =
    207.0808 98.8385 -4.7827 153.9192 
        98.8385 193.7827 -102.4596 68.2423 6.7827 -56.7827
        -4.7827 -102.4596 153.0808 -66.7577 -5.6769 -13.7827
        153.9192 68.2423 -66.7577 262.0808 50.3231 -29.2423
        64.2981 6.7827 -5.6769 50.3231 
    -89.9192 -56.7827 -13.7827 -29.2423 -9.7019 183.7019
>> rank(S)
ans =
    5
```


## References

[1] G. H. Golub, C. F. van Loan, Matrix Computations, The Johns Hopkins University Press, Baltimore, 1996.


[^0]:    ${ }^{1}$ This work was supported with institutional support RVO:67985807.
    ${ }^{2}$ Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4] x_{1}+[-2,1] x_{2}=[-2,2],[-1,2] x_{1}+[2,4] x_{2}=[-2,2]$ (Barth and Nuding [?])).

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    ${ }^{* *}$ Dedicated to Professor Ilja Cerný on the occasion of his 90th birthday.
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[^2]:    ${ }^{1}$ E.g. by the file available at http://uivtx.cs.cas.cz/~rohn/other/regising.m.
    ${ }^{2}$ E.g. by the interval halving (bisection) method.

