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# **Institute of Computer Science** Academy of Sciences of the Czech Republic

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Jiří Rohn http://uivtx.cs.cas.cz/~rohn

Technical report No. V-1268

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## Does a Singular Symmetric Interval Matrix Contain a Symmetric Singular Matrix?

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We consider the conjecture formulated in the title concerning existence of a symmetric singular matrix in a singular symmetric interval matrix. We show by means of a counterexample that it is generally not valid, and we prove that it becomes true under an additional assumption of positive semidefiniteness of the midpoint matrix. The proof is constructive.<sup>2</sup>



Keywords: Symmetric interval matrix, singularity, positive semidefiniteness.

<sup>&</sup>lt;sup>1</sup>This work was supported with institutional support RVO:67985807.

<sup>&</sup>lt;sup>2</sup>Above: logo of interval computations and related areas (depiction of the solution set of the system  $[2, 4]x_1 + [-2, 1]x_2 = [-2, 2], [-1, 2]x_1 + [2, 4]x_2 = [-2, 2]$  (Barth and Nuding [?])).

### Does a Singular Symmetric Interval Matrix Contain a Symmetric Singular Matrix?

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#### Abstract

We consider the conjecture formulated in the title concerning existence of symmetric singular matrix in a singular symmetric interval matrix. We show by means of a counterexample that it is generally not valid, and we prove that it becomes true under an additional assumption of positive semidefiniteness of the midpoint matrix. The proof is constructive.

Keywords: symmetric interval matrix, singularity, positive semidefiniteness 2010 MSC: 15A09, 65G40

#### 1. Introduction

A square interval matrix  $\mathbf{A} = [A - D, A + D]$  is called singular if it contains a singular matrix, and it is said to be symmetric if both A and D are symmetric. Thus unless D = 0,  $\mathbf{A}$  contains nonsymmetric matrices as well. This context – namely, presence of both symmetric and nonsymmetric matrices within  $\mathbf{A}$  – leads to a natural question: if a symmetric  $\mathbf{A}$  is singular, does it necessarily contain a symmetric singular matrix?

In Section 2 we show by means of a  $2 \times 2$  counterexample that this conjecture is not true; but then in Section 3 we prove that under an additional assumption of positive semidefiniteness of the midpoint A it becomes valid. The proof is constructive, and in Section 4 we translate it into the form of an algorithm. It is interesting that it is a two-stage process: first we must find an arbitrary (generally nonsymmetric) singular matrix in A, and then we

\*\*Dedicated to Professor Ilja Černý on the occasion of his 90th birthday. Email address: rohn@cs.cas.cz (Jiri Rohn)

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exploit the sign structure of its null vector to construct a symmetric singular matrix in A.

#### 2. Counterexample

The symmetric interval matrix

$$\boldsymbol{A} = \left(\begin{array}{cc} -1 & [-1, 1] \\ [-1, 1] & 1 \end{array}\right)$$

is obviously singular since it contains the singular matrix

$$\left(\begin{array}{cc} -1 & 1 \\ -1 & 1 \end{array}\right),$$

yet each symmetric matrix in A is of the form

$$A_t = \begin{pmatrix} -1 & t \\ t & 1 \end{pmatrix}, \quad t \in [-1, 1]$$

and it satisfies  $det(A_t) = -1 - t^2 < 0$ , i.e., it is nonsingular. Hence, a singular symmetric interval matrix does *not* contain a symmetric singular matrix in the general case.

#### 3. Existence of a symmetric singular matrix

We shall show, however, that under an additional assumption the conjecture becomes true.

**Theorem 1.** A singular symmetric interval matrix [A - D, A + D] with positive semidefinite A contains a symmetric singular matrix.

PROOF. By assumption there exists a singular matrix  $S_0 \in [A - D, A + D]$ and thus also a vector  $x \neq 0$  satisfying  $S_0 x = 0$ . Then we have

$$x^{T}Ax \le |x^{T}(A - S_{0})x| \le |x|^{T}|A - S_{0}||x| \le |x|^{T}D|x|.$$
(1)

Define a diagonal matrix T by  $T_{ii} = 1$  if  $x_i \ge 0$  and  $T_{ii} = -1$  otherwise, then |x| = Tx and substituting into (1) we obtain

$$x^T (A - TDT) x \le 0.$$

Because A - TDT is symmetric, by the Courant-Fischer theorem [1] we have

$$\lambda_{\min}(A - TDT) = \min_{x' \neq 0} \frac{x'^T (A - TDT)x'}{x'^T x'} \le \frac{x^T (A - TDT)x}{x^T x} \le 0.$$

Now define a function f of one real variable by

$$f(t) = \lambda_{\min}(A - tTDT), \quad t \in [0, 1].$$

Then  $f(0) = \lambda_{\min}(A) \ge 0$  because A is positive semidefinite by assumption,  $f(1) = \lambda_{\min}(A - TDT) \le 0$  as proved above, and, moreover, f is continuous in [0, 1] since by the Wielandt-Hofman theorem [1] for each  $t_1, t_2 \in [0, 1]$  we have

$$|f(t_1) - f(t_2)| \le ||(t_1 - t_2)TDT||_F \le |t_1 - t_2|||D||_F,$$

where  $\|\cdot\|_F$  is the Frobenius norm. In this way the assumptions of the intermediate value theorem are met, hence there exists a  $t^* \in [0, 1]$  such that  $f(t^*) = 0$ . Then

$$S = A - t^*TDT$$

is a symmetric singular matrix in [A - D, A + D].

#### 4. Computation of a symmetric singular matrix

We may now sum up the construction given in the proof into the form of an algorithm. Notice that first a singular matrix  $S_0$  must be constructed (by arbitrary means; we recommend the MATLAB file mentioned in the footnote) and then the sign structure of its null vector x is exploited to construct a real function f whose zero on the interval [0, 1] must be found (we recommend to use the classical bisection method which works well despite the lack on any additional information about f).

- 1. Find a singular matrix  $S_0 \in [A D, A + D]$ .
- 2. Find an  $x \neq 0$  satisfying  $S_0 x = 0$ .
- 3. T = I; set  $T_{ii} = -1$  whenever  $x_i < 0$ .
- 4. C = TDT.
- 5. Construct a function  $f(t) = \lambda_{\min}(A tC), t \in [0, 1].$
- 6. Find a zero<sup>2</sup>  $t^*$  of f(t) in [0, 1].

<sup>&</sup>lt;sup>1</sup>E.g. by the file available at http://uivtx.cs.cas.cz/~rohn/other/regising.m.

<sup>&</sup>lt;sup>2</sup>E.g. by the interval halving (bisection) method.

7.  $S = A - t^*C$ .

Consider a randomly generated symmetric positive semidefinite integer matrix A and a symmetric nonnegative integer matrix D.

#### A =

208	97	-8	153	62	-89
97	197	-102	71	10	-60
-8	-102	154	-64	-2	-17
153	71	-64	263	54	-32
62	10	-2	54	35	-12
-89	-60	-17	-32	-12	186

2	4	7	2	5	2
4	7	1	6	7	7
7	1	2	6	8	7
2	6	6	2	8	6
5	7	8	8	6	5
2	7	7	6	5	5

The computed matrix  $S_0$  is not yet symmetric, but it contains a symmetric integer submatrix A(2:5,2:5). This nice integer substructure is however destroyed while computing the symmetric singular matrix S which contains no more integer entry. Finally we compute the rank of S to demonstrate its singularity.

SO =

208.5947	98.1894	-5.9186	153.5947	63.4867	-88.4053
93.0000	190.0000	-103.0000	65.0000	3.0000	-67.0000
-15.0000	-103.0000	152.0000	-70.0000	-10.0000	-24.0000
151.0000	65.0000	-70.0000	261.0000	46.0000	-38.0000
57.0000	3.0000	-10.0000	46.0000	29.0000	-17.0000
-87.0000	-53.0000	-10.0000	-26.0000	-7.0000	191.0000

```
S =
  207.0808
             98.8385
                        -4.7827
                                 153.9192
                                             64.2981
                                                      -89.9192
   98.8385
                                              6.7827
                                                      -56.7827
            193.7827 -102.4596
                                  68.2423
   -4.7827 -102.4596
                      153.0808
                                 -66.7577
                                             -5.6769
                                                      -13.7827
             68.2423
                                 262.0808
                                                       -29.2423
  153.9192
                       -66.7577
                                             50.3231
   64.2981
              6.7827
                        -5.6769
                                  50.3231
                                             32.2423
                                                       -9.7019
            -56.7827
                                             -9.7019
                                                      183.7019
  -89.9192
                      -13.7827
                                 -29.2423
>> rank(S)
ans =
     5
```

#### References

 G. H. Golub, C. F. van Loan, Matrix Computations, The Johns Hopkins University Press, Baltimore, 1996.