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Abstract:

This report collects preliminary ideas on rendering classical mathematical notions in formal theories over suitable deductive fuzzy logics in such a way that references to real numbers are eliminated from their definitions and removed to the semantics of fuzzy logic. The conceptual simplification of the theory (in exchange for non-classical reasoning) is demonstrated on several examples. The framework employed for the number-free formalization of mathematical concepts is that of higher-order fuzzy logic, also known as Fuzzy Class Theory.

Keywords:
Fuzzy mathematics, fuzzy logic, Fuzzy Class Theory, distribution function, continuity, limit, metric, fuzzy Dedekind cut, similarity.

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1 Introduction

In the standard semantics of t-norm fuzzy logics [19], truth values are represented by real numbers from the unit interval $[0, 1]$; the truth functions of n-ary propositional connectives are interpreted as real-valued functions on $[0, 1]^n$; and the quantifiers $\exists, \forall$ of first-order t-norm logics as the suprema and infima of sets of truth values. Read inversely, the logical apparatus of t-norm fuzzy logics expresses, by means of its standard semantics, certain constructions over real numbers. Which constructions over reals can in this way be represented depends on which particular t-norm logic is being used and what is its expressive power. Expressively rich propositional logics such as $\text{Ł}\Pi\frac{1}{2}$ of [14] are capable of representing a large class of operations on reals [24], including all basic arithmetical operations. First-order variants of such logics [11] can even express basic second-order constructions, as the standard semantics of the quantifiers $\forall, \exists$ operates on sets of reals. The axioms and rules of t-norm logics are designed to capture basic properties of those operations on reals that are expressible in their language and to ensure the soundness[1] of formally derivable theorems on such operations.

In this way, many theorems and even theories of classical real-valued mathematics can be 'encoded' in suitable systems of t-norm fuzzy logic. A theory in fuzzy logic then captures some of the properties of classical mathematical notions that involve real numbers, and allows deriving theorems on these notions by means of logical deductions in fuzzy logic. Since the real numbers appear in the standard semantics of such a theory, they no longer need be explicitly mentioned in the theory itself. A classical mathematical notion involving real numbers is thus represented by another notion of fuzzy mathematics whose definition does not involve real numbers at all: real numbers are only implicitly present behind the logical axioms that govern reasoning in fuzzy mathematics and behind the axioms and definitions of the fuzzy-mathematical theory that describes the classical notion.

In what follows, this way of eliminating numbers from the theory in favor of reasoning by means of t-norm fuzzy logic will be called the *number-free* (or with a pun analogical to that of "pointless topology", *numberless*) approach.

Number-free formalization of mathematical concepts is not new and has implicitly been around since the beginning of the theory of fuzzy sets: in fact, the notion of fuzzy set itself can be understood as a number-free rendering in fuzzy logic of the notion of real-valued function (see Section 3 below). However, it was only after the development of t-norm fuzzy logics, mostly in the past decade, that number-free notions could be treated rigorously in formal theories over suitable fuzzy logics. An early example of the number-free treatment of a classical notion is the formalization of additive probability as a modality *Probable* in Łukasiewicz logic (see Section 4). A number-free rendering of more advanced mathematical notions, however, requires more complex concepts of formal fuzzy mathematics (esp. higher-order set-constructions and a formal theory of fuzzy relations). The latter prerequisites have only recently been developed in the framework of higher-order fuzzy logic [5, 2, 8], which made it possible to apply the number-free approach systematically to various classical mathematical notions. Thus, even though many number-free notions (e.g., that of fuzzy equivalence relation, cf. Section 8) have been around for years, only now it is possible to treat them with the rigor necessary for the development of number-free fuzzy mathematical disciplines based on formal fuzzy logic.

[1] Usually not completeness, though. Since higher-order fuzzy logics (and many first-order fuzzy logics as well) are in general essentially incomplete with respect to their standard semantics (due to the interpretability of true arithmetic), in logic-based fuzzy mathematics we are usually not interested in completeness. Instead we seek to use a sufficiently strong system of axioms that allows deriving enough theorems for practical needs.

The profit we gain under the number-free approach in exchange for having to use non-classical rules of reasoning is, in the first place, conceptual simplicity: usually, the number-free fuzzy notion is conceptually much simpler than the original classical notion. For example, the notion of *pseudometric,* which in classical mathematics is a two-argument function to reals, becomes in the number-free fuzzy rendering just an *equivalence relation* (more examples will be given later). Secondly, the number-free rendering often reveals a new perspective upon the notion, exposing the gradual quality of the classical construct. From this perspective, the classical representation by means of real-valued functions often looks artificial—as a derivative model of the notion by means of crisp mathematics, whereas the fuzzy version appears to be a direct rendering of the primitive graded predicate. Formal fuzzy mathematics can thus be viewed as a more natural setting for dealing with gradual notions than classical mathematics, which is only able to address them indirectly, by a detour through real-valued functions. Thirdly, many theorems of classical mathematics are under this approach detected as provable by simple (often, propositional) logical derivations in a suitable fuzzy logic, instead of complex classical proofs involving arithmetic, infima, functions, etc. Finally, adopting a non-standard semantics (e.g., taking Chang's MV-algebra instead of the standard real numbers) or a different interpretation of the logical symbols involved (e.g., taking another t-norm for conjunction) provides an effortless generalization that might be harder to find in the classical language of crisp mathematics. Moreover, the many-valuedness of all formulae in fuzzy logic enables to consider another kind of graded generalization, by admitting partial satisfaction of the axioms for the reduced notion (e.g., a metric to degree .99, cf. [6, 7]).

The aim of this report is only to introduce the number-free approach as a distinct paradigm of formalization, rather than to develop particular number-free theories in depth. Therefore it only gives definitions of and a few observations on several number-free notions and discusses the merits of such formalization. A detailed development of number-free theories is left for future work.

2 Preliminaries

The particular formal framework in which number-free formalization of classical mathematical concepts is carried out in this report is that of higher-order fuzzy logic, also known as Fuzzy Class Theory (FCT) [5, 6], over suitable expansions of the logic MTL (the specification of which particular expansion is being used will always be clear from the context). A working knowledge of FCT and relevant systems of t-norm fuzzy logics, esp. those of [19, 13], is assumed. We shall use the notation of [6], plus the following definitions:

Definition 2.1. We define the following derived propositional connectives (whose standard and linear semantics is that of comparison between truth values):

$$\begin{aligned}
\varphi = \psi &\quad \equiv_{\mathrm{df}} \quad \Delta(\varphi \leftrightarrow \psi) \\
\varphi \neq \psi &\quad \equiv_{\mathrm{df}} \quad \neg(\varphi = \psi) \\
\varphi \leq \psi &\quad \equiv_{\mathrm{df}} \quad \Delta(\varphi \to \psi) \\
\varphi < \psi &\quad \equiv_{\mathrm{df}} \quad (\varphi \leq \psi) \mathbin{\&} (\varphi \neq \psi)
\end{aligned}$$

and similarly for $\geq$ and $>$. The priority of these connectives in formulae is the same as that of implication. Like in classical mathematics, we may chain the comparison connectives, writing, e.g., $\varphi = \psi = x < \sigma$ for $(\varphi = \psi) \mathbin{\&} (\psi = \chi) \mathbin{\&} (\chi < \sigma)$.

3 Real-valued functions

The very first notion of fuzzy mathematics, namely Zadeh's [26] notion of fuzzy set, can be regarded as a number-free reduction of the classical notion of $[0, 1]$-valued function to a non-classical (namely, fuzzy) notion of set. Even though the formal apparatus of first-order fuzzy logic, which enables to cast fuzzy sets as a primitive notion instead of representing them by classical real-valued functions, was developed years later, the tendency of regarding fuzzy sets as a number-free rendition of real-valued functions has partly been present since the very beginning of the fuzzy set theory, as witnessed by the vocabulary and notation employed. For example, the function $x \mapsto \min(A(x), B(x))$ is in the traditional fuzzy set theory denoted by $A \cap B$ and called the *intersection* of A and B: that is, the functions $A, B \colon X \to [0, 1]$ are regarded as (non-classical) *sets* rather than real-valued functions (as the intersection of real-valued functions is a different thing). The terminological shift towards the number-free discourse is expressed by the very term "fuzzy set" and its informal motivation of an unsharp collection of elements.[2]

In the same way, the notion of n-argument $[0, 1]$-valued function was reduced to that of (non-classical, namely fuzzy) n-ary relation. In the theory of fuzzy relations there was similarly strong inclination to using a number-free discourse even before the era of formal fuzzy logic, as witnessed by defining such notions as fuzzy relational composition or the image of a fuzzy set under a fuzzy relation—which would make little sense if the functions involved were not regarded as (a non-classical kind of) *relations* and *sets*.

A certain part of the talk about *real-valued functions* and their properties was thus replaced by a talk about *sets* and *relations* that behave non-classically (e.g., do not follow the rule of excluded middle). The properties of real-valued functions were rephrased as properties of such non-classical relations and sets, which eliminated the references to numbers at least from the wording of some theorems.[3] This provided a conceptual simplification of the theory, giving its definitions and theorems a compact form and a new conceptual meaning (as they became theorems on non-classical *sets* rather than classical *real-valued functions*).

This very first number-free fuzzy reduction has thus proved fruitful even in its pre-formal and semi-formal form of traditional fuzzy mathematics: its merits are numerous and well known, therefore we need not dwell on them here (enough to say that all fuzzy mathematics ultimately stems from this original numberless reduction). The formal apparatus of logic-based fuzzy mathematics has now enabled accomplishing the long-present idea and developing a fully fledged number-free approach to fuzzy sets and fuzzy relations.[4]

4 Finitely additive probability measures

Another numberless reduction, which was already based on formal fuzzy logic, is the axiomatization of finitely additive probability measures as a fuzzy modality *Probable* over propositional Łukasiewicz logic (with rational truth constants) by Hájek, Godo, and Esteva [20].

[2] In his original paper [26], Zadeh kept the distinction between fuzzy sets and the membership functions that represent them. Since, however, fuzzy sets are characterized by their membership functions, they can be identified with them (as was a common practice later). The representation of fuzzy sets by membership functions provides exactly the translation between the languages of (unsharp) sets and real-valued functions.

[3] The elimination of numbers also from proofs would have required a consistent use of first-order fuzzy logic, which approach, however, was not embraced in the early works on fuzzy set theory (it is, nevertheless, available now, and for particular systems of fuzzy logic was conceivable even earlier, cf., e.g., [21]).

[4] Cf. the papers [5, 2], which develop the formal theory of fuzzy sets and relations without making any reference to real numbers in definitions, theorems, or proofs (they do so only in illustrative semantic examples).

Later it was elaborated in a series of papers [18, 23, 17, 16, 15] by Flaminio, Marchioni, Montagna, and the latter authors of [20]. We shall briefly recapitulate the original axiomatization (adapted from [18]) as another example of the number-free approach. For simplicity, we shall not discuss Pavelka-style completeness theorems, conditional probability, probability of fuzzy events, and other interesting topics treated in the cited papers, and restrict our attention to unconditional probability of crisp events as given in [18], since it is sufficiently illustrative for our purposes.

Consider a classical probability space $(\Omega, \mathcal{B}, \pi)$, where Ω is a set of elementary events, $\mathcal{B}$ is a Boolean algebra of subsets of Ω, and π is a finitely additive probability measure on $\mathcal{B}$, i.e., a function $\pi\colon \mathcal{B} \to [0,1]$ satisfying the following conditions:

$$\begin{aligned} &\pi(\Omega) = 1 \\ &\text{if } A \subseteq B, \text{ then } \pi(A) \leq \pi(B) \\ &\text{if } A \cap B = \emptyset, \text{ then } \pi(A \cup B) = \pi(A) + \pi(B) \end{aligned} \tag{4.1}$$

A numberless representation of π draws on the fact that a $[0,1]$-valued function on a Boolean algebra can be understood as the standard model of a fuzzy modality P over an algebra of crisp propositions. The above conditions on π can be transformed into the axioms for P, which (due to the additivity) are expressible in Łukasiewicz logic:[5]

Definition 4.1. The axioms and rules of the logic FP(Ł) are those of Łukasiewicz propositional logic plus the following axioms and rules, for non-modal φ, ψ:

$\varphi \vee \neg\varphi$	(crispness of events)
from φ infer $P\varphi$	(certain events)
$P\varphi$, for all Boolean tautologies φ	(tautologically certain events)
$P(\varphi \to \psi) \to (P\varphi \to P\psi)$	(monotony w.r.t. $\to$)
$P(\neg\varphi) \leftrightarrow \neg P\varphi$	(complementary events)
$P(\varphi \vee \psi) \leftrightarrow ((P\varphi \to P(\varphi \wedge \psi)) \to P\psi)$	(finite additivity)

Recall that in standard models, a propositional fuzzy modality is represented by a standard fuzzy set on the propositional algebra; here, the propositional algebra is a Boolean algebra, due to the axiom of the crispness of events. Since furthermore every Boolean algebra is isomorphic to an algebra of crisp sets, standard models of the fuzzy modality have the form of a classical measure space $(\Omega, \mathcal{B}, \pi)$. The axioms of FP(Ł) ensure the following representation theorem (adapted from [20]):

Theorem 4.2. *Any probabilistic space* $(\Omega, \mathcal{B}, \pi)$ *is a standard model of* FP(Ł)*. Vice versa, all standard models of* FP(Ł) *are probabilistic spaces.*[6]

The representation theorem shows that the number-free reduction faithfully captures the original notion of finitely additive probabilistic measure. Moreover, by the completeness

[5] Ł with rational truth constants is used in [20], but the truth constants are inessential for our account. The language is two-layered, admitting only non-modal formulae and propositional combinations of non-nested modal formulae.

[6] In [18], such models are called *probabilistic Kripke models.* Since Kripke frames with the full accessibility relation are sometimes called *logical spaces,* they could also be called *probabilistic logical spaces* (more precisely, *generalized* ones, as variables are allowed to range over a subalgebra of subsets, rather than all subsets of the space, exactly like in generalized Kripke frames).

theorem of FP(Ł) w.r.t. probabilistic spaces proved in [18], all valid laws of finitely additive probability that are expressible in the language of FP(Ł) can in FP(Ł) be also (number-freely) proved.

In a given probabilistic space $(\Omega, \mathcal{B}, \pi)$, i.e., a standard model of FP(Ł) with $\|P\| = \pi$, the truth value of $P\varphi$ is the probability of the event φ: $\|P\varphi\| = \pi(\|\varphi\|)$. Since the probability of φ has got replaced by a truth value of $P\varphi$, the latter formula can be understood as representing the proposition "φ is probable". Numerical calculations with probabilities are thus in FP(Ł) replaced by *logical derivations* with the modality "is probable".

5 Number-free distribution functions

5.1 Classical distribution functions

Classical distribution functions present a special way how to define a measure on Borel sets, i.e., on the σ-algebra $\mathcal{B}$ of subsets of the real line generated by all intervals $(-\infty, x]$. Since the prototypical examples are probability distributions, we shall assume that the range of the distribution function is $[0, 1]$; other ranges can be normalized to $[0, 1]$ first. A function $f\colon \mathbb{R} \to [0, 1]$ defines a measure on $\mathcal{B}$ with $\mu(-\infty, x] = f(x)$ and $\mu(\mathbb{R}) = 1$ iff it satisfies the following conditions, which can thus be taken as the axioms for distribution functions:

1. *Monotony:* if $x \le y$ then $f(x) \le f(y)$, for all $x, y \in \mathbb{R}$,
2. *Margin conditions:* $\lim_{x\to-\infty} f(x) = 0$, and $\lim_{x\to+\infty} f(x) = 1$,
3. *Right-continuity:* $\lim_{x\to x_0^+} f(x) = f(x_0)$ for all $x_0 \in \mathbb{R}$.

Remark. The right-continuity condition is necessitated by the σ-additivity of μ, by which

$$f(a) = \mu(-\infty, a] = \mu\Big(\bigcap_{n=1}^{\infty} (-\infty, a_n]\Big) = \lim_{n\to\infty} f(a_n)$$

for $a_n \searrow a$. Choosing $f(x) = \mu[x, +\infty)$ or $\mu(x, +\infty)$ rather than $\mu(-\infty, x]$ or $\mu(-\infty, x)$ would change monotony to antitony and swap the limits in $\pm\infty$, and choosing $f(x) = \mu(-\infty, x)$ or $\mu(x, +\infty)$ rather than $\mu(-\infty, x]$ or $\mu[x, +\infty)$ would change right-continuity to left-continuity. The numberless account can be easily adapted for any of these four possibilities, therefore we shall only deal with the one described above. Other domains for distribution functions besides $\mathbb{R}$ can also be considered (e.g., $[0, 1]$ or $\mathbb{Q}$): here we use $\mathbb{R}$ for the sake of simplicity, although the results can easily be generalized to many other domains.

5.2 Number-free distribution functions

A function $f\colon \mathbb{R} \to [0, 1]$ represents a standard fuzzy set of reals, i.e., in FCT over any expansion of MTL, it is the standard model of a predicate A on reals. (Recall that $\mathbb{R}$ as well as other crisp mathematical structures are available in FCT by means of the Δ-interpretation, see [5, §7] and [12, §4].) We shall describe how the defining conditions of distribution functions can be translated into number-free conditions on the predicate A in fuzzy logic. In the rest of this section, we shall understand all first-order quantifications relativized to $\mathbb{R}$, unless specified otherwise.

The condition of *monotony* of f w.r.t. $\leq$ on $\mathbb{R}$ obviously translates as the condition[7]

$$(\forall x, y)(x \leq y \rightarrow (Ax \leq Ay)), \tag{5.1}$$

which is, due to the crispness of $\leq$ and the commutativity of Δ and $\forall$ in MTL, equivalent to the number-free axiom[8]

$$(\forall x, y)(x \leq y \rightarrow (Ax \rightarrow Ay)),$$

i.e., the condition of *upperness* of A in $\mathbb{R}$.

Due to the monotony assumed, the *margin conditions* reduce to

$$\inf_x f(x) = 0 \quad \text{and} \quad \sup_x f(x) = 1,$$

which translate into the number-free conditions

$$(\exists x)Ax \quad \text{and} \quad \neg(\forall x)Ax,$$

i.e., the condition of the *full height* and *null plinth* of A.

For the number-free rendering of the *right-continuity* condition we shall employ the fact that in the presence of monotony it reduces to

$$f(x_0) \geq \inf_{x > x_0} f(x),$$

which translates into the number-free condition

$$(\forall x_0)[(\forall x > x_0)Ax \rightarrow Ax_0],$$

i.e., the *left-closedness* of the fuzzy upper class A.

In sum, number-free distribution functions are left-closed upper sets in $\mathbb{R}$ with full height and null plinth, i.e., (weakly bounded) *fuzzy Dedekind cuts on* $\mathbb{R}$. (The representation theorem is immediate by the above considerations and the standard semantics of t-norm logics.) It can be observed that if the conditions are required to the full degree (as the axioms of number-free distribution functions), then (due to the crispness of $\leq$), they are independent of the particular left-continuous t-norm used (particular t-norms start playing a role only if the conditions are taken as graded, i.e., possibly satisfied to partial degrees; however, the right-continuity condition would have to be rendered differently without full monotony, see Section 6 below).

The number-free rendering of distribution functions as fuzzy Dedekind cuts corresponds to the known fact that distribution functions represent Hutton fuzzy reals (cf., e.g., [22]). A use of fuzzy Dedekind cuts for the development of a logic-based theory of fuzzy intervals (which are often called fuzzy numbers, too) is hinted at in [1].

The question whether, or in which sense, number-free distribution functions define number-free measures in an analogous way as in classical mathematics, is left for future research. (The notion of number-free measure needs to be specified first, including the rather complicated condition of σ-additivity, or at least continuity.)

[7]Notice that the first $\leq$ in (5.1) is the crisp ordering of reals (i.e., an object-language predicate), while the second is the propositional connective introduced in Definition 4.1. Since they are always distinguished by the context, we use the same sign for both. The two meanings of $\leq$ coincide under the number-free rendering of the domains of functions as in Section 7.

[8]As we require the validity of axioms to degree 1, we can drop initial Δ's.

6 Number-free continuity of functions on reals

In Section 5 we abused the presence of monotony for the number-free rendering of right-continuity. If we want, however, to develop for instance a graded theory of continuous t-norms whose monotony can be satisfied to partial degrees, we need a different number-free characterization of continuity that does not rely on monotony. Again we shall work with functions $\mathbb{R} \to [0,1]$ only, even though various generalizations are easy to obtain. Again, therefore, the first-order quantifiers are assumed to be relativized to $\mathbb{R}$, unless specified otherwise. For simplicity, we shall mainly consider *left*-continuity; the adaptations for right-continuity and both-sided continuity are trivial.

For a number-free rendering of left-continuity, we shall use the following classical characterization: a function $f\colon \mathbb{R} \to [0,1]$ is left-continuous in x_0 iff

$$\limsup_{x\to x_0^-} f(x) = \liminf_{x\to x_0^-} f(x) = f(x_0),$$

where

$$\begin{aligned}
\limsup\nolimits_{x\to x_0^-} f(x) &= \inf\nolimits_{x_1<x_0} \sup\nolimits_{x_1<x<x_0} f(x)\\
\liminf\nolimits_{x\to x_0^-} f(x) &= \sup\nolimits_{x_1<x_0} \inf\nolimits_{x_1<x<x_0} f(x)
\end{aligned}$$

This translates into the following number-free definitions in MTL:

$$\begin{aligned}
\mathrm{LimSup}^-(A,x_0) &\equiv_{\mathrm{df}} (\forall x_1 < x_0)(\exists x \in (x_1,x_0))Ax\\
\mathrm{LimInf}^-(A,x_0) &\equiv_{\mathrm{df}} (\exists x_1 < x_0)(\forall x \in (x_1,x_0))Ax\\
\Delta\,\mathrm{Cont}^-(A,x_0) &\equiv_{\mathrm{df}} Ax_0 = \mathrm{LimSup}^-(A,x_0) = \mathrm{LimInf}^-(A,x_0)
\end{aligned} \tag{6.1}$$

The right-sided predicates LimSup^+ and LimInf^+ are defined analogously. The both-sided classical notions of limit superior and inferior then obviously translate into the number-free language as follows:

$$\begin{aligned}
\mathrm{LimSup}(A,x_0) &\equiv_{\mathrm{df}} \mathrm{LimSup}^-(A,x_0) \vee \mathrm{LimSup}^+(A,x_0)\\
\mathrm{LimInf}(A,x_0) &\equiv_{\mathrm{df}} \mathrm{LimInf}^-(A,x_0) \wedge \mathrm{LimInf}^+(A,x_0)
\end{aligned}$$

These definitions reconstruct the classical notions in a number-free way in the framework of FCT; the representation theorems follow directly from the above considerations and the standard semantics of MTL. Examples of theorems below show that many properties of the classical notions can be reconstructed in FCT as well. Again it can be observed that the definitions are independent of a particular t-norm and are the same in all expansions of MTL.

Since the definitions reconstruct classical notions, in the notation employed we retain the classical terminology and speak of limits and continuity, even though these regard *membership functions* rather than the fuzzy classes themselves. Recall however that in FCT, fuzzy classes are treated as a primitive notion rather than being represented by membership functions (which in FCT are derivative), and that the formulae of FCT express facts about non-classical (namely, fuzzy) sets in the non-classical mathematical theory. From this point of view it is natural to regard the property expressed by a formula φ of FCT as a non-classical (viz, fuzzy) counterpart of the classical property that would be expressed by the same formula in classical mathematics. (If all sets involved are crisp, then φ defines this classical property even in FCT.) In this sense, the number-free fuzzy predicate $\mathrm{LimSup}^-(A, x_0)$ expresses the fuzzy-mathematical fact that x_0 is a *left-limit point* of the fuzzy class A, and $\mathrm{LimInf}^-(A, x_0)$ that x_0 is an *interior point* of $A \cup \{x_0\}$ in the right half-open interval topology,[9] as these are the properties expressed by the defining formulae if all sets involved are crisp. An interpretation of left-continuity in these terms would be more complex; nevertheless, under monotony of f (i.e., upperness of A) it reduces to the right-closedness of A w.r.t. $\leq$ (see Section 5).

Observation 6.1. By shifts of crisp relativized quantifiers valid in MTL∀, one can easily prove in FCT for instance the following theorems:[10]

1. $\mathrm{LimInf}(A, x_0) \leq \mathrm{LimSup}(A, x_0)$
2. $A \subseteq B \rightarrow (\mathrm{LimSup}(A, x_0) \rightarrow \mathrm{LimSup}(B, x_0))$,
   and analogously for LimInf.
3. $\mathrm{LimInf}(\backslash A, x_0) \leq \neg\, \mathrm{LimSup}(A, x_0)$,
   $\mathrm{LimSup}(\backslash A, x_0) \geq \neg\, \mathrm{LimInf}(A, x_0)$.
   (Equality holds in logics with involutive negation, but not generally in MTL.)
4. $\mathrm{LimInf}(A \sqcap B, x_0) = \mathrm{LimInf}(A, x_0) \wedge \mathrm{LimInf}(B, x_0)$,
   $\mathrm{LimInf}(A \sqcup B, x_0) = \mathrm{LimInf}(A, x_0) \vee \mathrm{LimInf}(B, x_0)$,
   and analogously for LimSup.

Notice that by the above considerations, the theorems of Observation 6.1 have double meanings: they can be understood either as graded variants of the classical theorems on (membership) functions, or as fuzzy-mathematical theorems on (fuzzy) sets. In particular:

1. Observation 6.1(1) on the one hand number-freely reconstructs the fact that the limit superior is at least as large as the limit inferior, and on the other hand it expresses the fuzzy-mathematical fact that interior points of fuzzy sets of reals are also their limit points.
2. Observation 6.1(2) can be understood either as a graded version of the classical theorem on the limit superior or inferior of a pointwise larger (membership) function, or as the graded fuzzy-mathematical theorem that a limit point of a fuzzy set is also a limit point of a larger fuzzy set (and analogously for interior points).
3. Observation 6.1(3) can either be understood as a rather unattractive theorem estimating the limits of functions transformed by the residual negation, or as a fuzzy-mathematical theorem on the behavior of limit and interior points under fuzzy set complementation, saying that in a fuzzy interval topology on reals (for which see Remark 6.2 below), (i) an interior point of the complement of a fuzzy set A is not a limit point of A, and (ii) a limit point of the complement of a fuzzy set A is not an interior point of A (where 'P is not Q' is a natural-language expression of the fuzzy-logic formula $Pa \rightarrow \neg Qa$ for fuzzy predicates P, Q).
4. Similarly, Observation 6.1(4) reconstructs in the numberless manner the limits of the maxima and minima of functions, and at the same time it is a graded theorem on inner

[9] I.e., the topology with the open base of all right half-open intervals $(a, b]$, which is also known as the upper-limit topology or the Sorgenfrey line.

[10] The theorems are stated for both-sided limits only, but hold equally well for one-sided limits.

and limit points of the weak union and intersection of two fuzzy sets. (Viz, x_0 is a limit point of $A \sqcap B$ exactly to the degree it is a limit point of A *and*[11] it is a limit point of B, etc.)

Remark 6.2. Since the predicate $\mathrm{LimInf}(A, x_0)$ expresses the fact that x_0 is an interior point of $A \cup \{x_0\}$, the formula $(\forall x_0)(Ax_0 \to \mathrm{LimInf}(A, x_0))$ expresses the *openness* of A in a fuzzy interval topology on $\mathbb{R}$. Similarly, the formula $(\forall x_0)(Ax_0 \to \mathrm{LimSup}(A, x_0))$ expresses a notion of closedness in this topology. (Thus by Observation 6.1(3), the continuity of membership function corresponds to clopennes in logics with involutive negation.) The study of this notion of fuzzy interval topology and of its relationship to the fuzzy interval topologies of [10, 9] is left for future work.

7 Number-free operations on reals

It can be noticed that in the previous examples, only the codomain of real-valued functions of reals was rendered numberless. Obviously, the domain $\mathbb{R}$ (or more conveniently, $\mathbb{R} \cup \{\pm\infty\}$) can be re-scaled into $[0, 1]$ and regarded as the standard set of truth degrees as well. The functions $\mathbb{R}^n \to \mathbb{R}$ then become n-ary functions from truth values to truth values, i.e., the truth functions of *fuzzy-logical connectives.*

An apparatus for internalizing truth values and logical connectives in FCT was developed in [8, §3]. As shown there, the truth values can be internalized as the elements of the crisp class $\mathrm{L} = \mathrm{Ker}\,\mathrm{Pow}\{a\}$, i.e., subclasses of a fixed crisp singleton. The class L of internalized truth values is ordered by crisp inclusion $\subseteq^{\Delta}$, and the correspondence between internal and semantical truth values is given as follows:[12] $\alpha \in \mathrm{L}$ corresponds to the semantic truth value of $\emptyset \in \alpha$, and the semantic truth value of φ is represented by the class $\overline{\varphi} =_{\mathrm{df}} \{a \mid \varphi\}$; the correspondences $\varphi \leftrightarrow (a \in \overline{\varphi})$ and $\overline{\varphi} \subseteq \overline{\psi} \leftrightarrow (\varphi \to \psi)$ then hold.

Logical connectives are then internalized by crisp n-ary operations on L, i.e., functions $\mathrm{L}^n \to \mathrm{L}$. Definable connectives c of the logic are represented by their corresponding class operations $\overline{c} = \mathrm{L} \cap \{x \mid c(x \in X_1, \ldots, x \in X_n)\}$ on L (e.g., & by $\cap$, $\vee$ by $\sqcup$, etc.). All crisp functions $\mathbf{c}\colon \mathrm{L}^n \to \mathrm{L}$ can be called *internal* (or *inner,* or *formal*) connectives.

Since the n-ary internal connectives are functions valued in L, they can as well be regarded as fuzzy subsets of L^n, i.e., n-ary fuzzy relations on L. Usual class relations and operations can thus be applied to them, for instance the graded inclusion $\mathbf{c} \subseteq \mathbf{d} \equiv_{\mathrm{df}} (\forall x_1, \ldots, x_n)(\mathbf{c}(x_1, \ldots, x_n) \to \mathbf{d}(x_1, \ldots, x_n))$.

The number-free theory of functions $\mathbb{R}^n \to \mathbb{R}$ is thus the fuzzy-logical theory of internal logical connectives (which is not surprising, the operations on reals being the standard semantics of t-norm connectives), or equivalently, of fuzzy relations on internal truth values. Depending on the expressive power of the underlying logic, various properties of the internal connectives are (number-freely) expressible and various particular connectives c are definable (in the sense that in standard models they are realized by the truth function of c) in the theory. Like in all number-free representations, the theory admits also non-standard models (e.g., Chang algebra or non-standard reals instead of $[0, 1]$), which also satisfy all theorems provable on internal connectives in FCT.

A theory of unary and binary internal connectives (which by the above considerations can also be regarded as a numberless theory of unary and binary operations on $\mathbb{R}$) has been considered in [3, 4]. These preliminary papers focus on the defining properties of *t-norms* (i.e.,

[11] Understood here as $\wedge$.

[12] See, however, [8, Rem. 3.3] for certain metamathematical qualifications regarding this correspondence.

monotony, commutativity, associativity, and the unit) and the relation of domination between internal connectives, where the latter properties have been made graded by reinterpreting the defining formulae in fuzzy logic (cf. [5, §7], [6, §2.3], or [7, §4]) and a graded theory of t-norms has thus been developed. A full paper on the topic (by the authors of [4]) is currently under construction.

8 Number-free metrics

8.1 Classical metrics

Recall that a *pseudometric* on a set X is a function $d\colon X^2 \to [0, +\infty)$ such that

$$d(x, x) = 0 \tag{8.1}$$
$$d(x, y) = d(y, x)$$
$$d(x, z) \le d(x, y) + d(y, z) \tag{8.2}$$

The definition of a *metric* strengthens (8.1) to

$$d(x, y) = 0 \text{ iff } x = y$$

while that of a *(pseudo)ultrametric* strengthens (8.2) to

$$d(x, z) \le \max(d(x, y), d(y, z))$$

In order to avoid exceptions in theorems and definitions, we shall work throughout with *extended* pseudometrics, allowing the function d to take also the value $+\infty$, but omitting the word "extended" for the sake of brevity.

8.2 Number-free metrics

The numberless reduction will first need to normalize the range of pseudometrics from $[0, +\infty]$ to $[0, 1]$. A suitable function for this is, e.g., 2^{-x}, so we shall set

$$c(x, y) = 2^{-d(x,y)} \tag{8.3}$$

(Notice that since the function 2^{-x} reverses the order, the fuzzy relation $c\colon X^2 \to [0, 1]$ expresses *closeness* rather than distance.) The defining conditions on pseudometric then become the following equivalent conditions on c:

$$c(x, x) = 1$$
$$c(x, y) = c(y, x)$$
$$c(x, z) \ge c(x, y) \cdot c(y, z)$$

These conditions are nothing else but the defining conditions of *fuzzy equivalences,* also known as *similarity relations* [27], in the standard semantics of product fuzzy logic [19]. The stronger conditions on metrics are then equivalent to c being a *fuzzy equality* (also called *separated* similarity), i.e., $c(x, y) = 1$ iff $x = y$.

We can thus equate *number-free pseudometrics* with *product similarities,* i.e., standard models of the following axioms in product fuzzy logic:

$$Cxx$$
$$Cxy \to Cyx$$
$$Cxy \,\&\, Cyz \to Cxz$$

Number-free metrics replace the first axiom by $Cxy \leftrightarrow x = y$, and (pseudo)ultrametrics replace the third by $Cxy \wedge Cyz \to Cxz$ (their definition thus works already in MTL).

8.3 Generalized number-free metrics

The notion of similarity keeps a connection to pseudometrics also for some other t-norms. E.g., with the minimum t-norm, the similarity c represents a pseudoultrametric d under the same transformation (8.3). With the Łukasiewicz t-norm, the similarity represents a *bounded* pseudometric d under a different transformation

$$c(x, y) = \frac{1 - d(x, y)}{d_{\max}}, \tag{8.4}$$

where $d_{\max} < +\infty$ is an upper bound on the distances in the pseudometric space. (The obvious both-way representation theorems are left to the reader.) Therefore a generalization of number-free pseudometrics as similarities over any expansion of MTL seems quite natural, as left-continuous t-norms just represent different ways of combining the distances $d(x, y)$ and $d(y, z)$ in the triangle inequality (8.2). Many theorems on number-free metrics hold generally over MTL and their proofs never refer to the axioms specific for product logic; the generalization is therefore quite effortless. In the context of numberless metrics, the specific axioms of product logic play just the role of capturing some of the properties of the real line that are employed when measuring distances (esp. that adding distances is cancellative).

8.4 Comparison with classical metrics

The numberless rendering reveals that a metric is in fact a kind of similarity (of 'location' or some more general quality), or a graded equivalence relation. The axioms for closeness have a rather intuitive meaning, which can in natural language be expressed as: (i) "(only) x is fully close to x", (ii) "If x is close to y, then y is close to x", and (iii) "If x is close to y and y to z, then x is close to z". These clauses can be understood as primitive observations on the property of closeness, which in (product) fuzzy logic get a precise (formalized) meaning and their correctness can be proved by the interpretation in the classical theory of metric spaces as shown above, i.e., by using standard models of first-order product logic and the transformation (8.3).

It can be observed that fuzzy logic avoids Poincaré's paradox while keeping the transitivity of closeness.[13] In this way fuzzy logic provides an account closer to intuition than classical mathematics, in which the paradox is avoided only by discarding the predicate of closeness

[13] Poincaré's paradox states that the relation of indistinguishability is intuitively transitive (if x is indistinguishable from y and y from z, then x is indistinguishable from z), but assuming its transitivity leads to a contradiction with the distinguishability of extremals in a sufficiently long series in which each two neighbors are indistinguishable. The paradox is naturally solved by using fuzzy transitivity, as it is an instance of the Sorites paradox.

and replacing it with the notion of a metric (a function to reals), which mangles the natural property of transitivity by transforming it into the triangle inequality.

The numberless account also captures the intuitive idea that "is close" is a *predicate,* albeit one with no sharp boundaries (as no precise bound to closeness can be set), but rather one which is gradual (gradually decreases). In all these respects the numberless fuzzy account fares better than the classical definition of metric.

8.5 Limits in number-free metrics

Various notions based on number-free metrics can be defined and their properties investigated in the framework of FCT. Here, only a few observations on number-free limits are given as a further illustration of the numberless approach.

Fix a metric d rendered in the numberless way by a closeness predicate C under the transformation (8.3). The *limit* $\lim_{n\to\infty} x_n$ of a sequence $\{x_n\}_{n\in\mathbb{N}}$ (abbreviated $\vec{x}$) under this numberless metric can be defined as follows:[14]

Definition 8.1. $\Delta\mathrm{Lim}_C(\vec{x}, x) \equiv_{\mathrm{df}} \Delta(\exists n_0)(\forall n > n_0)Cxx_n$

Theorem 8.2. *Standard models over product logic validate* $\Delta\mathrm{Lim}_C(\vec{x}, x)$ *iff* $x = \lim_{n\to\infty} x_n$ *under* d*.*

Proof: $\lim x_n = x$ under d iff $\limsup d(x, x_n) = 0$, iff $\liminf 2^{-d(x,x_n)} = 1$, iff $\sup_{n_0} \inf_{n>n_0} c(x, x_n) = 1$, which is the semantics of $\Delta\mathrm{Lim}_C(\vec{x}, x)$. □

The meaning of $\Delta\,\mathrm{Lim}_C$ is natural not only in product logic, in which it by Theorem 8.2 corresponds directly to the classical notion of limit, but also in other t-norm logics, esp. if the relation C is interpreted as *indistinguishability* rather than mere closeness. (Cf. footnote 13 on page 11 for the connection to Poincaré's paradox.) The definition of a limit then expresses the fact that *from somewhere on,* x_n *is indistinguishable from* x*.* Similarly, Theorem 8.4 below expresses the fact that all limits of $\vec{x}$ are indistinguishable, to the degree the indistinguishability relation is symmetric and transitive.

Discarding the Δ in the definition of $\Delta\mathrm{Lim}_C$ yields a graded number-free notion of limit:

Definition 8.3. In FCT, we define the following notions:

$$\mathrm{Lim}_C(\vec{x}, x) \equiv_{\mathrm{df}} (\exists n_0)(\forall n > n_0)Cxx_n$$
$$\lim_C \vec{x} =_{\mathrm{df}} \{x \mid \mathrm{Lim}_C(\vec{x}, x)\}$$
$$\mathrm{Convg}_C(\vec{x}) \equiv_{\mathrm{df}} (\exists x)\,\mathrm{Lim}_C(\vec{x}, x), \text{ i.e., } \mathrm{Hgt}(\lim_C \vec{x})$$

We can omit the subscript C wherever clear from the context.

Interestingly, $\mathrm{Lim}_C(\vec{x}, x)$ is exactly the graded notion of *similarity-based fuzzy limit* introduced and investigated by Gültekin Soylu in [25]. Even though he does not explicitly employ the formalism of t-norm fuzzy logic, graded theorems such as

$$\mathrm{Lim}_C(\vec{x}, x) \,\&\, \mathrm{Lim}_C(\vec{y}, y) \to \mathrm{Lim}_C(\vec{x} + \vec{y}, x + y)$$

are proved in his paper, see [25, Prop. 3.5].

With the apparatus of FCT, the gradedness of Soylu's results can be extended even further by not requiring the full satisfaction of the defining properties of the similarity C (this conforms to the standard methodology of constructing graded theories [7, §7]). An example of such graded results is the following theorem on the fuzzy uniqueness of the limit:

[14] The Δ in $\Delta\mathrm{Lim}_C$ refers to the Δ in the defining formula, which will later be dropped.

Theorem 8.4. FCT *over* MTL *proves:*

$$\mathrm{Sym}\, C \,\&\, \mathrm{Trans}\, C \to (\mathrm{Lim}_C(\vec{x}, x) \,\&\, \mathrm{Lim}_C(\vec{x}, x') \to Cxx')$$

Proof. By Trans C we obtain $Cxx_n \,\&\, Cx_nx' \to Cxx'$; thus $Cx_nx \,\&\, Cx_nx' \to Cxx'$ by Sym C, whence $((n \geq n_0) \to Cx_nx) \;\&\; ((n \geq n_0) \to Cx_nx') \to Cxx'$ follows propositionally. By generalization on n and distribution of the quantifier,

$$(\forall n \geq n_0)Cx_nx \;\&\; (\forall n \geq n_0)Cx_nx' \to Cxx'$$

is obtained (as in the consequent the quantification is void). Generalization on n_0 and shifting the quantifier to the antecedent (as $\exists$) then yields the required formula. □

A more detailed investigation of convergence based on fuzzy indistinguishability in the formal framework of FCT exceeds the scope of the present report, and is therefore left for future research.

9 Conclusions

The number-free reductions discussed in this paper are summarized in Table 9.1.

Classical notions *(in classical logic)*	**Number-free reconstructions** *(in fuzzy logic)*
function $X \to \mathbb{R}$ or $[0,1]$	fuzzy set on X
n-ary function $X^n \to \mathbb{R}$	n-ary fuzzy relation on X
finitely additive probabilistic measure	fuzzy modality axiomatized by FP(Ł)
monotony (of a function wrt crisp $\leq$)	upperness (of a fuzzy set wrt crisp $\leq$)
right-continuity (of a monotone function)	left-closedness (of an upper fuzzy set)
distribution function	fuzzy Dedekind cut
lim sup in x_0 (of a function)	x_0 is a limit point (of a fuzzy set)
lim inf in x_0 (of a function)	x_0 is an interior point (of a fuzzy set)
n-ary operation on reals	(internal) n-ary propositional connective
pseudometric	(product or Łukasiewicz) similarity
metric	(product or Łukasiewicz) equality
ultra(pseudo)metric	Gödel similarity (equality)
limit	indistinguishability-based limit

Table 9.1: Summary of number-free reductions described in this paper

Even though we have only discussed the numberless approach over t-norm logics, some other classes of logics are also suitable for the number-free rendering of classical mathematical notions, for instance uninorm logics or suitable expansions of Abelian logic. Here we have restricted our attention to t-norm logic as their properties are known sufficiently well for our purposes.

9.1 Triviality issues

Since number-free notions correspond in standard models exactly to the classical notions, a question of triviality of the whole number-free approach arises. Take, for instance, the case of number-free probability (Section 4). Even though it is a nice (but easy) observation that the laws of probability can be translated into fuzzy logic, what else do we gain in exchange for restricting the language to the operations expressible in FP(Ł) and having to use non-classical rules for reasoning? The question was already partly addressed in the Introduction. Here we shall just recall that:

1. It provides a different viewpoint on additive probability: we deal with a propositional fuzzy modality *probable* instead of a real-valued probability measure.

2. It facilitates several kinds of generalization. First, the generalization to models over non-standard algebras for Łukasiewicz logic: thus we can have, e.g., probability valued in Chang's MV-algebra, or probability valued in non-standard reals (as in [15]). Second, the generalization to measures with only partially satisfied additivity (by the graded approach to axioms, see [7]). And third, the generalization to the probability of fuzzy events, where one discards the axiom of crispness for events and adapts the finite additivity axiom for fuzzy events, e.g., as $P(\varphi \oplus \psi) \leftrightarrow ((P\varphi \rightarrow P(\varphi \,\&\, \psi)) \rightarrow P\psi)$. This approach has been taken in [16], though only over finitely-valued Łukasiewicz logic of events, as the authors strove for the completeness of the logic; in [15] this was generalized to infinitely-valued Łukasiewicz logic of events, with completeness w.r.t. non-standard reals.

3. The key difference between classical calculations with probabilities and logical derivations in FP(Ł) is the fact that the latter make inference *salva probabilitate* (i.e., salvo *probabilitatis gradu,* in the sense of P). It can be, for instance, observed that numberless probability is transmitted by modus ponens, as $P\varphi \,\&\, P(\varphi \rightarrow \psi) \rightarrow P\psi$, i.e., "if φ is probable and $\varphi \rightarrow \psi$ is probable, then ψ is probable", is a theorem of FP(Ł).

Mutatis mutandis this justification can be applied to other number-free notions as well. The assessment whether this is enough to justify the reduction has to be left to the reader and to further investigation of number-free theories

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