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Possibilistic Distributions Taking Values in an Incomplete Lattice

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Abstract:

Since the first presentation of fuzzy sets with non-numerical membership degrees by J. A. Goguen in 1969 (cf, [5]) in the greatest part of works dealing with this subject complete lattices have been considered as the structures in which these non-numerical fuzzy sets, called also possibility or possibilistic distributions in [8] and later on, take their values. The reasons for such a choice are to simplify mathematical procedures when processing fuzzy or possibility degrees, nevertheless, in what follows, we will investigate possibilistic distributions taking their values in incomplete lattices defined over the systems of all finite subsets of an infinite basic set. This structure is completed in a nonstandard (non-boolean) way in order to minimize the resulting ontologically independent inputs into the obtained complete lattice. Some results are introduced and proved, dealing with complete maxitivity of the corresponding lattice-valued possibilistic measures and with the convergence of sequences of values of such measures given a converging sequence of subsets for which these measures are defined (continuity of the measures in question).

Keywords:
Complete lattice, possibilistic distribution

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1 Introduction, Motivation, Preliminaries

The notion of fuzzy set (fuzzy subset of a crisp basic space, to be more correct) was conceived in 1965 by L. A. Zadeh in his famous pioneering paper [7]. The aim was to propose a real-valued, and easy to handle within the framework of standard mathematical analysis, characteristic for uncertainty quantification, alternative to probability measure and oriented rather to process uncertainty of the kind of fuzziness and vagueness than the uncertainty in the sense of randomness. The dominating role when processing fuzzy sets (degrees of fuzziness) is played by the operations of supremum and infimum, which may be defined not only in the unit interval $[0, 1]$ of real numbers equipped by their standard linear ordering, but also in other structures including some non-numerical ones. Moreover, there is no immediate analogy of the notion of relative frequency related to fuzzy degrees by something like a law of large numbers in probability theory. Hence, the idea of fuzzy sets with non-numerical degrees of fuzziness emerged as soon as in the paper [5] by J. A. Goguen who conceived the idea of fuzzy sets taking their degrees of fuzziness in a complete lattice. Before analyzing the Goguen's idea and approach in more detail, a very brief re-calling of the most elementary notions seems to be of use.

Let T be a nonempty set. A binary relation $\leq$ on T (i.e., a subset $\leq$ of the Cartesian product $T \times T$) is called a *partial pre-ordering* on T, if it is reflexive and transitive, i.e., if $t_1 \leq t_1$ holds for each $t_1 \in T$ and if $t_1 \leq t_2$ together with $t_2 \leq t_3$ yields $t_1 \leq t_3$ for each $t_1, t_2, t_3 \in T$. If $\leq$ is moreover, antisymmetric in the sense that, for each $t_1, t_2 \in T, t_1 \leq t_2$ and $t_2 \leq t_1$ hold together only when $t_1 = t_2$, then the relation $\leq$ is called *partial ordering* on T and the pair $\mathcal{T} = \langle T, \leq \rangle$ is called *partially ordered set (p.o.set* or *poset).*

Let $\leq$ be a partial pre-ordering on T, set $t_1 \equiv t_2$ for each $t_1, t_2 \in T$ such that $t_1 \leq t_2$ and $t_2 \leq t_1$ hold together. As can be easily seen, $\equiv$ defines an equivalence relation on T, so that we may define the factor-space $T^* = T/\equiv$ the elements of which are equivalence classes $[t]_\equiv$; here $[t]_\equiv = \{t_1 \in T : t_1 \equiv t\}$. Given $t_1, t_2 \in T$, set $[t_1]_\equiv \leq^* [t_2]_\equiv$ if and only if $t_1 \leq t_2$ holds, as can be easily seen, this relation does not depend on which representants of the equivalence classes $[t_1]_\equiv$ and $[t_2]_\equiv$ are chosen. Consequently, $\leq^*$ defines a binary relation on the factor-space $T/\equiv$ and, as can be easily proved, $\leq^*$ is a partial ordering on $T/\equiv$, so that $\mathcal{T}^* = \langle T/\equiv, \leq^* \rangle$ defines a p.o.set.

Let $\mathcal{T} = \langle T, \leq \rangle$ be a p.o.set, let $\emptyset \neq A \subset T$ hold. The *supremum* of A w.r.to $\mathcal{T}$ (if it exists) is defined as the smallest upper bound of A and it is denoted by $\bigvee^{\mathcal{T}} A$, the *infimum* of A w.r.to $\mathcal{T}$ (if it exists) is defined as the greatest lower bound of A and it is denoted by $\bigwedge^{\mathcal{T}} A$. If $A = \{t_1, t_2\}$, or $A = \{t_1, t_2, \ldots, t_n\}, n \in N$, we write $t_1 \vee^{\mathcal{T}} t_2$ or $\bigvee_{i=1}^{\mathcal{T} n} t_i$, and similarly for infimum, the index $\mathcal{T}$ being omitted, if no misunderstanding menaces.

As a matter of fact, if $\mathcal{T} = \langle T, \leq \rangle$ is a p.o.set, then in general $\bigvee^{\mathcal{T}} A$ and/or $\bigwedge^{\mathcal{T}} A$ need not be defined for some $A \subset T$. P.o.set $\mathcal{T} = \langle T, \leq \rangle$ is called a *lower semi-lattice* (an *upper semi-lattice*, resp.) if $\bigwedge^{\mathcal{T}} A (\bigvee^{\mathcal{T}} A$, resp.) is defined for each *finite* $A \subset T$. If this condition holds for *each* $A \subset T$, then $\mathcal{T}$ is called a *complete lower semi-lattice* (a *complete upper semi-lattice,* resp.). If $\mathcal{T} = \langle T, \leq \rangle$ defines, at the same time, a (complete) lower semilattice and a (complete) upper semilattice, then $\mathcal{T}$ is a (complete) lattice. Hence, p.o.set $\mathcal{T} = \langle T, \leq \rangle$ defines a complete lattice, if for each $\emptyset \neq A \subset T$, $\bigvee^{\mathcal{T}} A$ and $\bigwedge^{\mathcal{T}} A$ are defined (if $A = \emptyset$, the convention $\bigvee^{\mathcal{T}} \emptyset = \bigwedge T = \oslash_{\mathcal{T}}$ and $\bigwedge^{\mathcal{T}} \emptyset = \bigvee T = \mathbf{1}_{\mathcal{T}}$ applies). Both the most often used structures for quantification, the unit interval with the standard linear ordering of real numbers as well as the power-set of all subsets of a non-empty space partially ordered by the set inclusion, obviously define complete lattices.

Hence, the Goguen's decision to limit the investigation of non-numerical fuzzy sets to those taking their values in complete lattices, followed by the greatest part of researchers oriented toward this or a close field of investigation, was quite a reasonable simplification enabling to eliminate cumbersome technical difficulties following from the necessity to analyze in particular each supremum and infimum operation occurring in particular steps of our mathematical reasoning (for a more detailed discussion the reader could be referred to [2], [3], [5] or elsewhere).

However, the aim or each rational research effort consists not only in looking for conditions strong enough in order to deduce some rich, important or interesting results, but also in analyzing which of these consequences remain (and which not) to be valid when weakening the conditions imposed on the input structures. In what follows, we will investigate fuzzy sets taking their values in a particular incomplete lattice to be specified in Section 2. A more detailed discussion on the sensefulness and appropriateness of this particular choice will be given in the last section, having already at hand some results on which the consequences of the discussion should be based on.

The following terminological note should be re-called. Given a nonempty space Ω, a fuzzy set $\pi : \Omega \to [0,1]$ and a crisp subset $A \subset \Omega$, Zadeh defines, in [8], by $\sup_{\omega \in A} \pi(\omega)$ the portion of the total fuzziness, defined by π, located in the subset A. This values is denoted by $\Pi(A)$ and the mapping $\Pi : \mathcal{P}(\Omega) \to [0,1]$ is called the *possibilistic* (or *possibility*) *measure* defined by π; the mapping π is, in this context, called a (*real-valued*) *possibilistic distribution* on Ω. We will continually apply this terminology in what follows.

For a more detailed and systematic information dealing with p.o.sets, lattices, Boolean algebras, and structures in general, the reader is recommended to consult either the already classical monographs [1], [4], or [6] or some more recent monographs and textbooks.

2 An Incomplete Set-Valued Lattice and Its Non-Boolean Completion

Let X be an infinte set, let $\mathcal{P}(X)$ denote the system of all subsets of X (the power-set induced by X), let $\mathcal{P}_f(X) \subset \mathcal{P}(X)$ denote the system of all *finite* subsets of X including the empty set $\emptyset$, let $\subset$ denote the set inclusion on $\mathcal{P}(X)$. We will write $A \subset_f X$, if X is a finite subset of X so that $A \subset_f X$ denotes the same as $A \in \mathcal{P}_f(X)$. Setting $\mathcal{T} = \langle \mathcal{P}_f(X), \subseteq \rangle$ we obtain easily that $\mathcal{T}$ is a lattice on $T = \mathcal{P}_f(X)$ induced by the partial ordering $\subseteq$. Moreover, $\mathcal{T}$ is a complete lower semilattice, however, for an infinite $\mathcal{A} \subset \mathcal{P}_f(X)$ the supremum $\bigvee^{\mathcal{T}} \mathcal{A}$ is not defined in $\mathcal{T}$.

Consider the following binary relation $\leq$ on $\mathcal{P}(X)$ (i.e., $\leq \subset \mathcal{P}(X) \times \mathcal{P}(X)$) : (i) if $A, B \subset_f X$ holds, then $A \leq B$ coincides with the standard set inclusion $A \subseteq B$ on $\mathcal{P}_f(X)$, and (ii) if B is infinite, i.e., if $B \in \mathcal{P}(X) - \mathcal{P}_f(X)$ is the case, then $A \leq B$ holds for each $A \subset X$ (no matter whether the inclusion $A \subseteq B$ in the standard sense is valid). As can be easily seen, the binary relation $\leq$ defines a partial pre-ordering on the power-set $\mathcal{P}(X)$. Hence, we can define an equivalence relation $\equiv$ on $\mathcal{P}(X)$, setting $A \equiv B$ iff $A \leq B$ and $B \leq A$ holds together and, consequently, to consider the factor-space $\mathcal{P}(X)| \equiv$ with the uniquely defined binary relation $\leq^*$ between equivalence classes $[A]_\equiv, [B]_\equiv \in \mathcal{P}(X)| \equiv$.

Analyzing this construction in more detail, we obtain that for each $A \subset_f X$ the equivalence class $[A]$ reduces to the singleton $\{A\}$ (singleton when related to $\mathcal{P}(X)$), but all infinite subsets of X are covered by one equivalence class which may be denoted by $[X]$, hence $[X] = \mathcal{P}(X) - \mathcal{P}_f(X)$. The corresponding partial ordering $\leq^*$ on $\mathcal{P}(X)| \equiv$ then reads that $[A] \leq^* [B]$ holds if $A, B \subset_f X$ and $A \subset B$ is the case, or if $[B] = [X]$. Simplifying our notation and hoping that no misunderstanding menaces, we may replace the structure $\langle \mathcal{P}(X)| \equiv, \leq^* \rangle$ by an isomorphic structure $\mathcal{T}^* = \langle \mathcal{P}_f(X) \cup \{X\}, \subseteq \rangle$, where $\subseteq$ may be taken as the standard set inclusion on $\mathcal{P}_f(X) \cup \{X\}$.

The following fact is easy to prove.

Fact 1. For $T^* = \mathcal{P}_f(X) \cup \{X\}$ the structure $\mathcal{T}^* = \langle T^*, \subseteq \rangle$ defines a complete lattice with the empty set $\emptyset$ as the *minimum* or *zero (element)* of $\mathcal{T}^*$ (denoted also by $\emptyset_{\mathcal{T}^*}$) and with X as the *maximum* or *unit (element)* of $\mathcal{T}^*$, denoted also by $\mathbf{1}_{\mathcal{T}^*}$. The index $\mathcal{T}^*$ will be omitted, if no misunderstanding menaces. In particular, this will be the case in the next section when only the complete lattice $\mathcal{T}^*$ will be considered.

3 $\mathcal{T}^*$-Lattice-Valued Possibilistic Distributions and Measures

Let Ω be a nonempty set, let $\pi : \Omega \to \mathcal{P}_f(X) \cup \{X\}$ be a mapping ascribing to each $\omega \in \Omega$ a finite subset of X or the set X itself. Set, for each $A \subset \Omega$, $\sigma(E, \pi) = \{\pi(\omega) : \omega \in E\}$, hence, $\sigma(E, \pi)$ denotes the set of values taken by $\pi(\omega)$ when ω ranges over E. If $\sigma(E, \pi)$ is finite, then the set E is called π-*finite* (in particular, it is trivially the case when E is finite). Hence, if E is π-finite, then there exists a finite subset $E_0 \subset E$ such that $\sigma(E, \pi) = \sigma(E_0, \pi)$ even if E itself is an infinite subset of Ω.

Definition 1 *Given* $\Omega \neq \emptyset$ *and* $\pi : \Omega \to T^*$, *set* $\Pi(E) = \bigvee^{\mathcal{T}^*}_{\omega \in \sigma(E,\pi)} \pi(\omega)$ *for each* $\emptyset \neq E \subseteq \Omega$. *The mapping* π *is called a* $\mathcal{T}^*$*-(valued) possibilistic distribution on* Ω, *if* $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}^*}(= X)$ *holds; if this* *is the case, then the mapping* $\Pi : \mathcal{P}(\Omega) \to T^*$ *is called the* $\mathcal{T}^*$*-(valued) possibilistic measure on* $\mathcal{P}(\Omega)$ *induced by* π *on* Ω.

If $E \subset \Omega$ *is not* π*-finite, we say that* E *is* π*-infinite.*

Lemma 1 *Let* π *be a* $\mathcal{T}^*$*-possibilistic distribution on a nonempty set* Ω*. Then* $\Pi(E) = \mathbf{1}_{\mathcal{T}}(= X)$ *for each* σ*-infinite* $E \subset \Omega$.

Proof. Let $E \subset \Omega$ be π-infinite, so that π takes infinitely many different values from T^* for ω ranging over E. Hence, the relation $\pi(\omega) \leq \bigvee_{\omega_1 \in E}^{\mathcal{T}^*} \pi(\omega)$ must be valid for infinite number of different values from T^*. However, the only element in T^* meeting this condition is the unit element $\mathbf{1}_{\mathcal{T}^*} = X$, so that $\bigvee_{\omega \in E}^{\mathcal{T}^*} \pi(\omega) = \Pi(E) = \mathbf{1}_{\mathcal{T}^*}$ follows.

Consequently, if π is a $\mathcal{T}^*$-possibilistic distribution on Ω and Ω is π-finite, then there exists $\omega_0 \in \Omega$ such that $\pi(\omega_0) = X(= \mathbf{1}_{\mathcal{T}^*})$. □

Theorem 1 *Let* π *be a* $\mathcal{T}^*$*-possibilistic distribution on a nonempty set* Ω*. Then the* $\mathcal{T}^*$*-possibilistic measure* Π *on* $\mathcal{P}(\Omega)$*, induced by* π*, is completely maxitive in the sense that for each nonempty system* $\mathcal{E}$ *of nonempty subsets of* Ω *the relation*

$$\Pi\left(\bigcup \mathcal{E}\right) = \Pi\left(\bigcup_{E \in \mathcal{E}} E\right) = \bigvee_{E \in \mathcal{E}}^{\mathcal{T}^*} \Pi(E) \tag{3.1}$$

is valid.

Proof. For each $\emptyset \neq E \subset \Omega$, the relation $\Pi(E) = \bigvee^{\mathcal{T}^*}\{x : x \in \sigma(E, \pi)\}$ holds by definition. For each $\emptyset \neq \mathcal{E} \subset \mathcal{P}(\Omega)$ and each $x \in T^* = \mathcal{P}_f(X) \cup \{X\}, x \in \sigma(\cup\mathcal{E}, \pi)$ holds iff $x \in \sigma(E, \pi)$ is the for some $E \in \mathcal{E}$, consequently, the relation

$$\sigma\left(\bigcup \mathcal{E}, \pi\right) = \bigcup_{E \in \mathcal{E}} \sigma(E, \pi) \tag{3.2}$$

holds. If $X \in \sigma(E_1, \pi)$ for some $E_1 \in \mathcal{E}$, then $X \in \sigma(\bigcup \mathcal{E}, \pi)$ holds as well, so that

$$\Pi\left(\bigcup \mathcal{E}\right) = X = \bigvee_{E \in \mathcal{E}}^{\mathcal{T}^*} \Pi(E) \tag{3.3}$$

trivially results.

Let $\sigma(E, \pi) \subset_f X$ hold for each $A \subset \mathcal{E}$, let $\bigcup \mathcal{E}$ be π-finite. Then $\bigcup_{E \in \mathcal{E}} \sigma(E, \pi) \subset_f X$ holds, so that

$$\Pi\left(\bigcup \mathcal{E}\right) = \bigvee^{\mathcal{T}^*}\{x : x \in \bigcup_{E \in \mathcal{E}} \sigma(E, \pi)\} = \bigvee_{E \in \mathcal{E}}\left(\bigvee^{\mathcal{T}^*}\{x : x \in \sigma(E, \pi)\}\right) = \bigvee_{E \in \mathcal{E}}^{\mathcal{T}^*} \Pi(E) \tag{3.4}$$

holds. Finally, let $\bigcup \mathcal{E}$ be π-infinite, let $\sigma(E, \pi) \subset_f X$ hold for each $E \in \mathcal{E}$. Then the set $\bigcup_{E \in \mathcal{E}} \sigma(E, \pi)$ is infinite, so that the only element of T^* containing every $\sigma(E, \pi), E \in \mathcal{E}$, is the set X, so that also in this case the relation

$$\begin{aligned}
&\bigvee_{E \in \mathcal{E}}^{\mathcal{T}^*} \Pi(E) = \bigvee_{E \in \mathcal{E}}^{\mathcal{T}^*} \{x : x \in \sigma(E, \pi)\} = X = \bigvee^{\mathcal{T}^*}\left\{x : x \in \bigcup_{E \in \mathcal{E}} \sigma(E, \pi)\right\} = \\
= \quad &\bigvee^{\mathcal{T}^*}\left\{x : x \in \sigma\left(\bigcup \mathcal{E}, \pi\right)\right\} = \Pi\left(\bigcup \mathcal{E}\right)
\end{aligned} \tag{3.5}$$

is valid. The assertion is proved. □

Corollary 1 *Let the notations and conditions of Theorem 1 hold. Then* $\mathcal{T}^*$*-possibilistic measure* Π *on* $\mathcal{P}(\Omega)$ *is monotonous in the sense that* $\Pi(A) \leq \Pi(B)$ *holds for each* $A \subset B \subset \Omega$.

Proof. Obviously, for each $A \subset B \subset \Omega$ we obtain that

$$\Pi(A) \leq \Pi(A) \vee \Pi(B) = \Pi(A \cup B) = \Pi(B)$$

holds. $\square$

4 Modified $\mathcal{T}^*$-Possibilistic Distribution

The following alternative definition of $\mathcal{T}^*$-valued possibilistic measure purposely neglects the difference between finite and π-finite subsets of Ω, playing an important role in Definition 1.

Definition 2 *Consider the complete lattice $\mathcal{T}^* = \langle \mathcal{P}_f(X) \cup \{X\}, \subseteq \rangle$ as above, a nonempty set Ω and a mapping $\pi_0 : \Omega \to T^* = \mathcal{P}_f(X) \cup \{X\}$. Set, for each $E \subset \Omega$, $\Pi_0(E) = \bigvee_{\omega \in E} \pi(\omega)$, if E is finite, $\Pi_0(E) = X$ for each infinite $E \subset \Omega$. If $\Pi(\Omega) = X$, then π_0 is called modified $\mathcal{T}^*$-possibilistic distribution on Ω (m-$\mathcal{T}^*$-distribution, abbreviately) and Π_0 is called the modified $\mathcal{T}^*$-possibilistic measure (m-$\mathcal{T}^*$-measure, abbreviately) induced by π_0 on $\mathcal{P}(\Omega)$.*

Hence, if Ω is infinite, then each $\pi_0 : \Omega \to T^*$ defines an m-$\mathcal{T}$ defines an m-$\mathcal{T}^*$-distribution on Ω, if Ω is finite, then this is the case iff $\pi_0(\omega_0) = X$ for some $\omega_0 \in \Omega$.

Lemma 2 *Let π_0 be an m-$\mathcal{T}^*$-possibilistic distribution on Ω, then the induced m-$\mathcal{T}^*$-measure Π_0 on $\mathcal{P}(\Omega)$ is finitely maxitive in the sense that $\Pi_0(E \cup F) = \Pi_0(E) \vee \Pi_0(F)$ holds for each $E, F \subset \Omega$.*

Proof. Let E or F (or both) be infinite subset(s) of Ω, then either $\Pi_0(E)$ or $\Pi_0(F) = X$, but $E \cup F$ is also an infinite subset of Ω, so that $X = \Pi_0(E \cup F) = \Pi_0(E) \vee \Pi_0(F)$ follows. If both E and F are finite subsets of Ω, then

$$\Pi_0(E \cup F) = \bigvee_{\omega \in E \cup F} \pi(\omega) = \bigvee_{\omega \in E} \pi(\omega) \vee \bigvee_{\omega \in F} \pi(\omega) = \Pi_0(E) \vee \Pi_0(F) \tag{4.1}$$

holds. The assertion is proved. $\square$

Contrary to the properties of $\mathcal{T}^*$-possibilistic measure Π introduced and investigated above, m-$\mathcal{T}^*$-measure Π_0 need not be completely maxitive on $\mathcal{P}(\Omega)$. Indeed, let $\{\Omega_i\}_{i=1}^{\infty}$ be a disjoint decomposition of an infinite countable set Ω, let each Ω_i be finite. Let Y be a finite system of finite subsets of X, let $\pi(\omega) \in Y$ for each $\omega \in \Omega$. Then $\Pi_0(\Omega_i) = \bigvee_{\omega \in \Omega_i} \pi_0(\omega) \leq \bigvee Y = \bigvee_{x \in Y} x < X$ holds for each Ω_i, hence, $\bigvee\{\Pi_0(\Omega_i) : i = 1, 2, \dots\} \leq \bigvee Y < X = \Pi_0(\Omega)$ follows, consequently, Π_0 is not completely maxitive on $\mathcal{P}(\Omega)$.

Let us note that taking the same $\pi_0 : \Omega \to T^*$ but applying the definition of Π instead of that of Π_0 we obtain that the set Ω is π_0-finite, as the set $\sigma(\Omega, \pi_0) \subset Y$ is finite. Hence, $\Pi(\Omega) \leq \bigvee Y < X$ holds and π_0 is not a $\mathcal{T}^*$-possibilistic distribution on Ω in the sense of Definition 1 Obviously, each $\mathcal{T}^*$-possibilistic distribution on Ω defines also an m-$\mathcal{T}^*$-distribution on Ω, as $\Pi(\Omega) = X$ holds, but the inverse implication is not the case in general, as we have just observed.

5 Convergence and Continuity of $\mathcal{T}^*$-Valued Possibilistic Measures

According to the standard definition, a sequence $\mathcal{E} = \{E_1, E_2, \dots\}$ of subsets of a space Ω tends (converges) to a set $E_0 \subset \Omega$, if the identity

$$\liminf\{E_n\} =_{df} \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} E_j = \limsup\{E_n\} =_{df} \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j = E_0 \tag{5.1}$$

holds. We will write $\mathcal{E} \to E_0$ or $\{E_n\} \to E_0$, if (5.1) holds.

This standard definition of convergence can be weakened as follows.

Definition 3 *Let Y be a nonempty set, let $\approx$ denote an equivalence relation on the power-set $\mathcal{P}(Y)$, hence, for each $E_1, E_2, E_3 \subseteq Y$, (i) $E_1 \approx E_1$, (ii) if $E_1 \approx E_2$, then $E_2 \approx E_1$, and (iii) if $E_1 \approx E_2$ and $E_2 \approx E_3$ holds, then $E_1 \approx E_3$ follows. A sequence $\{E_1, E_2, \dots\}$ of subsets of Y tends (converges) to $E_0 \subset Y$ with respect to equivalence relation $\approx$ ($\{E_n\} \to_{\approx} E_0$, in symbols), if there exists, for each $n = 0, 1, 2, \dots,$ a subset $E_n^* \subset Y$ such that $E_n \approx E_n^*$ holds and $\{E_n^*\}$ tends to E_0^* in the standard sense of (5.1).*

As can be easily seen, if the identity relation on $\mathcal{P}(Y)$ is taken as the equivalence relation $\approx$ between subsets of Y, then the convergence w.r.to $\approx$ reduces to the standard convergence in the sense of (5.1).

Theorem 2 *Let $\mathcal{T}^* = \langle \mathcal{P}_f(X) \cup \{X\}, \subseteq \rangle$ be the complete lattice defined above, let Ω be a nonempty set, let $\pi : \Omega \to \mathcal{T}^* = \mathcal{P}_f(X) \cup \{X\}$ be a $\mathcal{T}^*$-possibilistic distribution Ω, so that $\Pi(\Omega) = X$, let $E_1 \subset E_2 \subset \dots$ be a sequence of subsets of Ω such that $\{E_n\} \to \Omega$ holds, let $\equiv$ be the equivalence relation on $\mathcal{P}(X)$ such that $[x]_{\equiv} = \{x\}$ for each $x \in \mathcal{P}_f(X)$, and $[x]_{\equiv} = [X] = \mathcal{P}(X) - \mathcal{P}_f(X)$ for each infinite $x \subseteq X$. Then $\Pi(E_n) \to_{\equiv} X = \Pi(\Omega)$ holds, hence, $\Pi(\bigcup_{j=1}^{n} E_j)$ tends to $\Pi(\bigcup_{j=1}^{\infty} E_j) = \Pi(\Omega) = X$ with respect to the equivalence relation $\equiv$ on $\mathcal{P}(X)$.*

Proof. Let $\pi(\omega_0) = X$ for some $\omega_0 \in \Omega$. As $\{E_n\} \to \Omega$ holds (in the standard sense), then there exists $n_0 \in \{1, 2, \dots\}$ such that $\omega_0 \in E_n$ for each $n \geq n_0$, hence, $\pi(\omega_0) = X = \sigma(E_n, \pi)$ for each $n \geq n_0$, consequently, $\Pi(E_n) = X = \Pi(\Omega)$ holds for each $n \geq n_0$ so that the assertion $\Pi(E_n) \to_{\equiv} X$ holds trivially. If some $E_i \subset \Omega$ is π-infinite, then $\Pi(E_j) = X$ holds for each $j \geq i$, as we suppose that $E_i \subset E_j$ is valid for each $j \geq i$. Hence, $\Pi(E_j) \to X$ in the standard sense of (5.1) so that $\Pi(E_j) \to_{\approx} X$ holds for each equivalence relation $\approx$ on $\mathcal{P}(X)$ including the relation $\equiv$.

What remains to be analyzed is the case when each E_i in the sequence $\{E_i\}$ of subsets of Ω under consideration is π-finite and that $\pi(\omega) \neq X$, $\omega \in \Omega$, as the opposite case has been already solved. As $\Pi(\Omega) = X$ due to the assumptions imposed on π, it follows that the space Ω is π-infinite, hence $\sigma(\Omega, \pi) = \{\pi(\omega) : \omega \in \Omega\} = \bigcup_{j=1}^{\infty} \sigma(E_j, \pi)$ is an infinite subset of X. Indeed, each $\omega \in \Omega$ is in each E_j for $j \geq n_0(\omega) \in \{1, 2, \dots\}$, consequently, each $x \in \mathcal{P}_f(X), x = \pi(\omega)$ for some $\omega \in \Omega$, is in each $\sigma(E_j, \pi)$, $j \geq n_0(\omega) \in \{1, 2, \dots\}$ for some $n_0(\omega)$. Moreover, as each E_i is π-finite, the relation $\Pi(E_i) = \bigvee_{\omega \in E_i} \pi(\omega) = \bigcup_{\omega \in E_i} \pi(\omega) = \sigma(E_i, \pi)$ is valid. Consequently, $\Pi(E_i)$ tends to $\bigcup_{j=1}^{\infty} \sigma(E_j, \pi)$ in the standard sense of convergence of subsets of X and the equivalence relation $\bigcup_{j=1}^{\infty} \sigma(E_j, \pi) \equiv X$ is valid, as both the last sets are infinite subsets of X. Hence, $\Pi(E_n) \to_{\equiv} X = \Pi(\Omega)$ follows and the assertion is proved. □

The result just proved does not hold, in general, for other equivalence relations $\approx$ on $\mathcal{P}(X)$, in particular, it does not hold for the relation of identity of subsets of X. In other terms, the sets $\sigma(E_n, \pi)$ tend in the standard sense to $\bigcup_{n=1}^{\infty} \sigma(E_n, \pi)$, but this subset of X, even if infinite, need not be identical with X. Indeed, let $\Omega = \{\omega_1, \omega_2, \dots\}$ and $X = \{x_1, x_2, \dots\}$ be infinite countable sets, let $\pi(\omega_i) = \{x_{2i}\} \subset \mathcal{P}_f(X)$ hold for each $i = 1, 2, \dots$, let $E_n = \{\omega, \omega_2, \dots, \omega_n\} \subset \Omega$ for each $n = 1, 2, \dots$. Then $\{E_n\}$ tends to $\bigcup_{n=1}^{\infty} E_n = \Omega$ in the standard sense and $\sigma(E_n, \pi) = \{\pi(\omega) : \omega \in E_n\} = \{x_2, x_4, \dots, x_{2n}\}$ for each $n = 1, 2, \dots$. So, the space Ω is π-infinite, as $\sigma(\Omega, \pi) = \bigcup_{n=1}^{\infty} \sigma(E_n, \pi) = \{x_2, x_4, \dots\}$ is an infinite subset of X, hence, $\bigvee_{n=1}^{\infty} \sigma(E_n, \pi) = X$, as the supremum is defined in the sense of complete lattice $\mathcal{T}^* = \langle \mathcal{P}_f(X) \cup \{X\}, \subseteq \rangle$. However, $\bigcup_{n=1}^{\infty} \sigma(E_n, \pi) = \{x_2, x_4, \dots\} \neq X$, so that Theorem 2 does not hold when replacing the equivalence relation $\equiv$ by identity $=$ on $\mathcal{P}(X)$.

Let us consider the problem of convergence and continuity for the case of the modified $\mathcal{T}^*$-possibilistic measure Π_0 on $\mathcal{P}(\Omega)$. Let Ω be an infinite countable set, let $\pi_0(\omega) = x_0 \subset_f X$ for each $\omega \in \Omega$, so that π_0 defines an m-$\mathcal{T}^*$-possibilistic distribution on Ω (let us recall that in the modified case $\Pi(\Omega) = X$ due to the simple fact that the set Ω is infinite no matter which the values $\pi_0(\omega)$ may be). Let $\{E_n\}$ be a sequence of finite subsets of Ω such that $\{E_n\}$ tends to Ω in the standard sense. So, $\Pi_0(E_n) = x_0$ for each $n = 1, 2, \dots,$ so that $\Pi_0(E_n)$ tends to x_0 in the standard sense. However, $x_0 \subset_f X$ yields that $x_0 \neq X, x_0 \not\equiv X$ is the case. So, $\Pi_0(E_n)$ does not tend to $\Pi_0(\Omega)$ neither in the standard sense nor (contrary to the case of $\mathcal{T}^*$-possibilistic measure Π) with respect to the equivalence relation $\equiv$ on $\mathcal{P}(X)$.

6 Comments and Conclusions

Our approach to the problems arising when weakening the assumptions imposed on the structures in which uncertainty degrees take their values seems to be rather conservative in the sense that we tried to complete somehow the incomplete lattice under consideration, keeping in mind that this completion should be ontologically as weak as possible. In other terms expressed, the new inputs enriching the original incomplete lattice should be just the minimum and weakest ones leading to a completion of the original lattice, but implying as small as possible portion of further properties in which both the lattices differ from each other (this is obviously not the case when completing $\mathcal{T} = \langle \mathcal{P}_f(X), \subseteq \rangle$ by extending $\mathcal{P}_f(X)$ to $\mathcal{P}(X)$). The shift from $\mathcal{P}_f(X)$ to $\mathcal{P}_f(X) \cup \{X\}$ seems to meet much better the ontological minimization principle sketched above.

Taking the problem of completion at its most abstract level, we could consider an abstract incomplete lattice $\mathcal{T}_0 = \langle T_0, \leq_0 \rangle$ subsequently enriched by the abstract entity $\mathbf{1}_{\mathcal{T}_0}$ endowed by the only property that $x \leq_0 \mathbf{1}_{\mathcal{T}_0}$ holds for each $x \in T_0$. The reason for choosing just the particular case $\mathcal{T} = \langle \mathcal{P}_f(X), \subseteq \rangle$ of $\langle T_0, \subseteq \rangle$ as the starting point of our considerations and constructions is that in this case also some entities beyond the support $\mathcal{P}_f(X)$ of $\mathcal{T}$, namely, infinite subsets of X, can be analyzed and processed, using the tools of elementary set theory, even if those entities (infinite subsets of X) cannot be taken as fuzziness degrees ascribed to some subsets of the space Ω.

Given a complete lattice $\mathcal{T}_1 = \langle T_1, \leq_1 \rangle$, a $\mathcal{T}_1$-possibilistic distribution π on a nonempty space Ω, and a subset A of Ω, the relation $\Pi(A) = \bigvee_{\omega \in A}^{\mathcal{T}_1} \pi(\omega)$ may be taken either as a definition of the value $\Pi(A)$, but also as the only tool enabling to compute the value $\Pi(A)$ given the value $\pi(\omega)$ for each $\omega \in A$. From this point of view, if A is finite (or at least if the set $\sigma(A, \pi) = \{\pi(\omega : \omega \in A\}$ is finite) we are able, at least in principle, to compute the supremum value $\Pi(A) = \bigvee_{\omega \in A}^{\mathcal{T}_1} \pi(\omega)$. If the set A (or the set $\sigma(A, \pi)$ is infinite, such a computation is impossible and we have to accept that $\Pi(A) = \mathbf{1}_{\mathcal{T}_1} (= X$, if $\mathcal{T}_1 = \mathcal{T}^*)$ as it describes the case with the greatest nonspecificity degree for $\Pi(A)$. Both these approaches are formalised by $\mathcal{T}^*$-possibilistic measures Π and their modified versions Π_0 introduced and analyzed above.

The idea according to which the values $\Pi(A)$ can be effectively computed only when the set $\sigma(A, \pi)$ (or the set A itself in the modified case) is finite may be strenghened in the sense that the value $\Pi(A)$ is effectively accessible only when the cardinality of the set $\sigma(A, \pi)$ or A does not exceed on apriori given natural number R. Hence, if $card(\sigma(A, \pi)) > R$ (in the case of $\mathcal{T}^*$-possibilistic measure Π) or if $card(A) > R$ (in the modified case of $\mathcal{T}^*$-possibilistic measure Π_0) holds, we set $\Pi(A) = X$ (or $\Pi_0(A) = X$), so that the sets the cardinality of which exceeds R are processed as infinite subsets of X (or of Ω) in both the models introduced and analyzed above. Another way how to introduce a finite restriction into our former models may read as follows: given a $\mathcal{T}^*$-possibilistic distribution π on a space Ω and given a fixed finite subset $Y \subset X$, we reduce π to $\pi^Y : \Omega \to \mathcal{P}(Y)$, setting $\pi^Y(\omega) = \pi(\omega) \cap Y$ for each $\omega \in \Omega$; the resulting mapping then defines a $\mathcal{T}^Y$-possibilistic distribution of $\mathcal{T}^Y = \langle \mathcal{P}(Y), \subseteq \rangle$.

Both these finite restrictions of the complete lattice $\mathcal{T}^* = \langle \mathcal{P}_f(x), \subseteq \rangle$, as well as their mutual relations, deserve perhaps a more detailed analysis, in particular, from the point of view whether, and in which sense and degree, these finite restrictions approximate the complete lattice $\mathcal{T}^*$, e.g., with the value R of the free parameter increasing. A further investigation in this direction, as well as the looking for some qualitatively new approaches, different from the idea of a conservative completion of the original incomplete lattice ($\mathcal{T} = \langle \mathcal{P}_f(X), \subseteq \rangle$, in our case) as applied above, seems to be useful and interesting, and the author hopes to have an opportunity to go on in this effort at an appropriate future occasion.

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