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Abstract:

Graded properties of binary and unary connectives valued in $\mathrm{MTL}_\triangle$ algebras are studied employing the apparatus of Fuzzy Class Theory (or higher-order fuzzy logic) as a tool for easy derivation of graded theorems on the connectives. The paper focuses on graded properties of conjunctive connectives such as t-norms, uninorms, aggregation operators, or quasicopulas. The properties studied include graded monotony, Lipschitz property, commutativity, associativity, unit elements, and dominance.

Keywords:
Connectives, Fuzzy Class Theory, Dominance

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1 Introduction

We propose to study a 'graded' generalization of connectives, or operators on truth degrees, including copulas, t-norms, uninorms, and other aggregation operators. We restrict our attention to unary and binary connectives here. As a tool we use the framework of higher-order fuzzy logic, also known as Fuzzy Class Theory (FCT) introduced in [4]. FCT is specially designed to allow a quick and sound development of graded, lattice-valued generalizations of the notions of traditional 'fuzzy mathematics' and is a backbone of a broader program of logic-based foundations for fuzzy mathematics, described in [5].

This short abstract is to be understood as just a 'teaser' of the broad and potentially very interesting area of graded properties of fuzzy connectives. We sketch basic definitions and properties present a few examples of results in the area of equivalence and order relations (in particular, we show interesting graded generalization of basic results from [9]). Also some of our theorems are, for expository purposes, stated in a less general form here and can be further generalized substantively.

Ad motivation: In the traditional, non-graded theory of domination, a theorem says nothing if one of its precondition fails. Often, however, a precondition is *almost* satisfied (e.g., the difference of the actual membership degree from the required one is less than .001). Although the traditional non-graded theorem is not applicable in such cases, it seems obvious (and indeed is provably the case, as we shall show) that if the premises of the theorem are *almost* true, so will be the conclusion. In practice, however, the imperfection of a premise can influence the imperfection of a conclusion to different degree (e.g., the truth degree of some premise may be much more important for the conclusion than that of another premise). We shall study the transmission of guaranteed truth degrees from premises to conclusions in such cases, and our theorems will express the thresholds guaranteed for conclusions given the truth degrees of premises.

Such studies are not uncommon in traditional mathematics—cf., e.g., error analysis in probability theory and statistics, the study of defects of mathematical properties in [1], or the usage of quantifiers like *almost everywhere* in various parts of mathematics (where measure theory provides the apparatus for giving the estimates as to the scope of validity of conclusions). In fuzzy mathematics, graded properties have been studied esp. in the theory of fuzzy relations [11, 12, 3] and fuzzy topology [17, 18].

Since we are concerned with the transmission of partial degrees of validity, which is the primary motivation for *deductive fuzzy logics* (cf. [2]), we shall use deductive fuzzy logic as a tool for our study. In particular, as we need a sufficient expressive power of the logical apparatus, we shall use *higher-order* fuzzy logic, also known as *Fuzzy Class Theory* FCT [4].[2]

Besides deriving general results in the axiomatic framework of FCT, we shall also present some semantic results for particular classes of aggregation operators, t-norms, and other truth-value operators on the real unit interval $[0,1]$. In the derivation of the latter kind of results we shall not employ the apparatus of FCT, but rather traditional methods of reasoning about *standard models* of FCT, which consist exactly of Zadeh's fuzzy sets (and fuzzy relations) of all orders over some fixed crisp domain.

2 Preliminaries

Unless stated otherwise, we work in Fuzzy Class Theory (FCT) over the logic $\mathrm{MTL}_\triangle$ of all left-continuous t-norms [10]. The apparatus of FCT and its standard notation is explained in detail in the primer [6], which is freely available online. Furthermore we shall use the following useful definitions and conventions.

[2]For the deductive framework of FCT to be applicable, the structure of truth degrees has to be at least a (linear) $\mathrm{MTL}_\triangle$-algebra (see [2]), possibly with additional operators. The logical connectives of the logic $\mathrm{MTL}_\triangle$ (in the standard case, a left-continuous t-norm for conjunction and its residuum for implication) will thus be used for logical deductions to get the estimates of the graded properties. However, the class of operations studied will be much broader: since the theorems themselves are valid in standard Zadeh models of FCT (consisting of all usual $[0,1]$-valued fuzzy sets of all orders and arities over a fixed crisp domain), they actually apply to *all* (at most binary) operations on $[0,1]$.

Definition 2.1 *We define the following derived propositional connectives of* MTL$_\triangle$*:*

$$\begin{aligned}\varphi \le \psi &\equiv_{\mathrm{df}} \triangle(\varphi \to \psi)\\ \varphi = \psi &\equiv_{\mathrm{df}} \triangle(\varphi \leftrightarrow \psi)\\ \varphi \ne \psi &\equiv_{\mathrm{df}} \neg(\varphi = \psi)\\ \varphi < \psi &\equiv_{\mathrm{df}} (\varphi \le \psi) \mathbin{\&} (\varphi \ne \psi)\end{aligned}$$

and analogously for $\ge, >$*. The priority of these connectives is the same as that of implication.*

Obviously, the semantics of these connectives corresponds to the crisp ordering and equality of truth degrees as expected.

Convention 2.2 *We shall use the following abbreviations in the formulae of* FCT*:*

$$\begin{aligned}\varphi^n &\equiv_{\mathrm{df}} \varphi \mathbin{\&} \dots \mathbin{\&} \varphi \quad (n \text{ times, with } \varphi^0 \equiv_{\mathrm{df}} 1)\\ \varphi^\triangle &\equiv_{\mathrm{df}} \triangle\varphi\\ x_1 \dots x_n &=_{\mathrm{df}} \langle x_1, \dots, x_n\rangle\end{aligned}$$

Chains of implications $\varphi_1 \to \varphi_2, \varphi_2 \to \varphi_3, \dots, \varphi_{n-1} \to \varphi_n$ *can be written as* $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \dots \longrightarrow \varphi_n$*, and similarly for the equivalence connective.*

Definition 2.3 *In* FCT*, we introduce the following defined notions:*

$$\begin{aligned}A \sqsubseteq B &\equiv_{\mathrm{df}} \triangle(A \subseteq B) && \textit{crisp inclusion}\\ R^{\mathrm{T}} &=_{\mathrm{df}} \{xy \mid Ryx\} && \textit{converse relation}\end{aligned}$$

For better readability of some complex theorems, we shall adopt the following convention:

Convention 2.4 *Formulae of* FCT *of the form* $(\varphi_1 \mathbin{\&} \dots \mathbin{\&} \varphi_n) \to (\psi_1 \mathbin{\&} \dots \mathbin{\&} \psi_k)$ *can also be written as* $\varphi_1, \dots, \varphi_n \;\Rightarrow\; \psi_1, \dots, \psi_k$ *(and similarly for* $\leftrightarrow$ *and* $\Leftrightarrow$ *as the middle symbol). Chains of such implications will use* $\Longrightarrow$ *as the middle sign (standing for* $\longrightarrow$*). If no confusion can arise, we shall also write conjunctions* $\varphi_1 \mathbin{\&} \dots \mathbin{\&} \varphi_n$ *simply as lists* $\varphi_1, \dots, \varphi_n$*.*

Internal truth values An important feature of FCT is the absence of variables for truth degrees: in FCT, truth degrees are the semantic values of formulae rather than object of the theory (see [5] for an explanation of methodological advantages of this approach). However, many theorems of traditional fuzzy mathematics do speak about truth values or quantify over operators on truth values like aggregation operators, copulas, t-norms, etc. In order to be able to speak of truth values within FCT, truth values need be *internalized* in the theory. This is done in [7] by a rather standard technique, by representing truth values by subclasses of a crisp singleton.[3]

The details of the representation are not important in the present paper; we refer the interested readers to [7, Sect. 3]. For our present purposes it is fully sufficient to assume that we do have variables $\alpha, \beta, \dots$ for truth values in FCT, and that the ordering of truth values and the usual propositional connectives and the quantifiers $\forall, \exists$ are definable in FCT. The class of the internal truth values will denoted by L.

Internal connectives Binary operators on truth values (including propositional connectives $\&, \wedge, \vee, \dots$) can be regarded as functions $\mathbf{c}\colon \mathrm{L} \times \mathrm{L} \to \mathrm{L}$ or as fuzzy relations $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$. Consequently, graded class relations can be applied to such operators, e.g., fuzzy inclusion $\mathbf{c} \subseteq \mathbf{d} \equiv (\forall\alpha\beta)(\alpha\, \mathbf{c}\, \beta \to \alpha\, \mathbf{d}\, \beta)$, which means $\bigwedge_{\alpha,\beta}(\alpha\, \mathbf{c}\, \beta \Rightarrow_* \alpha\, \mathbf{d}\, \beta)$ in models.

Similarly, unary operators on truth values can be regarded as functions $\mathbf{u}\colon \mathrm{L} \to \mathrm{L}$, i.e., fuzzy classes $\mathbf{u} \sqsubseteq \mathrm{L}$. Again, graded properties are applicable to unary truth-operators, e.g., graded inclusion $\mathbf{u} \subseteq \mathbf{v} \equiv (\forall\alpha\beta)(\mathbf{u}\alpha \to \mathbf{u}\beta)$. Nullary operators on truth values can be identified with truth values

[3]Cf. [15] for an analogous construction in a set theory over a variant of Gödel logic.

themselves. Instead of "operators on truth values" we shall simply speak of *connectives* (always meaning at most binary). By convention, we shall always use Greek variables for truth values,[4] the letters $\mathbf{u}, \mathbf{v}, \mathbf{w}$ for unary connectives, and the letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ for binary connectives. In formulae, binary inner connectives will by convention have the same priority as &: thus, e.g., $\neg\alpha\,\mathbf{c}\,\beta \to \gamma$ means $((\neg\alpha)\,\mathbf{c}\,\beta) \to \gamma$.

3 Graded properties of connectives

Many crisp classes of truth-value operators (e.g., t-norms, uninorms, copulas, negations, etc.) can be defined by formulae of FCT. The apparatus, however, enables also *partial* satisfaction of such conditions. In the following, we therefore give several *fuzzy* conditions on truth-value operators and use them as graded preconditions of theorems which need not be satisfied to the full degree. This yields a completely new *graded* theory of truth-value operators and allows non-trivial generalizations of well-known theorems on such operators, including their consequences for properties of fuzzy relations.

Definition 3.1 *In* FCT*, we define the following graded properties of a unary connective* $\mathbf{u} \sqsubseteq \mathrm{L}$*:*

$$\begin{aligned}
\mathrm{Mon}(\mathbf{u}) &\equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \le \beta) \to (\mathbf{u}\alpha \to \mathbf{u}\beta)) && \textit{Monotony}\\
\mathrm{Ant}(\mathbf{u}) &\equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \le \beta) \to (\mathbf{u}\beta \to \mathbf{u}\alpha)) && \textit{Antitony}\\
\mathrm{Lip}(\mathbf{u}) &\equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \leftrightarrow \beta) \to (\mathbf{u}\alpha \leftrightarrow \mathbf{u}\beta)) && \textit{Lipschitz}\\
\mathrm{PosLip}(\mathbf{u}) &\equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \to \beta) \to (\mathbf{u}\alpha \to \mathbf{u}\beta)) && \textit{Positive Lipschitz}\\
\mathrm{NegLip}(\mathbf{u}) &\equiv_{\mathrm{df}} (\forall\alpha\beta)((\alpha \to \beta) \to (\mathbf{u}\beta \to \mathbf{u}\alpha)) && \textit{Negative Lipschitz}
\end{aligned}$$

Furthermore, we define the identity connective $\mathbf{id} \sqsubseteq \mathrm{L}$ *by* $\mathbf{id}\,\alpha \equiv_{\mathrm{df}} \alpha$*.*

The name *Lipschitz,* or more accurately *1-Lipschitz property wrt* $\leftrightarrow$*,* comes from the fact that in standard models over Łukasiewicz logic, the full satisfaction of this property expresses the 1-Lipschitz property of $\mathbf{c}$. The stronger property called *positive Lipschitz* expresses the same property wrt the 'distance' given by $\to$ rather than $\leftrightarrow$.

Theorem 3.2 *FCT proves the following graded properties of truth-value operators:*

(C1) $\mathrm{Lip}(\mathbf{u}) \wedge \mathrm{Mon}(\mathbf{u}) \leftrightarrow \mathrm{PosLip}(\mathbf{u})$

(C2) $\mathrm{Lip}(\mathbf{u}) \wedge \mathrm{Ant}(\mathbf{u}) \leftrightarrow \mathrm{NegLip}(\mathbf{u})$

Proof: We prove (C1), the proof of the second claim is analogous. From $\alpha \le \beta$ by $\mathrm{Mon}(\mathbf{u})$ we get $(\mathbf{u}\alpha \to \mathbf{u}\beta)$ and so by weakening we obtain $(\alpha \to \beta) \to (\mathbf{u}\alpha \to \mathbf{u}\beta)$. From $\beta \le \alpha$ we get $(\alpha \to \beta) \to (\alpha \leftrightarrow \beta)$ and so by $\mathrm{Lip}(\mathbf{u})$ we obtain $(\alpha \to \beta) \to (\mathbf{u}\alpha \to \mathbf{u}\beta)$. The SLP completes the proof.

The converse direction, first observe that trivially $\mathrm{PosLip}(\mathbf{u}) \to \mathrm{Mon}(\mathbf{u})$. Second, from $\mathrm{PosLip}(\mathbf{u})$ we get both $(\alpha \leftrightarrow \beta) \longrightarrow (\alpha \to \beta) \longrightarrow (\mathbf{u}\alpha \to \mathbf{u}\beta)$ and $(\alpha \leftrightarrow \beta) \longrightarrow (\beta \to \alpha) \longrightarrow (\mathbf{u}\beta \to \mathbf{u}\alpha)$. The rest is simple.

QED

Analogously we proceed for binary connectives.

[4] By a harmless abuse of language, we shall not distinguish between inner and semantic truth values, as otherwise it would complicate notation too much. In the rigorous notation of [2, Sect. 3], we should, e.g., write $\underline{0} \in (\overline{\forall}\alpha\beta\gamma)(\overline{\underline{0} \in (\alpha\,\mathbf{c}\,\beta)}\,\mathbf{c}\,\gamma) \mathrel{\overline{\leftrightarrow}} (\alpha\,\mathbf{c}\,\overline{\underline{0} \in (\beta\,\mathbf{c}\,\alpha)})$ instead of the simple formula defining associativity in Definition 3.3. Similarly we shall not notationally distinguish between inner and semantical logical connectives.

Definition 3.3 *In* FCT*, we define the following graded properties of a binary connective* $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$*:*

$\mathrm{Com}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall \alpha\beta)(\alpha \,\mathbf{c}\, \beta \to \beta \,\mathbf{c}\, \alpha)$		*Commutativity*
$\mathrm{Ass}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall \alpha\beta\gamma)((\alpha \,\mathbf{c}\, \beta) \,\mathbf{c}\, \gamma) \leftrightarrow (\alpha \,\mathbf{c}\, (\beta \,\mathbf{c}\, \alpha))$		*Associativity*
$\mathrm{Idem}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall \alpha)(\alpha \,\mathbf{c}\, \alpha \leftrightarrow \alpha)$		*Idempotence*
$\mathrm{UnL}(\mathbf{c}, \eta) \equiv_{\mathrm{df}} (\forall \alpha)(\eta \,\mathbf{c}\, \alpha \leftrightarrow \alpha)$		*Left-argument unit element*
$\mathrm{NulL}(\mathbf{c}, \nu) \equiv_{\mathrm{df}} (\forall \alpha)(\nu \,\mathbf{c}\, \alpha \leftrightarrow \nu)$		*Left-argument null element*
$\mathrm{MonL}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall \alpha\beta\gamma)((\alpha \leq \beta) \to (\alpha \,\mathbf{c}\, \gamma \to \beta \,\mathbf{c}\, \gamma))$		*Left-argument monotony*
$\mathrm{AntL}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall \alpha\beta\gamma)((\alpha \leq \beta) \to (\beta \,\mathbf{c}\, \gamma \to \alpha \,\mathbf{c}\, \gamma))$		*Left-argument antitony*
$\mathrm{LipL}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall \alpha\beta\gamma)((\alpha \leftrightarrow \beta) \to (\alpha \,\mathbf{c}\, \gamma \leftrightarrow \beta \,\mathbf{c}\, \gamma))$		*Left-argument Lipschitz*
$\mathrm{PosLipL}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall \alpha\beta\gamma)((\alpha \to \beta) \to (\alpha \,\mathbf{c}\, \gamma \to \beta \,\mathbf{c}\, \gamma))$		*Left-argument positive Lipschitz*
$\mathrm{NegLipL}(\mathbf{c}) \equiv_{\mathrm{df}} (\forall \alpha\beta\gamma)((\alpha \to \beta) \to (\beta \,\mathbf{c}\, \gamma \to \alpha \,\mathbf{c}\, \gamma))$		*Left-argument negative Lipschitz*

The analogous right-argument properties UnR*,* NulR*,* MonR*,* LipR*,* PosLipR*, and* NegLipR *are defined as the corresponding left-argument properties for the converse connective* $\mathbf{c}^{\mathrm{T}}$*, e.g.,* $\mathrm{MonR}(\mathbf{c}) \equiv_{\mathrm{df}} \mathrm{MonL}(\mathbf{c}^{\mathrm{T}})$*, i.e.,* $\mathrm{MonR}(\mathbf{c}) \equiv (\forall \alpha\beta\gamma)((\alpha \leq \beta) \to (\alpha \,\mathbf{c}\, \gamma \to \beta \,\mathbf{c}\, \gamma))$*. For convenience, we also define*

$$\mathrm{Mon}(\mathbf{c}) \equiv_{\mathrm{df}} \mathrm{MonL}(\mathbf{c}) \mathbin{\&} \mathrm{MonR}(\mathbf{c})$$
$$\mathrm{wMon}(\mathbf{c}) \equiv_{\mathrm{df}} \mathrm{MonL}(\mathbf{c}) \wedge \mathrm{MonR}(\mathbf{c})$$

and analogously for Un*,* Nul*,* Lip *and* PosLip*.*

It can be observed that the traditional non-graded classes of truth-value operators can be defined by requiring the full satisfaction of some of the properties defined in Definition 3.3. In particular, a connective $\mathbf{c}$ is a (non-graded)

$$\textit{t-norm} \text{ iff } \triangle\,\mathrm{Com}(\mathbf{c}), \triangle\,\mathrm{Ass}(\mathbf{c}), \triangle\,\mathrm{Mon}(\mathbf{c}), \triangle\,\mathrm{Un}(\mathbf{c}, 1)$$
$$\textit{uninorm} \text{ iff } \triangle\,\mathrm{Com}(\mathbf{c}), \triangle\,\mathrm{Ass}(\mathbf{c}), \triangle\,\mathrm{Mon}(\mathbf{c}), (\exists \eta)\triangle\,\mathrm{Un}(\mathbf{c}, \eta)$$
$$\textit{binary aggregation operator} \text{ iff } \triangle\,\mathrm{Mon}(\mathbf{c}), \triangle(1 \,\mathbf{c}\, 1), \triangle\,\neg(0 \,\mathbf{c}\, 0)$$

Furthermore, in standard Łukasiewicz logic, $\mathbf{c}$ is a (non-graded)

$$\textit{quasicopula} \text{ iff } \triangle\,\mathrm{Un}(\mathbf{c}, 1),\ \triangle\,\mathrm{Nul}(\mathbf{c}, 0), \triangle\,\mathrm{PosLip}(\mathbf{c}).$$

Idempotent binary aggregation operators are those which additionally satisfy $\triangle\,\mathrm{Idem}(\mathbf{c})$, commutative quasicopulas those which also satisfy $\triangle\,\mathrm{Com}(\mathbf{c})$; etc. The conditions $\triangle(1\mathbf{c}1), \triangle\,\neg(0\mathbf{c}0)$ in the definition of aggregation operators are shorter equivalents of the usual conditions $1 \,\mathbf{c}\, 1 = 1$ and $0 \,\mathbf{c}\, 0 = 0$, respectively. Quasicopulas can in our setting not only be generalized in a graded manner, but also to analogous operators that satisfy PosLip or Lip wrt an equivalence $\leftrightarrow$ other than standard Łukasiewicz as a measure of distance.

The following theorem provides us with samples of basic graded results.

Theorem 3.4 *FCT proves the following graded properties of truth-value operators:*

(C3) $\mathrm{wMon}(\mathbf{c}), \mathrm{wUn}(\mathbf{c}, 1) \;\Rightarrow\; \mathbf{c} \subseteq \wedge$

(C4) $\mathrm{wMon}(\mathbf{c}), \mathrm{Idem}(\mathbf{c}) \;\Rightarrow\; \wedge \subseteq \mathbf{c}$

(C5) $\mathrm{wMon}(\mathbf{c}), \mathrm{wUn}(\mathbf{c}, 1) \;\Rightarrow\; (\alpha \,\mathbf{c}\, \alpha \leftrightarrow \alpha) \leftrightarrow (\forall \beta)((\alpha \,\mathbf{c}\, \beta) \leftrightarrow (\alpha \wedge \beta))$

(C6) $\mathrm{MonR}(\mathbf{c}), \mathrm{UnR}(\mathbf{c}, 1) \;\Rightarrow\; \mathrm{NulL}(\mathbf{c}, 0)$

(C7) $\mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; (\alpha \leq \beta) \mathbin{\&} (\gamma \leq \delta) \to (\alpha \,\mathbf{c}\, \gamma \to \beta \,\mathbf{c}\, \delta)$

Proof:

(C3) From $\alpha \leq 1$ by $\mathrm{MonL}(\mathbf{c})$ we get $\alpha \,\mathbf{c}\, \beta \to 1 \,\mathbf{c}\, \beta$. Further by $\mathrm{UnL}(\mathbf{c})$ we get $\alpha \,\mathbf{c}\, \beta \to \beta$. Analogously using $\mathrm{MonR}(\mathbf{c})$ and $\mathrm{UnR}(\mathbf{c})$ we get $\alpha \,\mathbf{c}\, \beta \to \alpha$. The rest is simple.

(C4) From $\alpha \leq \beta$ be MonR($\mathbf{c}$) we get $\alpha \,\mathbf{c}\, \alpha \rightarrow \alpha \,\mathbf{c}\, \beta$. Using Idem($\mathbf{c}$) and $\alpha \leq \beta$ again we get $\alpha \wedge \beta \longrightarrow \alpha \longrightarrow \alpha \,\mathbf{c}\, \beta$. Analogously we get $\alpha \wedge \beta \rightarrow \alpha \,\mathbf{c}\, \beta$ from $\beta \leq \alpha$, MonL($\mathbf{c}$), and Idem($\mathbf{c}$). The rest is simple.

(C5) One direction of the equivalence in the conclusion is trivial (set $\beta = \alpha$). The converse one again consists of proving two implications: using (C3) we get the left-to right direction for free. The right-to-left one: inspect the proof of the previous part and instead of full Idem($\mathbf{c}$) use just $(\alpha \,\mathbf{c}\, \alpha \leftrightarrow \alpha)$.

QED

The three assertions above are generalizations of well-known basic properties of t-norms. Theorem 1.1 corresponds to the fact that the minimum is the greatest (so-called strongest) t-norm. Theorem 1.2 generalizes the basic fact that the minimum is the only idempotent t-norm, while 1.3 is a graded characterization of the idempotents of $\mathbf{c}$ [13].

4 Composition of operations

The operations, being functions $\mathbf{c}\colon \mathrm{L}^2 \rightarrow \mathrm{L}$ (binary), $\mathbf{u}\colon \mathrm{L} \rightarrow \mathrm{L}$ (unary), and $\alpha\colon \mathrm{L}^0 \rightarrow \mathrm{L}$ (nullary, for $\mathrm{L}^0 = \{a\}$ an arbitrary fixed crisp singleton), can be composed whenever their domains and codomains match. In this section we shall investigate the transmission of graded properties under compositions. Recall the standard definitions for crisp functions:

Definition 4.1 *For $f\colon X \rightarrow Y$, $g\colon Y \rightarrow Z$, and $x \in X$, the composition $gf\colon X \rightarrow Z$ is defined as $(gf)(x) =_{\mathrm{df}} g(f(x))$.*

Given the projections $p_1\colon X \times Y \rightarrow X$ and $p_2\colon X \times Y \rightarrow Y$, for $f\colon Z \rightarrow X$ and $g\colon Z \rightarrow Y$, the product function $(f, g)\colon Z \rightarrow X \times Y$ is defined so that $(f, g)(z) = (f(z), g(z))$, i.e., $p_1((f, g)(z)) = f(z)$ and $p_2((f, g)(z)) = g(z)$.

The constant function $X \rightarrow Y$ assigning a fixed $y \in Y$ to all $x \in X$ will be denoted by $\underline{y}_X$. The subscript X may be omitted if known from the context.

The projections $\mathbf{p_1}, \mathbf{p_2} : \mathrm{L}^2 \rightarrow \mathrm{L}$ are defined as $\mathbf{p_1}(\alpha, \beta) =_{\mathrm{df}} \alpha$ and $\mathbf{p_2}(\alpha, \beta) =_{\mathrm{df}} \beta$. The identity on L is denoted by $\mathbf{id}$, i.e., $\mathbf{id}\,\alpha =_{\mathrm{df}} \alpha$ (see already Def. 3.1).

Various constructions by composition are frequently used, e.g.,

$$\begin{array}{rcl}
\mathbf{c}(\underline{\alpha}, \mathbf{v})\colon \mathrm{L} \rightarrow \mathrm{L} & \ldots & \mathbf{c}(\underline{\alpha}, \mathbf{v})\,(\beta) = \alpha \,\mathbf{c}\, (\mathbf{v}\beta) \\
\mathbf{c}(\underline{\alpha}, \mathbf{id})\colon \mathrm{L} \rightarrow \mathrm{L} & \ldots & \mathbf{c}(\underline{\alpha}, \mathbf{id})\,(\beta) = \alpha \,\mathbf{c}\, \beta \\
\mathbf{c}(\mathbf{u}, \underline{\beta})\colon \mathrm{L} \rightarrow \mathrm{L} & \ldots & \mathbf{c}(\mathbf{u}, \underline{\beta})\,(\alpha) = (\mathbf{u}\alpha) \,\mathbf{c}\, \beta \\
\mathbf{c}(\mathbf{u}, \mathbf{id})\colon \mathrm{L}^2 \rightarrow \mathrm{L} & \ldots & \mathbf{c}(\mathbf{u}, \mathbf{id})\,(\alpha, \beta) = (\mathbf{u}\alpha) \,\mathbf{c}\, \beta \\
\mathbf{c}(\mathbf{u}, \mathbf{v})\colon \mathrm{L} \rightarrow \mathrm{L} & \ldots & \mathbf{c}(\mathbf{u}, \mathbf{v})\,(\alpha) = (\mathbf{u}\alpha) \,\mathbf{c}\, (\mathbf{v}\alpha) \\
\mathbf{c}(\mathbf{u}\mathbf{p_1}, \mathbf{v}\mathbf{p_2})\colon \mathrm{L}^2 \rightarrow \mathrm{L} & \ldots & \mathbf{c}(\mathbf{u}\mathbf{p_1}, \mathbf{v}\mathbf{p_2})\,(\alpha, \beta) = (\mathbf{u}\alpha) \,\mathbf{c}\, (\mathbf{v}\beta) \\
\mathbf{c}(\mathbf{id}, \mathbf{id})\colon \mathrm{L} \rightarrow \mathrm{L}^2 & \ldots & \mathbf{c}(\mathbf{id}, \mathbf{id})\,(\alpha) = \alpha \,\mathbf{c}\, \alpha \\
\mathbf{c}(\mathbf{p_1}, \mathbf{p_1})\colon \mathrm{L}^2 \rightarrow \mathrm{L} & \ldots & \mathbf{c}(\mathbf{p_1}, \mathbf{p_1})\,(\alpha, \beta) = \alpha \,\mathbf{c}\, \alpha \\
\mathbf{c}(\mathbf{p_2}, \mathbf{p_1})\colon \mathrm{L}^2 \rightarrow \mathrm{L} & \ldots & \mathbf{c}(\mathbf{p_2}, \mathbf{p_1})\,(\alpha, \beta) = \beta \,\mathbf{c}\, \alpha, \text{ i.e., } \mathbf{c}\,(\mathbf{p_2}, \mathbf{p_1}) = \mathbf{c}^{\mathrm{T}} \\
\mathbf{c}(\mathbf{d}, \mathbf{e})\colon \mathrm{L}^2 \rightarrow \mathrm{L}^2 & \ldots & \mathbf{c}(\mathbf{d}, \mathbf{e})\,(\alpha, \beta) = (\alpha \,\mathbf{d}\, \beta) \,\mathbf{c}\, (\alpha \,\mathbf{e}\, \beta), \text{ etc.}
\end{array}$$

Convention 4.2 *For the sake of more compact formulation of the following theorems, let us write* $\mathrm{PosLip} = \mathrm{Lip}_{+1}$, $\mathrm{NegLip} = \mathrm{Lip}_{-1}$, $\mathrm{Mon} = \mathrm{Mon}_{+1}$, $\mathrm{Ant} = \mathrm{Mon}_{-1}$.

Theorems (C1) and (C2) thus can be jointly formulated as $\mathrm{Lip}\,\mathbf{u}, \mathrm{Mon}_i\,\mathbf{u} \Leftrightarrow \mathrm{Lip}_i\,\mathbf{u}$.

Theorem 4.3 FCT *proves:*

(C8) $\mathrm{Mon}_i(\mathbf{u}), \triangle\,\mathrm{Mon}_j(\mathbf{v}) \;\Rightarrow\; \mathrm{Mon}_{i\cdot j}(\mathbf{uv})$. *The $\triangle$ cannot be omitted (counterexamples are easy to construct).*

(C9) $\mathrm{Lip}(\mathbf{u}), \mathrm{Lip}(\mathbf{v}) \;\Rightarrow\; \mathrm{Lip}(\mathbf{uv})$

(C10) $\mathrm{Lip}_i(\mathbf{u}), \mathrm{Lip}_j(\mathbf{v}) \;\Rightarrow\; \mathrm{Lip}_{i\cdot j}(\mathbf{uv})$

Proof:

(C8) We shall only prove the case $i = j = -1$, the other cases are analogous. We obtain $(\alpha \leq \beta) \to (\mathbf{v}\beta \leq \mathbf{v}\alpha)$ by $\triangle\,\mathrm{Mon}_{-1}(\mathbf{v})$ and $(\mathbf{v}\beta \leq \mathbf{v}\alpha) \to (\mathbf{uv}\alpha \leq \mathbf{uv}\beta)$ by $\mathrm{Mon}_{-1}(\mathbf{u})$. Transitivity of implication and generalization completes the proof.

(C9) Analogous.

(C10) Analogous.

QED

Similar theorems can be proved for compositions of binary connectives.

5 Graded dominance

Applying the definition of dominance between at most binary aggregation operators and making them graded by replacing crisp $\leq$ by $\to$ (i.e., deleting the $\triangle$ hidden by Definition 2.1 in $\leq$ that appears in the non-graded definition), we obtain the following notions of graded dominance. As usually, the traditional notion of dominance is expressible as the graded notion satisfied to degree 1, i.e., prepended by $\triangle$.

Definition 5.1 *The graded relation* $\ll$ *of* dominance *between binary connectives is defined as follows:*

$$\mathbf{c} \ll \mathbf{d} \equiv_{\mathrm{df}} (\forall\alpha\beta\gamma\delta)((\alpha\,\mathbf{d}\,\gamma)\,\mathbf{c}\,(\beta\,\mathbf{d}\,\delta) \to (\alpha\,\mathbf{c}\,\beta)\,\mathbf{d}\,(\gamma\,\mathbf{c}\,\delta))$$

The graded relation of dominance (denoted by the same sign $\ll$*) involving unary connectives is defined as follows:*

$$\mathbf{u} \ll \mathbf{c} \equiv_{\mathrm{df}} (\forall\alpha\beta)(\mathbf{u}(\alpha\,\mathbf{c}\,\beta) \to (\mathbf{u}\alpha\,\mathbf{c}\,\mathbf{u}\beta))$$
$$\mathbf{c} \ll \mathbf{u} \equiv_{\mathrm{df}} (\forall\alpha\beta)((\mathbf{u}\alpha\,\mathbf{c}\,\mathbf{u}\beta) \to \mathbf{u}(\alpha\,\mathbf{c}\,\beta))$$
$$\mathbf{u} \ll \mathbf{v} \equiv_{\mathrm{df}} (\forall\alpha)(\mathbf{uv}\alpha \to \mathbf{vu}\alpha)$$

Observation 5.2 $\mathbf{u} \ll \mathbf{u}$

Remark 5.3 Formally, we can also define dominance for nullary connectives as follows:

$$\alpha \ll \mathbf{c} \equiv_{\mathrm{df}} \alpha \to \alpha\,\mathbf{c}\,\alpha$$
$$\mathbf{c} \ll \alpha \equiv_{\mathrm{df}} \alpha\,\mathbf{c}\,\alpha \to \alpha$$
$$\alpha \ll \mathbf{u} \equiv_{\mathrm{df}} \alpha \to \mathbf{u}\alpha$$
$$\mathbf{u} \ll \alpha \equiv_{\mathrm{df}} \mathbf{u}\alpha \to \alpha$$
$$\alpha \ll \beta \equiv_{\mathrm{df}} \alpha \to \beta$$

These definitions can sometimes be notationally useful. For example the properties $\neg(0\,\mathbf{c}\,0)$ and $1\,\mathbf{c}\,1$ occurring in the definition of aggregation operator can be expressed as $\mathbf{c} \ll 0$ and $1 \ll \mathbf{c}$, respectively (or shortly $0 \ll \mathbf{c} \ll 1$). Various observations can be made on these notions, e.g., $\mathbf{u} \subseteq \mathbf{id} \leftrightarrow (\forall\alpha)(\alpha \ll \mathbf{u})$ and $\mathbf{id} \subseteq \mathbf{u} \leftrightarrow (\forall\alpha)(\mathbf{u} \ll \alpha)$.

Theorem 5.4 FCT *proves the following graded properties of dominance:*

(D1) $\triangle\,\mathrm{Com}(\mathbf{c}), \triangle\,\mathrm{Ass}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \mathbf{c}$

(D2) $\mathrm{Com}(\mathbf{c}), \mathrm{Ass}^4(\mathbf{c}), \mathrm{Lip}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \mathbf{c}$

(D3) $\triangle\,\mathrm{Com}(\mathbf{c}), \triangle\,\mathrm{Ass}(\mathbf{c}), \mathrm{Mon}(\mathbf{c}), \mathbf{d} \sqsubseteq \mathbf{c}, \mathbf{c} \subseteq \mathbf{d} \;\Rightarrow\; \mathbf{c} \ll \mathbf{d}$

(D4) $\mathrm{Com}(\mathbf{c}), \mathrm{Ass}^4(\mathbf{c}), \mathrm{Lip}(\mathbf{c}), \mathrm{Mon}(\mathbf{c}), \mathbf{d} \sqsubseteq \mathbf{c}, \mathbf{c} \subseteq \mathbf{d} \;\Rightarrow\; \mathbf{c} \ll \mathbf{d}$

(D5) $\triangle\,\mathrm{Com}(\mathbf{d}), \triangle\,\mathrm{Ass}(\mathbf{d}), \mathrm{Mon}(\mathbf{d}), \mathbf{d} \sqsubseteq \mathbf{c}, \mathbf{c} \subseteq \mathbf{d} \;\Rightarrow\; \mathbf{c} \ll \mathbf{d}$

(D6) $\mathrm{Com}(\mathbf{d}), \mathrm{Ass}^4(\mathbf{d}), \mathrm{Lip}(\mathbf{d}), \mathrm{Mon}(\mathbf{d}), \mathbf{d} \sqsubseteq \mathbf{c}, \mathbf{c} \subseteq \mathbf{d} \;\Rightarrow\; \mathbf{c} \ll \mathbf{d}$

Proof:

(D1) Notice that the proof is almost classical, but we use to get a fuzzy version:

From $\triangle\,\mathrm{Com}(\mathbf{c})$ we get $\gamma\,\mathbf{c}\,\beta = \beta\,\mathbf{c}\,\gamma$ and so also (by extensionality) $(\gamma\,\mathbf{c}\,\beta)\,\mathbf{c}\,\delta = (\beta\,\mathbf{c}\,\gamma)\,\mathbf{c}\,\delta$. By $\triangle\,\mathrm{Ass}(\mathbf{c})$ (twice) we get $\gamma\,\mathbf{c}\,(\beta\,\mathbf{c}\,\delta) = \beta\,\mathbf{c}\,(\gamma\,\mathbf{c}\,\delta)$ and so also (by extensionality) $\alpha\,\mathbf{c}\,(\gamma\,\mathbf{c}\,(\beta\,\mathbf{c}\,\delta)) = \alpha\,\mathbf{c}\,(\beta\,\mathbf{c}\,(\gamma\,\mathbf{c}\,\delta))$. $\triangle\,\mathrm{Ass}(\mathbf{c})$ twice completes the proof.

(D2) The same proof as above just use graded version of assumptions and $\mathrm{Lip}(\mathbf{c})$ instead of extensionality.

(D3) From $\mathbf{d} \sqsubseteq \mathbf{c}$ we get $(\alpha\,\mathbf{d}\,\gamma) \leq (\alpha\,\mathbf{c}\,\gamma)$ and $(\beta\,\mathbf{d}\,\delta) \leq (\beta\,\mathbf{c}\,\delta)$. Thus we get $(\alpha\,\mathbf{d}\,\gamma)\,\mathbf{c}\,(\beta\,\mathbf{d}\,\delta) \longrightarrow (\alpha\,\mathbf{c}\,\gamma)\,\mathbf{c}\,(\beta\,\mathbf{c}\,\delta) \longrightarrow (\alpha\,\mathbf{c}\,\beta)\,\mathbf{c}\,(\gamma\,\mathbf{c}\,\delta) \longrightarrow (\alpha\,\mathbf{c}\,\beta)\,\mathbf{d}\,(\gamma\,\mathbf{c}\,\delta)$, where the first implication follows from $\mathrm{Mon}(\mathbf{c})$ by (C7), the second implication is obtained from $\triangle\,\mathrm{Com}(\mathbf{c})$ and $\triangle\,\mathrm{Ass}(\mathbf{c})$ by (D1) and the last one follows from $\mathbf{c} \subseteq \mathbf{d}$.

(D4) Analogous, just use (D2) instead of (D1).

(D5) Analogous.

(D6) Analogous.

QED

Theorem 5.5 FCT *proves the following graded properties of dominance:*

(D7) $(\exists\eta)\,\triangle(\mathrm{Un}(\mathbf{c},\eta)\;\&\;\mathrm{Un}(\mathbf{d},\eta)), \mathbf{c} \ll \mathbf{d} \;\Rightarrow\; \mathbf{c} \subseteq \mathbf{d}$

(D8) $(\exists\eta)(\mathrm{Un}(\mathbf{c},\eta)\;\&\;\mathrm{Un}(\mathbf{d},\eta)), \mathrm{Lip}(\mathbf{c}), \mathrm{Lip}(\mathbf{d}), \mathbf{c} \ll \mathbf{d} \;\Rightarrow\; \mathbf{c} \subseteq \mathbf{d}$

Proof:

(D7) We assume $\mathbf{c} \ll \mathbf{d}$ i.e., in particular: $(\alpha\,\mathbf{d}\,\eta)\,\mathbf{c}\,(\eta\,\mathbf{d}\,\delta) \to (\alpha\,\mathbf{c}\,\eta)\,\mathbf{d}\,(\eta\,\mathbf{c}\,\delta)$. From the assumptions $\alpha\,\mathbf{d}\,\eta = \alpha\,\mathbf{c}\,\eta = \alpha$, $\eta\,\mathbf{d}\,\delta = \eta\,\mathbf{c}\,\delta = \delta$, we get $(\alpha\,\mathbf{c}\,\delta) \to (\alpha\,\mathbf{d}\,\delta)$. Quantifier shift completes the proof.

(D8) Again the proof is analogous.

QED

Theorem 5.6 FCT *proves the following graded properties of dominance:*

(D9) $\mathrm{Mon}(\mathbf{c}), \& \ll \mathbf{c} \;\Rightarrow\; (\alpha \to \beta)\,\mathbf{c}\,(\gamma \to \delta) \to (\alpha\,\mathbf{c}\,\gamma \to \beta\,\mathbf{c}\,\delta)$

(D10) $\mathrm{Mon}(\mathbf{c}), \& \ll \mathbf{c} \;\Rightarrow\; (\alpha \leftrightarrow \beta)\,\mathbf{c}\,(\gamma \leftrightarrow \delta) \to (\alpha\,\mathbf{c}\,\gamma \leftrightarrow \beta\,\mathbf{c}\,\delta)$

(D11) $\mathrm{Mon}(\mathbf{c}), \mathrm{UnR}(\mathbf{c},1), \& \ll \mathbf{c} \;\Rightarrow\; \mathrm{PosLipL}(\mathbf{c})$

(D12) $\mathrm{Mon}^2(\mathbf{c}), \mathrm{Un}(\mathbf{c},1), \& \ll^2 \mathbf{c} \;\Rightarrow\; \mathrm{PosLip}(\mathbf{c})$

(D13) $\mathrm{Mon}(\mathbf{c}), \mathrm{wUn}(\mathbf{c},1), \& \ll \mathbf{c} \;\Rightarrow\; \mathrm{wPosLip}(\mathbf{c})$

Proof:

(D9) $\mathrm{MTL}_\triangle$ proves $((\alpha \to \beta)\,\&\,\alpha) \leq \beta$ and $((\gamma \to \delta)\,\&\,\gamma) \leq \delta$. Thus by $\mathrm{Mon}(\mathbf{c})$ and (C7) we obtain $((\alpha \to \beta)\&\alpha)\mathbf{c}((\gamma \to \delta)\&\gamma) \to \beta\mathbf{c}\delta$. As $\& \ll \mathbf{c}$ we obtain $((\alpha \to \beta)\mathbf{c}(\gamma \to \delta))\&(\alpha\mathbf{c}\gamma) \to \beta\mathbf{c}\delta$, which completes the proof.

(D10) Analogous.

(D11) Follows from (D9) for $\gamma = \delta$, using $\mathrm{UnR}(\mathbf{c})$ to obtain $(\alpha \to \beta)\,\mathbf{c}\,(\gamma \to \gamma) \longleftrightarrow (\alpha \to \beta)\,\mathbf{c}\,1 \longleftrightarrow (\alpha \to \beta)$.

(D12) Analogous.

(D13) Analogous.

QED

Theorems (D1) and (D7) are generalizations of two basic facts, namely that every t-norm dominates itself and that dominance implies inclusion / pointwise order. Theorem (D1) can be informally explained as saying that self-domination (or Aczél's property of bisymmetry), holds not only for t-norms, but to a fair degree also for all fully commutative, very associative and fairly monotone connectives.

Theorems (D3)–(D6) have no correspondences among known results; they provide us with bounds for the degree to which $(\mathbf{c} \ll \mathbf{d})$ holds, where the assumption $(\mathbf{d} \sqsubseteq \mathbf{c})$ & $(\mathbf{c} \subseteq \mathbf{d})$ would be obviously useless in the crisp non-graded framework (as it necessitates that $\mathbf{c}$ and $\mathbf{d}$ coincide anyway). Theorem (D9) provides us with strengthened monotonicity of an aggregation operator $\mathbf{c}$ provided that $\mathbf{c}$ fulfills $\mathrm{Mon}(\mathbf{c})$ and dominates the conjunction of the underlying logic. Theorem (D10) is then a kind of "Lipschitz property" of $\mathbf{c}$ (if we view $\leftrightarrow$ as a kind of generalized closeness measure).

Theorem 5.7 *FCT proves the following graded properties of dominance w.r.t.* $\wedge$*:*

(D14) $\mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; \mathbf{c} \ll \wedge$

(D15) $\triangle\,\mathrm{Un}(\mathbf{c}, 1) \;\Rightarrow\; (\wedge \ll \mathbf{c}) \leq (\wedge \subseteq \mathbf{c})$

(D16) $\mathrm{Un}(\mathbf{c}, 1), \mathrm{Lip}(\mathbf{c}) \;\Rightarrow\; (\wedge \ll \mathbf{c}) \rightarrow (\wedge \subseteq \mathbf{c})$

(D17) $\triangle\,\mathrm{Mon}(\mathbf{c}), \triangle\,\mathrm{Un}(\mathbf{c}, 1) \;\Rightarrow\; (\wedge \subseteq \mathbf{c}) \leq (\wedge \ll \mathbf{c})$

(D18) $\triangle\,\mathrm{Mon}(\mathbf{c}), \triangle\,\mathrm{Un}(\mathbf{c}, 1) \;\Rightarrow\; (\wedge \subseteq \mathbf{c}) = (\wedge \ll \mathbf{c})$

(D19) $\mathrm{Mon}(\mathbf{c}) \;\Rightarrow\; (\wedge \ll \mathbf{c}) \leftrightarrow (\forall \alpha, \beta)((\alpha\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,\beta) \leftrightarrow (\alpha\,\mathbf{c}\,\beta))$

Proof:

(D14) From $\triangle(\alpha \wedge \gamma \rightarrow \alpha)$ and $\triangle(\beta \wedge \delta \rightarrow \beta)$ we obtain $(\alpha \wedge \gamma)\,\mathbf{c}\,(\beta \wedge \delta) \rightarrow (\alpha\,\mathbf{c}\,\beta)$ using $\mathrm{Mon}(\mathbf{c})$ and (C7). Analogously we obtain $(\alpha \wedge \gamma)\,\mathbf{c}\,(\beta \wedge \delta) \rightarrow (\gamma\,\mathbf{c}\,\delta)$ and the proof is done.

(D15) As clearly $\triangle\,\mathrm{Un}(\wedge, 1)$ we can use (D7).

(D16) As clearly $\mathrm{Un}(\wedge, 1)$ and also $\mathrm{Lip}(\wedge)$ we can use (D8).

(D17) From $\triangle\,\mathrm{Mon}(\mathbf{c})$ and $\triangle\,\mathrm{Un}(\mathbf{c}, 1)$ we obtain $\mathbf{c} \sqsubseteq \wedge$ using (C3). Further we observe that $\triangle\,\mathrm{Com}(\wedge), \triangle\,\mathrm{Ass}(\wedge), \mathrm{Mon}(\wedge)$ and thus we can use (D3).

(D18) Trivial.

(D19) From $\wedge \ll \mathbf{c}$ we obtain $(\alpha\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,\beta) \longrightarrow (\alpha \wedge 1)\,\mathbf{c}\,(1 \wedge \beta) \longrightarrow \alpha\,\mathbf{c}\,\beta$. From $\mathrm{MonR}(\mathbf{c})$ we get $\alpha\,\mathbf{c}\,\beta \rightarrow \alpha\,\mathbf{c}\,1$ and from $\mathrm{MonL}(\mathbf{c})$ we get $\alpha\,\mathbf{c}\,\beta \rightarrow 1\,\mathbf{c}\,\beta$, i.e., from $\mathrm{wMon}(\mathbf{c})$ we get $\alpha\,\mathbf{c}\,\beta \rightarrow (\alpha\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,\beta)$.

The second direction: assume that $\alpha \leq \beta$ and so $\alpha \leq \alpha \wedge \beta$. As $\gamma \leq 1$ we can use $\mathrm{Mon}(\mathbf{c})$ and (C7) to obtain the second implication in: $(\alpha\,\mathbf{c}\,\gamma) \wedge (\beta\,\mathbf{c}\,\delta) \longrightarrow \alpha\,\mathbf{c}\,\gamma \longrightarrow (\alpha \wedge \beta)\,\mathbf{c}\,1$. Observe that we can prove the same from $\beta \leq \alpha$. Thus we have shown $(\alpha\,\mathbf{c}\,\gamma) \wedge (\beta\,\mathbf{c}\,\delta) \rightarrow (\alpha \wedge \beta)\,\mathbf{c}\,1$.

Analogously we show $(\alpha\,\mathbf{c}\,\gamma) \wedge (\beta\,\mathbf{c}\,\delta) \rightarrow 1\,\mathbf{c}\,(\gamma \wedge \delta)$ Thus together we have $(\alpha\,\mathbf{c}\,\gamma) \wedge (\beta\,\mathbf{c}\,\delta) \rightarrow ((\alpha \wedge \beta)\,\mathbf{c}\,1) \wedge (1\,\mathbf{c}\,(\gamma \wedge \delta))$. Our assumption gives us $(\alpha\,\mathbf{c}\,\gamma) \wedge (\beta\,\mathbf{c}\,\delta) \rightarrow (\alpha \wedge \beta)\,\mathbf{c}\,(\gamma \wedge \delta)$

QED

Theorem 5.7(1) is a graded generalization of the well-known fact that the minimum dominates any aggregation operator [14]. Theorem 5.7(2) demonstrates a rather surprising fact: that the degree to which a monotonic binary operation with neutral element 1 dominates the minimum is nothing else but the degree to which it is larger. Theorem 5.7(3) is an alternative characterization of operators dominating the minimum; for its non-graded version see [14, Prop. 5.1].

Example 5.8 Assertion 2. of Theorem 5.7 can easily be utilized to compute degrees to which standard t-norms on the unit interval dominate the minimum. It can be shown easily that

$$(\wedge \subseteq \mathbf{c}) = \inf_{x\in[0,1]} (x \Rightarrow \mathbf{c}(x,x))$$

holds, i.e. the largest "difference" of a t-norm $\mathbf{c}$ from the minimum can always be found on the diagonal. In standard Łukasiewicz logic, this is, for instance, 0.75 for the product t-norm and 0.5 for the Łukasiewicz t-norm itself. So we can infer that the product t-norm dominates the minimum with a degree of 0.75 (assuming that the underlying logic is standard Łukasiewicz!); with the same assumption, the Łukasiewicz t-norm dominates the minimum to a degree of 0.5.

6 Graded dominance and graded properties of fuzzy relations

Definition 6.1 In FCT, we define basic properties of fuzzy relations as follows:

$$\begin{array}{rcll}
\mathrm{Refl}(R) & \equiv_{\mathrm{df}} & (\forall x)Rxx & \text{reflexivity} \\
\mathrm{Irrefl} & \equiv_{\mathrm{df}} & (\forall x)\neg Rxx & \text{irreflexivity} \\
\mathrm{Sym}(R) & \equiv_{\mathrm{df}} & (\forall x,y)(Rxy \to Ryx) & \text{symmetry} \\
\mathrm{Trans}_{\mathbf{c}}(R) & \equiv_{\mathrm{df}} & (\forall x,y,z)(Rxy\,\mathbf{c}\,Ryz \to Rxz) & \text{transitivity} \\
\mathrm{AntiSym}_{(E),\mathbf{c}}(R) & \equiv_{\mathrm{df}} & (\forall x,y)(Rxy\,\mathbf{c}\,Ryx \to Exy) & (E)\text{-antisymmetry} \\
\mathrm{ASym}_{\mathbf{c}}(R) & \equiv_{\mathrm{df}} & (\forall x,y)\neg(Rxy\,\mathbf{c}\,Ryx) & \text{asymmetry}
\end{array}$$

The following theorems show the importance of graded dominance for graded properties of fuzzy relations. Theorem 4 is a graded generalization of the well-known theorem by De Baets and Mesiar that uses dominance to characterize preservation of transitivity by aggregation [9, Th. 2].

Theorem 6.2 *FCT proves:*

$$\mathrm{Mon}(\mathbf{d}), \mathbf{c} \ll \mathbf{d} \;\Rightarrow\; (\forall E,F)(\triangle\,\mathrm{Trans}_{\mathbf{c}}(E) \;\&\; \triangle\,\mathrm{Trans}_{\mathbf{c}}(F) \to \mathrm{Trans}_{\mathbf{c}}(E\,\underline{\mathbf{d}}\,F))$$

and

$$\triangle\,\mathrm{Nul}(\mathbf{c},0), (\forall E,F)(\triangle\,\mathrm{Trans}_{\mathbf{c}}(E) \;\&\; \triangle\,\mathrm{Trans}_{\mathbf{c}}(F) \to \mathrm{Trans}_{\mathbf{c}}(E\,\underline{\mathbf{d}}\,F)) \;\Rightarrow\; \mathbf{c} \ll \mathbf{d}$$

where $\underline{\mathbf{c}}$ *is the class operation given by* $\mathbf{c}$*, i.e.,* $\langle x,y\rangle \in \underline{\mathbf{c}}(E,F) \equiv Exy \;\; \mathbf{c} \;\; Fxy$*. We use infix notation.*

Proof: Right-to-left direction: from $(\mathbf{c} \ll \mathbf{d})$ we obtain $(Exy\,\mathbf{d}\,Fxy)\,\mathbf{c}\,(Eyz\,\mathbf{d}\,Fyz) \to (Exy\,\mathbf{c}\,Eyz)\,\mathbf{d}\,(Fxy\,\mathbf{d}\,Fyz)$. From $\triangle\,\mathrm{Trans}_{\mathbf{c}}(E)$ and $\triangle\,\mathrm{Trans}_{\mathbf{c}}(F)$ we know that $(Exy\,\mathbf{c}\,Eyz) \leq Ezx$ and $(Fxy\,\mathbf{c}\,Fyz) \leq Fzx$. $\mathrm{Mon}(\mathbf{d})$ completes the proof.

The second direction: let us fix three elements $a \neq b \neq c \neq a$ and define two relations:

$$Exy = (x = a \wedge y = b \wedge \alpha) \vee (x = b \wedge y = c \wedge \beta) \vee (x = a \wedge y = b \wedge \alpha\,\mathbf{c}\,\beta)$$

and

$$Exy = (x = a \wedge y = b \wedge \gamma) \vee (x = b \wedge y = c \wedge \delta) \vee (x = a \wedge y = b \wedge \gamma\,\mathbf{c}\,\delta)$$

Observe that from $\triangle\,\mathrm{Nul}(\mathbf{c},0)$ we easily get $\triangle\,\mathrm{Trans}_{\mathbf{c}}(E)$ and $\triangle\,\mathrm{Trans}_{\mathbf{c}}(F)$. Thus we infer $\mathrm{Trans}_{\mathbf{c}}(E\,\underline{\mathbf{d}}\,F)$, which for $x = a$, $y = b$, and $z = c$ yields $(Eab\,\mathbf{d}\,Fab)\,\mathbf{c}\,(Ebc\,\mathbf{d}\,Fbc) \to (Eac\,\mathbf{d}\,Fac)$, to complete the proof use the definitions of E and F.

QED

The following theorem provides us with results on the preservation of various properties by symmetrizations of fuzzy relations.

Theorem 6.3 *FCT proves the following properties of the symmetrization of relations:*

1. $\mathrm{Com}(\mathbf{c}) \;\Rightarrow\; \mathrm{Sym}(R\,\underline{\mathbf{c}}\,R^{-1})$
2. $\& \subseteq \mathbf{c}, \mathrm{Refl}^2 R \;\Rightarrow\; \mathrm{Refl}(R\,\underline{\mathbf{c}}\,R^{-1})$

3. $\& \subseteq \mathbf{c} \;\Rightarrow\; \mathrm{AntiSym}_{R\,\underline{\mathbf{c}}\,R^{-1}}\, R$

4. $\mathrm{Mon}(\mathbf{c}), \& \ll \mathbf{c}, \triangle\, \mathrm{Trans}\, R \;\Rightarrow\; \mathrm{Trans}(R\,\underline{\mathbf{c}}\,R^{-1})$

In the crisp case, the commutativity of an operator trivially implies the symmetry of symmetrizations by this operator. In the graded case, Theorem 5.1 above states that the degree to which a symmetrization is actually symmetric is bounded below by the degree to which the aggregation operator **c** is commutative. Theorems 5.2–4 are also well-known in the non-graded case [8, 9, 16]. Obviously, 5.4 is a simple corollary of Theorem 6.2.

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