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## Absolute Value Mapping

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## Institute of Computer Science

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Technical report No. V-1266
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## Absolute Value Mapping

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## Abstract:

We prove a necessary and sufficient condition for an absolute value mapping to be bijective. This result simultaneously gives a characterization of unique solvability of an absolute value equation for each right-hand side. ${ }^{2}$


Keywords:
Absolute value mapping, bijectivity, interval matrix, regularity, absolute value equation, unique solvability.

[^0]
## 1 Introduction

The mapping

$$
\begin{equation*}
f_{A B}(x)=A x+B|x|, \tag{1.1}
\end{equation*}
$$

where $A, B \in \mathbb{R}^{n \times n}$, is called an absolute value mapping (the absolute value of a vector is understood entrywise). In this report we are solely interested in condition under which $f_{A B}$ is bijective, i.e., is a one-to-one mapping of $\mathbb{R}^{n}$ onto itself. We show below that the problem is closely connected with regularity of interval matrices.

Our result given in Theorem 3 can be also seen as a necessary and sufficient condition for unique solvability of an absolute value equation

$$
A x+B|x|=b
$$

for each right-hand side $b \in \mathbb{R}^{n}$, a property for which only sufficient conditions have been known so far.

## 2 Auxiliary results

For the proof of the main theorem we shall need two auxiliary results that are of independent interest. Let us recall that a square matrix is called a $P$-matrix if all its principal minors are positive. The first result is due to Murty [2, Thm. 4.2]; $x^{+}$and $x^{-}$are defined by $x^{+}=\max (x, 0), x^{-}=\max (-x, 0)$ (entrywise).

Theorem 1. Let $C \in \mathbb{R}^{n \times n}$. Then the mapping

$$
g_{C}(x)=x^{+}-C x^{-}
$$

is a bijection of $\mathbb{R}^{n}$ onto itself if and only if $C$ is a $P$-matrix.
The second result is due to Rump [4, Thm. 4.1]. The set of the form

$$
[F-G, F+G]=\{H \mid F-G \leq H \leq F+G\}
$$

where $F, G \in \mathbb{R}^{n \times n}, G \geq 0$, is called an interval matrix and it is said to be regular if each matrix $H$ contained therein is nonsigular.

Theorem 2. Let $C-I$ be nonsingular. Then $C$ is a $P$-matrix if and only if the interval matrix

$$
\left[(C-I)^{-1}(C+I)-I,(C-I)^{-1}(C+I)+I\right]
$$

is regular.
In the original Rump's formulation nonsingularity of both $C-I$ and $C+I$ was assumed; it was shown later in [3, Thm 2] that the second assumption is superfluous.

## 3 Characterization

Assume that $A+B$ is nonsingular; then we can define the matrix

$$
C=(A+B)^{-1}(A-B)
$$

which satisfies

$$
\begin{aligned}
& C-I=(A+B)^{-1}(A-B)-(A+B)^{-1}(A+B)=-2(A+B)^{-1} B, \\
& C+I=(A+B)^{-1}(A-B)+(A+B)^{-1}(A+B)=2(A+B)^{-1} A,
\end{aligned}
$$

and $C-I$ becomes nonsingular under an additional assumption of nonsingularity of $B$. Then we can introduce a matrix $D$ by

$$
D=(C-I)^{-1}(C+I)=-B^{-1}(A+B)(A+B)^{-1} A=-B^{-1} A
$$

Theorem 3. Let both $B$ and $A+B$ be nonsingular. Then the mapping (1.1) is a bijection of $\mathbb{R}^{n}$ onto itself if and only if the interval matrix

$$
[D-I, D+I]
$$

is regular.
Proof. Because $x$ and $|x|$ can be decomposed as $x=x^{+}-x^{-}$and $|x|=x^{+}+x^{-}$, we have

$$
\begin{aligned}
f_{A B}(x) & =A\left(x^{+}-x^{-}\right)+B\left(x^{+}+x^{-}\right)=(A+B) x^{+}-(A-B) x^{-} \\
& =(A+B)\left(x^{+}-C x^{-}\right)=(A+B) g_{C}(x)
\end{aligned}
$$

and since $A+B$ is nonsingular, $f_{A B}$ is a bijection of $\mathbb{R}^{n}$ onto itself if and only if $g_{C}$ possesses the same property which by Theorem 1 is the case if and only if $C$ is a $P$-matrix. Now, by Theorem 2, $C$ is a $P$-matrix if and only if the interval matrix

$$
[D-I, D+I]
$$

is regular which concludes the proof.

## 4 Checking

Thus checking bijectivity of $f_{A B}$ may be performed by the following MATLAB file whose subroutine can be downloaded from http://uivtx.cs.cas.cz/~rohn/other/regising.m.

```
function b=bijectivity(A,B)
%
% b== 1: the mapping x --> A*x + B*abs(x) is bijective,
% b==-1: the mapping is not bijective.
%
n=size(A,1); I=eye(n,n);
if rank(B)<n || rank(A+B)<n
    error('Condition not satisfied.')
end
D=-inv(B)*A;
S=regising(D,I);
if isempty(S), b=1; else b=-1; end
```


## Bibliography

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[2] K. G. Murty, On the number of solutions to the complementarity problem and spanning properties of complementary cones, Linear Algebra and Its Applications, 5 (1972), pp. 65108.
[3] J. Rohn, On Rump's characterization of P-matrices, Optimization Letters, 6 (2012), pp. 1017-1020.
[4] S. M. Rump, On P-matrices, Linear Algebra and Its Applications, 363 (2003), pp. 237250.


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    ${ }^{2}$ Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4] x_{1}+[-2,1] x_{2}=[-2,2],[-1,2] x_{1}+[2,4] x_{2}=[-2,2]$ (Barth and Nuding [1])).

