The Dynamics of Complex Logistic

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Abstract:

The new approach to study the dynamics of the complex valued deterministic logistic equation in the form of equivalent 2D mapping is presented. Numerical simulations then indicate surprisingly, that there is not the analogy of canonical Hnon map type deterministic chaos in this case. The important role of property "to be complex valued" in nonlinear dynamics in general with possible consequences, e. g., to quantum chaos, is considered.

Keywords:
Complex numbers in quantum mechanics; complex logistic equation; non-chaotic behavior; quantum chaos; quantum entanglement; quantum communication

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The dynamics of complex logistic equation and consequences

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Abstract

The new approach to study the dynamics of the complex valued deterministic logistic equation in the form of equivalent 2D mapping is presented. Numerical simulations then indicate surprisingly, that there is not the analogy of canonical Hénon map type deterministic chaos in this case. The important role of property "to be complex valued" in nonlinear dynamics in general with possible consequences, e. g., to quantum chaos, is considered.

1. Introduction

The complex numbers play a very important role in science in general, and especially in mathematics and physics. So one can say that they have a privileged status in mathematics. Let us note the fundamental Theorem of Algebra: Every polynomial of degree *n* with complex coefficients has *n* roots in the complex numbers domain. Similarly, complex numbers create profound ingredients in both most important parts of modern physics, namely in quantum theory and relativity. It is well known that quantum mechanics is invariably done in the complex domain, namely the complex Hilbert space [1]. Besides, the complex unit $i=\sqrt{-1}$ figures explicitly in the basic equation of the theory, namely the Schrödinger equation. Surprisingly enough, this complex unit i was mathematically built in the formalism of special theory of relativity by Einstein´s teacher Minkowski. Einstein accepted it saying: "we must replace the usual time co-ordinate t by an imaginary magnitude ict proportional to it. (The c is the light velocity in vacuum here.) Under these conditions, the natural laws satisfying the demands of the theory of relativity assume mathematical forms, in which the time co-ordinate plays exactly the same role as the three space co-ordinates" [2]. Einstein goes even further, saying almost mysteriously, that "the introduction of the complex unit played a very important role in discovery of general theory of relativity" [2]. Let us mention here, that to more deeply understand a role of complex numbers playing in above mentioned fields of science would require much more space, we have here. Alas, in fact it remains very open so far.

To understand such complicated matters the study of dynamics in a complex domain could be of help. In fact, there is a long tradition in studying iterated holomorphic maps in one complex variable, pioneered by French mathematicians Fatou and Julia at the beginning of 20th century [3]. Then followed a prolonged period till the 1980th when almost nothing was done in the field. The new era of progress in dynamics in a complex domain started mainly due to the alluring computer graphics of Mandelbrot

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[4], connected to the beautiful pictures of Julia and Mandelbrot sets, and also to fractals. But equally enticing mathematical work of Douady, Hubbard, and Sulivan [3] drawn again attention to the rich dynamical behaviour of elementary maps of the complex plane.

In this letter, we start like Mandelbrot [4] with the fully complex logistic equation (CLE). But our approach is totally different of that of Mandelbrot. From the very definition we get the 2D classical equivalent map of our treated CLE. Surprisingly, we show that such 2D map has qualitatively different dynamics of that of the canonical Hénon map, namely not possessing a chaos.

The organization of this Letter is as follows Section 2 gives the derivation of 2D classical equivalent map of standard CLE. Section 3 is devoted to detailed numerical analysis of such 2D map. In Section 4 a Conjecture is formulated according to which there is no (linear) affine transformation between the CLE and the canonical Hénon (2D) map. In Section 5, we present a summary of our main results with possible consequences, which the property "to be complex valued" can have on nonlinear dynamics in general, and on so called quantum chaos, in particular.

2. The derivation of 2D classical equivalent map of 1D complex logistic equation

We start, like Mandelbrot [4], with the fully complex valued logistic equation in the form

$$Z_{n+1} = AZ_n(1 - Z_n)\,, \tag{2.1}$$

where Z_n and A are complex numbers. Let us note that (2.1) has the unique normal form in the family of all quadratic polynomial maps, as anytime there exists an appropriate affine change of coordinates. A closely related normal form to (2.1) is the following famous conjugate form

$$W_{n+1} = W_n^2 + c\,, \tag{2.2}$$

where again W_n and c are complex numbers. Using such a normal form, we can make a computer picture in the parameter space consisting of all complex constants A or c. But we will not follow a big business started by Mandelbrot and followed by many others in producing beautiful pictures of Julia or Mandelbrot sets. Remember our main goal here is to study a possible role the property to be complex valued could play in the dynamics in general. To do this we prescribe (2.1) according to the definition of complex numbers on the plane. Namely, for the complex variable Z_n we put

$$Z_n = x_n \exp(i\, y_n)\,, \tag{2.3}$$

where x_n, y_n are real numbers, now. Similarly for the complex parameter A we use

$$A = a \exp(i\,\alpha)\,, \tag{2.4}$$

where again a and α are real. Now, substituting from (2.3) and (2.4) into (2.1) we derive after some arithmetic and comparing real and imaginary parts of both sides of (2.1), the following equations:

$$
\begin{aligned}
x_{n+1}\cos y_{n+1} &= a\,x_n\cos\left(\alpha+y_n\right)-a\,x_n^2\cos\left(\alpha+2y_n\right),\\
x_{n+1}\sin y_{n+1} &= a\,x_n\sin\left(\alpha+y_n\right)-a\,x_n^2\sin\left(\alpha+2y_n\right)\ .
\end{aligned}
\tag{2.5}
$$

And, by elementary reschuffling one gets finally

$$
\begin{aligned}
x_{n+1} &= a\,x_n\sqrt{1+x_n^2-2x_n\cos y_n}\ ,\\
y_{n+1} &= \tan^{-1}\frac{\sin\left(\alpha+y_n\right)-x_n\sin\left(\alpha+2y_n\right)}{\cos\left(\alpha+y_n\right)-x_n\cos\left(\alpha+2y_n\right)}\ .
\end{aligned}
\tag{2.6}
$$

We consider the dynamics of 2D mapping of (2.6) for standard values of parameters, i. e., $a\in\left[0,4\right],\alpha\in\left[0,\frac{\pi}{2}\right],x_n\in\left[0,1\right]$ and $y_n\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, where clearly $y_n:=\frac{\pi}{180}\cdot y_n$.

Let us note that in the real case where both Z and A are real, i.e., $y_n=0,\alpha=0$ one gets of (2.6) immediately as expected, the normal logistic equation

$$
\begin{aligned}
x_{n+1} &= a\,x_n\left(1-x_n\right),\\
y_{n+1} &= 0.
\end{aligned}
\tag{2.7}
$$

It means one has the standard 1D logistic equation possessing deterministic chaos for some values of parameter a, as is well known. Alas, to analyze the behaviour of 2D mapping (2.6) is a very complicated task. We do this in the following section by means of detailed numerical simulations.

3. Numerical results

To study the 2D dynamical system (2.6) analytically is a very difficult task. But following the main goal of this Letter, namely looking for a possible role which could be played by a property to be complex valued, we proceed by detailed numerical simulations of (2.6), as it is common in nonlinear dynamics.

To show that our 2D map of (2.6) has qualitatively different dynamics of that of the canonical Hénon map considered here in its standard form

$$
\begin{aligned}
x_{n+1} &= a-x_n^2-y_n\ ,\\
y_{n+1} &= b\,x_n\ ,
\end{aligned}
\tag{3.1}
$$

we proceed by numerical simulations of solutions of (2.6). Namely, in this section the bifurcation diagrams, typical attractors with their basins of attraction, as well as, the

Lyapunov exponents for appropriate values of parameters for (2.6) are numerically calculated.

Fig. 1. (a) shows the typical bifurcation diagram for the radial variable part x_n of the complex variable Z_n depending on the parameter values, for the fixed value of parameter *a,* in (2.6). The diagram looks rather complicated with a form of some cyclicity (periodicity) in. In Fig. 1. (b) the detailed view around the transient point is zoomed.

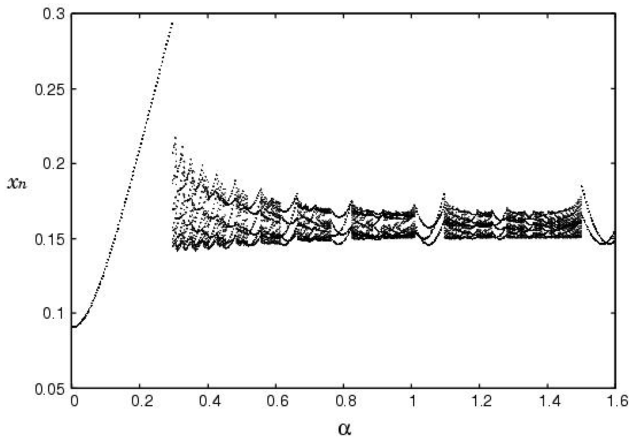


Fig. 1. (a)

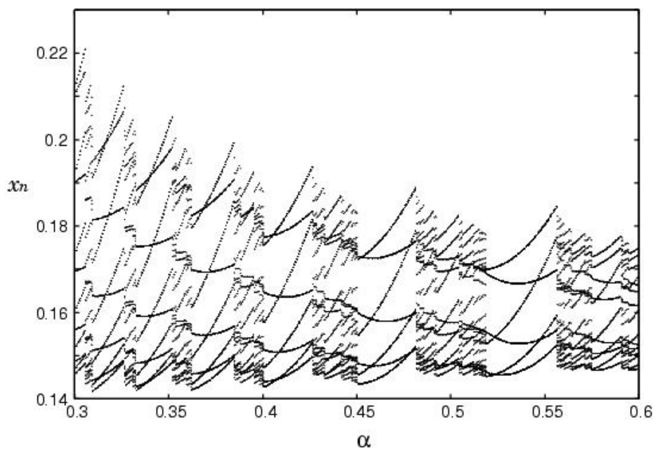


Fig. 1. (b)

Fig. 1. (a). The bifurcation diagram of radial variable x_n according to (2.6) for the fixed value of parameter *a= 1.1* and $\alpha \in [0, 1.6]$. (b) The detailed view of Fig. 1. (a) for the values of parameter α scaled around $\alpha \in [.0.3, 0.6]$. It figures for a possibility of the existence of multi-attractors as it will be seen also later (see Fig. 4), where the basin of attraction is presented.

The bifurcation diagram for the phase variable part y_n of the complex variable Z_n of (2.6) is depicted in Fig. 2 for the same value of parameter *a* as in Fig. 1.

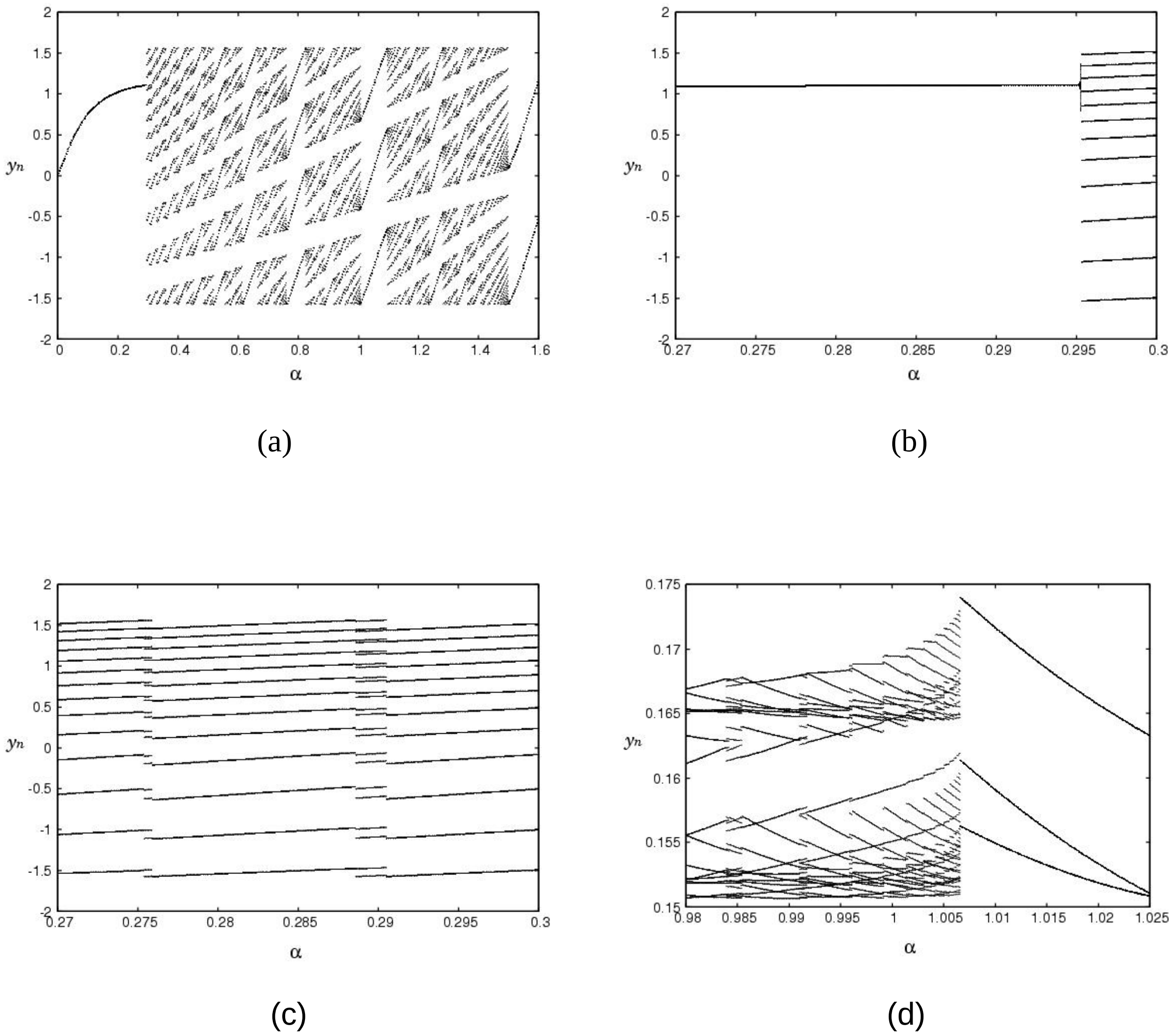
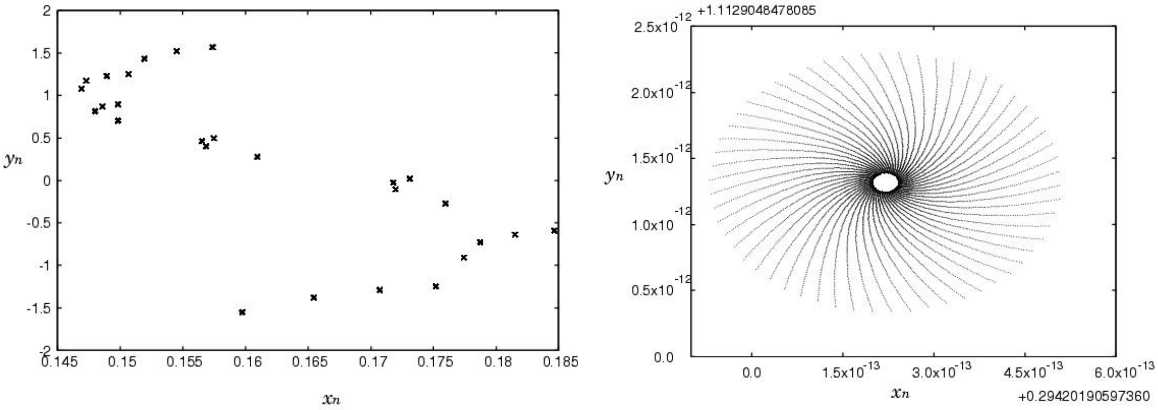


Fig. 2. (a). The bifurcation diagram of phase variable y_n of (2.6) for the fixed value of parameter $a = 1.1$ and $\alpha \in [0, 1.6]$ as in Fig. 1. Figs. 2. (b), (c) and (d) show the more detailed (zoomed) views of that of Fig. 2. (a).

Here again we can see the complicated structure of the diagram representing in a cyclicity of the phase variable. More detailed views in Figs. 2. (b), (c) and (d) uncover complicated bifurcation routes. Here again the existence of multi-attractors is indicated as mentioned above.

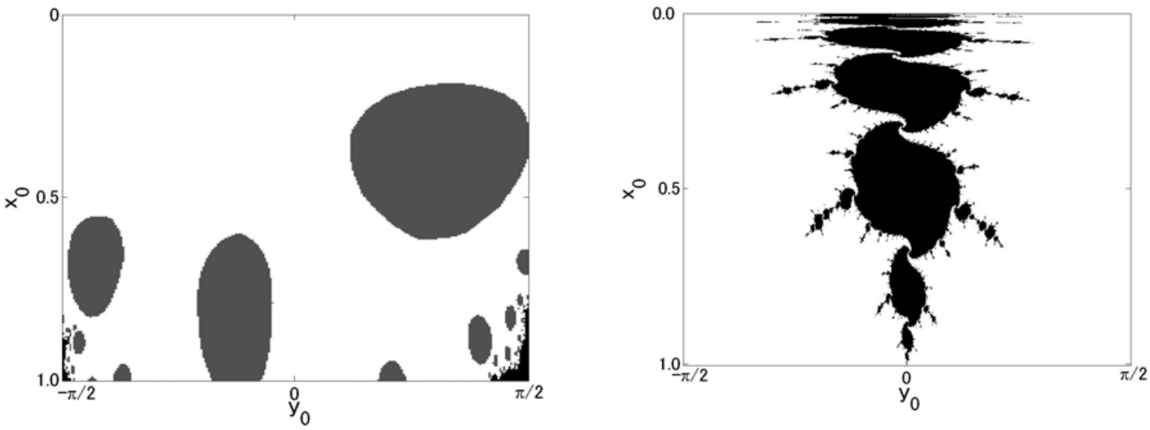
The phase space portrait (x, y) of the attractor of (2.6) for the values of parameters $a = 1.1$ and $\alpha = 0.485$ is seen in Fig. 3. (a). It is clear that the attractor is not chaotic. One can observe a transient process to the attractor (a circle here) on the bifurcation point as in Fig. 3. (b).



(a) (b)

Fig. 3. (a). The phase space portrait of attractor of (2.6) for *a = 1.1 and α = 0.485.* (b) The transient process to the attractor (a circle) on the bifurcation point for *a = 1.1 and α ≃ 0.2945*.

Remember that the real 2D dynamical system (2.6) is in fact equivalent to the 1D complex logistic equation (2.1). So it is natural to study by numerical simulations a basin of above mentioned attractors. The point is that we should obtain an analogy of very famous and beautiful pictures of Julia sets. The Fig. 4 brings pictures of basin of attraction for different values of adequate parameters *a* and *α* in (2.6).



(a) (b)

Fig. 4. (a) Basins of attractors of (2.6) for the *a = 1.1* and the parameter *α=0.29*. The grey and white regions correspond to the basins of two different attractors. (b) Basin of a single attractor of (2.6) for *a=3.1* and *α=0.0314.* The black region shows the basin.

To proceed further in our analysis of difference between the dynamics of complex logistic equation (2.1) and the canonical Hénon map (3.1) we make the calculation of the maximum Lyapunov exponent of the mapping (2.6) which is equivalent to the CLE of (2.1). The result can be seen in Fig. 5.

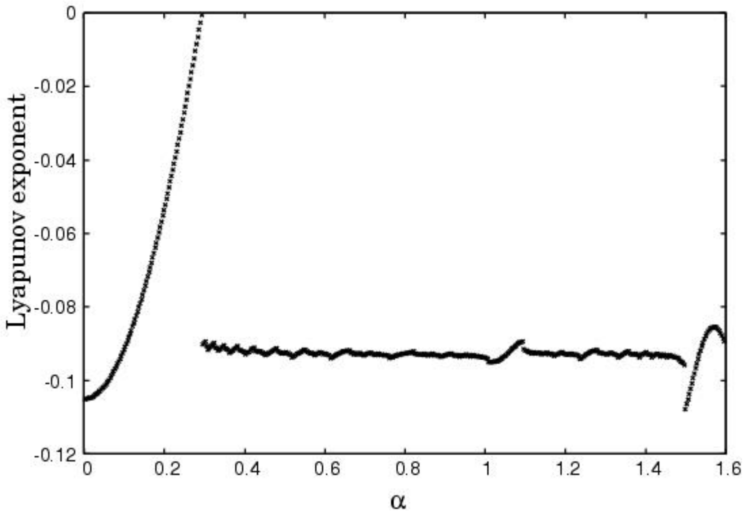


Fig. 5. The maximum Lyapunov exponent of (2.6) for the values of parameters *a=1.1* and *α ∈ [0, 1.6]*.

It is clear that for a given value of parameter *a*, the maximum Lyapunov exponent for *α ∈ [0, 1.6]* is non-positive. But that means there are no chaotic attractors as solutions of (2.6). In other words, it seems the dynamics of the 2D mapping (2.6) cannot be chaotic in contradiction to the Hénon map (3.1), possessing the chaotic dynamics.

4. The conjecture

As has been shown in the previous section there are indications the dynamics of complex logistic equation (2.1) is not equivalent to that of the canonical Hénon map (3.1). Here, we bring a further support for this assertion. We will use a similar hint as in the Section 2. But instead of (2.3) and (2.4) we use the standard definition of complex numbers, namely we put in (2.1)

$$Z_n = X_n + i\,Y_n\,, \tag{4.1}$$

$$A = a + i\,b\;. \tag{4.2}$$

Now, substituting (4.1) and (4.2) into (2.1) and by means of similar procedures as in the section 2, we arrive at the 2D real valued mapping as follows:

$$\begin{aligned} X_{n+1} &= a\,X_n\left(1-X_n\right) + aY_n^2 - bY_n\left(1-2\,X_n\right), \\ Y_{n+1} &= a\,Y_n\left(1-2\,X_n\right) + b\,X_n\left(1-X_n\right) + b\,Y_n^2\;. \end{aligned} \tag{4.3}$$

We are not going to analyze the 2D dynamical system (4.3) here. (It will be done elsewhere.) Let us note the numerical simulations of (4.3) lead to similar conclusions as those of the 2D mapping (2.6), i.e., chaotic dynamics in (4.3) is missing. On the other hand, it is well known the Hénon map (3.1) possesses the chaotic dynamics.

So again the question arrives why is the dynamics of (4.3) different from that of the Hénon map? To better understand this mysterious situation let us consider the case when *b = 0,* i.e., the parameter *A = a* is real. Then (4.3) has the following form

$$\begin{aligned} X_{n+1} &= a X_n \left(1 - X_n\right) + a Y_n^2 , \\ Y_{n+1} &= a Y_n \left(1 - 2 X_n\right) . \end{aligned} \tag{4.4}$$

But if we put *b = 0* in the Hénon map (3.1), we get just the ID logistic equation. So the situation seems principally different from that of 2D mapping (4.4). In other words, the complex valued variables lead even for the real parameter to the situation basically different from that of real variables. The analysis of the 2D mapping (4.3) seems rather complicated. There are some symmetries with exchanged parameters there. On the basis of what has been shown so far we can make the following Conjecture.

<u>Conjecture.</u> The complex logistic equation (2.1) is not conjugate with the canonical Hénon map (3.1). Or equivalently, there is not an affine transformation between the 2D mapping (4.3) and the Hénon map (3.1).

5. Discussion

We have presented the novel approach to study the dynamics of 1D complex valued logistic equation in the form of equivalent 2D real mapping. The purpose is to find out a possible role the property "to be complex valued" could play in nonlinear dynamics in general, and some possible consequences, e.g., to better understand the phenomenon of so called quantum chaos, in particular. To this end, we have studied in detail numerically the 2D real dynamical system (2.6), looking for bifurcation diagrams, phase space attractors with their basins of attraction, and finally maximum Lyapunov exponent. The results show there are no indications for the existence of chaotic dynamics in such a system. But we have started with the complex logistic equation which for the real case transforms to the classical 1D logistic equation, of course, possessing chaos. Besides, it is well known that a canonical 2D real map, so called the Hénon map, also possesses the chaotic dynamics. So the only possibility to understand logically this peculiar situation is to suppose the property of "to be complex valued" can remove the chaoticity of that of the analogical real dynamical system. To prove this in very general form we formulate our Conjecture according to which there is not any linear affine transformation of coordinates between the Hénon map and the 2D real transcription of complex valued 1D logistic map.

Summarising, we have seen so far that the property to be complex valued depresses in a somewhat mysterious way solutions of complex valued logistic equation not to possess chaos. Surprisingly, it is known that quantum mechanics prevails with periodic phenomena, too. On the other hand there have been many attempts to look for so called quantum chaos in the last three decades [5]. In such a way the peculiar situation has originated in which there prevails a general belief in the incompatibility between quantum and chaos. And in fact, there is no rigorous definition of quantum chaos, so far. Instead, by quantum chaos one means quantum mechanics, or quantization of classically chaotic systems. As the typical example the quantization of classical billiards may be of help. Quantum billiards have been constructed and studied, too. The output of all this huge business has been paradoxically the non-existence of chaos in quantum mechanical systems. We arrived here at the analogical situation with that of complex valued logistic equation. But the complexification is the very natural part of quantization, as we learned already. And in general, it does not matter if one considers the logistic equation or billiards. To this end we have brought the novel explanation of some paradoxes of so called quantum chaos. Alas, we can say with R. Penrose [6]: Quantum mechanics – undoubtedly one of the supreme intellectual achievements of the 20-th century – is still full of deep mysteries.

Acknowledgements

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References

[1] J. von Neumann, Mathematical Foundations of Quantum Mechanics. 12th Ed., Princeton UP, Princeton 1996.

[2] A. Einstein, Relativity: The Special and General Theory, Plume, 2006.

[3] J. Milnor, Dynamics in One Complex Variable. 3rd Ed., Princeton UP, Princeton and Oxford, 2006.

[4] B. Mandelbrot, Annals NY Acad. Sci. 357 (1980) 249.

[5] H.-J. Stockmann, Quantum Chaos: An Introduction, Cambridge UP, 2006.

[6] R. Penrose, In Foreword to the book: Quo Vadis Quantum Mechanics, A. Elitzur, et al (Eds.), Springer-Verlag, Berlin, 2005.