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Technical report No. 1027

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Abstract:

The big progress of novel quantum information science brings about a possibility to study the role the quantum entanglement may play in different fields, e.g., quantum computation, quantum communication, quantum teleportation, quantum cryptography, et cetera. It is proved rigorously here that every quantum operation generating quantum entanglement states in the multipartite quantum system of qubits leads to the reduction of quantum entropy of whole system. The novel quantum second law is then formulated on this basis. As the consequence it follows the quantum entanglement can also explain rather surprisingly many misleading and false expectations of so called challenges to the second law which have been proposed over the last ten years, or so.

Keywords:
Quantum information, quantum entanglement, quantum second law

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The quantum second law and quantum information

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It is proved rigorously that every quantum operation generating quantum entanglement in the multipartite quantum system of qubits leads to the reduction of quantum entropy of whole system. The quantum second law is then formulated on this basis.

If thermodynamics has been honoured as Queen of sciences, then according to A. Eddington (1), the second law of thermodynamics holds the supreme position among the laws of Nature. Surprisingly enough, for A. Einstein (2) the second law was the corn stone in his discoveries of special and general theory of relativity. The formulation of the second law in this spirit as the non existence (impossibility) of perpetuum mobile of second kind could have served also as the starting point to "no go theorems", as we know them today, at least intuitively (3).

Alas, a rigorous quantum mechanical formulation of the second law is still missing. Instead, between one and two dozen challenges to the second law have been proposed over the last ten years, or so (4). Among them, the most important role is played by so called quantum limits to the second law (5). But this seems to be a rather paradoxical situation as one is trying there to defeat something what has not been rigorously defined yet.

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To solve this delicate problem a possible role played by quantum entanglement in the entropy balance in quantum systems will be treated in detail here. In this sense the only meaning has a difference of entropy, i.e., changes of entropy during a given process. This will allow for a novel formulation of second law where quantum entanglement and quantum information enter the game.

The phenomenon of quantum entanglement (QET) was introduced almost unintentionally by A. Einstein in the famous EPR paper (6) to prove quantum mechanics is incomplete. The meaning of QET was soon after that disclosed by E. Schroedinger (7), who also gave the name QET to the phenomenon, saying: "When two systems, of which we know the states by their respective representations, enter into a temporary physical interaction due to known forces between them and when after a time of mutual influence the systems separate again, then they can no longer be described as before, viz., by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics". From this characterisation it is clear the QET is the process realized (thermodynamically) in an open quantum system. This is very important to know for to make a next step, namely, the concrete calculations of entropy changes during the process of entanglement generation.

To start with let us have a system of two qubits. Then the entropy change between entangled states and separable ones can be calculated as follows. Take, e.g., the entangled state of two qubits as an element of Bell´s base in the form

$$\left| \Psi_{12}^{ET} \right\rangle = \frac{1}{\sqrt{2}} \left(\left| 00 \right\rangle + \left| 11 \right\rangle \right) \tag{1}$$

where the standard notation is used (3)

$$\left| 00 \right\rangle = \left| 0 \right\rangle \otimes \left| 0 \right\rangle ; \quad \left| 11 \right\rangle = \left| 1 \right\rangle \otimes \left| 1 \right\rangle ; \quad \left| 0 \right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \left| 1 \right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \equiv \begin{pmatrix}1\\0\\0\\0\end{pmatrix} ; ...$$

Then one has for a density matrix $\hat{\rho}_{12}^{\mathrm{ET}}$

$$\hat{\rho}_{12}^{\mathrm{ET}} = |\Psi_{12}^{\mathrm{ET}}\rangle\langle\Psi_{12}^{\mathrm{ET}}| = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 1\end{pmatrix} \qquad (2)$$

For the von Neumann quantum entropy one has from the definition in general

$$\mathrm{H} = -\mathrm{Tr}\left(\hat{\rho}\ln\hat{\rho}\right) \qquad (3)$$

After substituting from (2) one gets

$$\mathrm{H}_{12}^{\mathrm{ET}} = -\mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 0\end{pmatrix}\ln\frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 0\end{pmatrix}\right] = 2\ln 2 \qquad (4)$$

On the other hand, for the separable states of two qubits $|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, namely

$$|\Psi_{12}\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \qquad (5)$$

So one has for a density matrix

$$\hat{\rho}_{12} = |\Psi_{12}\rangle\langle\Psi_{12}| = \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{pmatrix} \qquad (6)$$

and finally for the quantum von Neumann entropy for separable states

$$H_{12} = -\mathrm{Tr}\left[\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}\ln\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}\right] = 8\ \ln\ 2 \qquad (7)$$

One gets the inequality between entropies of separable and entangled states in the form

$$H_{12}^{ET} \langle\ H_{12} \qquad (8)$$

It is easy to show the same inequality holds for all elements of Bell´s bases of two entangled qubits $|\Psi_{12}^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right)$ and $|\Phi_{12}^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right)$.

For the system of two qubits the entropy change of entanglement generation is in general non positive, i.e.,

$$\Delta H^{ET}\left(\hat{\rho}_{12}\right) = H_{12}^{ET} - H_{12} \le 0. \qquad (9)$$

By other words, the entanglement of subsystems of given system leads to the reduction of entropy of the system. Or, put it in opposing way, the QET can serve as a source of quantum information. But this brings the novel explanation of the potentiality of power of QET in the quantum information processing in general and in so far known quantum algorithms of Shor (8), and Grover (9), in special.

Now, a question arises if the same holds in the case of system consisting of $N = 3$ qubits. Here one can exploit the same methodology as in the case of $N = 2$ qubits, but instead of calculations of quantum entropy in the $\mathcal{H}_{2^2} = \mathcal{H}_4$ Hilbert space formally more complicated calculations in the $\mathcal{H}_{2^3} = \mathcal{H}_8$ Hilbert space must be done. To begin with, the so-called GHZ (10) states as maximally entangled states of three qubits will be used, e.g.,

$$|\Psi_{GHZ}\rangle \equiv |\Psi_{123}^{ET}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \qquad (10)$$

where according to the previous notation $|000\rangle = |00\rangle \otimes |0\rangle$; $|111\rangle = |11\rangle \otimes |1\rangle$.

Not going to details of adequate matrix form, one has for the density matrix of given GHZ state (10)

$$\hat{\rho}_{\mathrm{GHZ}} \equiv \hat{\rho}_{123}^{\mathrm{ET}} = \left|\psi_{123}^{\mathrm{ET}}\right\rangle \left\langle \psi_{123}^{\mathrm{ET}}\right| \tag{11}$$

And finally for the quantum von Neumann entropy of such GHZ state one has

$$\mathrm{H}_{\mathrm{GHZ}} = \mathrm{H}_{123}^{\mathrm{ET}}\left(\hat{\rho}_{123}^{\mathrm{ET}}\right) = -\mathrm{Tr}\left(\hat{\rho}_{123}^{\mathrm{ET}} \ln \hat{\rho}_{123}^{\mathrm{ET}}\right) = 2 \ln 2 \tag{12}$$

In looking for a role of entanglement here let us have a system of three qubits, two of which are in entangled state but the third one is separable of this entangled state of two, e.g.,

$$\left|\Psi_{\overline{12},3}\right\rangle = \left|\Psi_{12}^{\mathrm{ET}}\right\rangle \otimes \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right) =$$
$$= \frac{1}{2}\left(\left|000\right\rangle + \left|001\right\rangle + \left|110\right\rangle + \left|111\right\rangle\right) \tag{13}$$

Then using the same methodology as for the GHZ states in the same notation one has for the quantum entropy of such partially entangled three qubits

$$\mathrm{H}_{\overline{12},3} = -\mathrm{Tr}\left[\,|\Psi_{\overline{12},3}\rangle\langle\Psi_{\overline{12},3}| \ln\left(|\Psi_{\overline{12},3}\rangle\langle\Psi_{\overline{12},3}|\right)\right] = 8 \ln 2 \tag{14}$$

To get the complete picture of the situation with three qubits, in a similar way, the quantum entropy of the totally separable state $|\Psi_{123}\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \otimes |\Psi_3\rangle$ of three qubits was calculated, to be

$$\mathrm{H}_{123} = 24 \ln 2 \tag{15}$$

So the following inequalities hold among calculated entropies in relation to the degree of entanglement

$$\mathrm{H}_{\mathrm{GHZ}} = \mathrm{H}_{123}^{\mathrm{ET}} \langle \mathrm{H}_{\overline{12},3} \langle \mathrm{H}_{123}\,, \tag{16}$$

where the inequalities hold for all variations of qubits and their entanglement. In analogy with (8) one has similar result for the entropy change of entanglement generation in the case of three qubits, namely

$$\Delta \mathrm{H}^{\mathrm{ET}}\left(\hat{\rho}_{123}\right) = \mathrm{H}_{123}^{\mathrm{ET}} - \mathrm{H}_{\overline{12},3}^{\mathrm{PET}} \leq 0 \quad (17)$$

where the acronym "PET" means partially entangled, containing also separable states.

Generalizing so far presented results allows to formulate the following finding.

Theorem: Every quantum mechanical operation generating entangled states in a multipartite quantum systems $Q = \{ q_1, \ldots, q_n \}$ of qubits $| q_i \rangle \left(i = 1, \ldots, n \geq 2 \right)$ leads to the reduction of quantum entropy of the whole system or by other words, a quantum entanglement can serve as a potential source of quantum information.

Still some open questions remain. One needs, e.g., to formulate the notion of quantum entropy production. In addition, in such systems the QET enters a game. So one has in general for the entropy change in quantum systems

$$\Delta \mathrm{H} = \Delta_i \mathrm{H} + \Delta_e \mathrm{H} + \Delta \mathrm{H}^{\mathrm{ET}} \quad (18)$$

where in analogy to classical systems $\Delta_i \mathrm{H}$ presents the internal entropy production in the system, $\Delta_e \mathrm{H}$ is the entropy flow, or the entropy exchanged with the surroundings, $\Delta \mathrm{H}^{\mathrm{ET}}$ is the quantum entropy reduction due to the process of entanglement formation, which is specifically quantum phenomenon.

One meets a very specific problem of quantum nature here. Namely, a possibility of very intricate interplay between the entropy production $\Delta_i \mathrm{H}$ and the reduction of entropy $\Delta \mathrm{H}^{\mathrm{ET}}$ due to the QET formation, i.e., the answer to the question

$$\mathrm{sign}\left(\Delta_i \mathrm{H} + \Delta \mathrm{H}^{\mathrm{ET}} \right) = ? \quad (19)$$

in different quantum systems.

These results shed a new light on the problem of rigorous formulation of the quantum second law. To understand a role the QET can play in a formulation of the quantum second law seems to be of crucial nature. E.g., to find explicitly an answer to the question (19) in concrete quantum systems may probably explain in a new way some above mentioned possible misunderstandings concerning so-called quantum challenges to the second law. Besides, our approach can bring a deeper view upon still pertaining problem of black hole information paradox (11), too.

Nevertheless one has to admit there still remain some principal open questions concerning the quantum second law as such. E.g., it is not clear where an intrinsic entropy production comes from in quantum systems.

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