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Lanzendörfer, Martin
2008

# A Note on Steady Flows of an Incompressible Fluid with Pressure- and Shear Rate-dependent Viscosity

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Field of Study:
Mathematical Modeling

This work was supported by GAČR 201/06/0352.

**Abstract**

A class of incompressible fluids whose viscosities depend on the pressure and the shear rate is considered. The existence of weak solutions for flows of such fluids under different settings was studied lately. In this short note, two recent existence results are adverted and their direct generalization into different setting is indicated; in this setting the corresponding energy estimates are derived showing the existence of a solution to an approximate system. A minor correction to one of the referred papers is also stated.

## 1. Introduction

The Newtonian homogeneous incompressible fluid is described by Navier-Stokes equations, where a linear relation between the stress tensor and the symmetric part of the velocity gradient is assumed, with a given constant called viscosity. However, in many important applications a non-Newtonian model is required. In this short note, the existence of a weak solution for steady flows of fluids with the viscosity increasing with the pressure and decreasing with the shear rate is addressed.

### 1.1. Fluid model

The theoretical analysis of the following problem is considered: Find the pressure and the velocity $(p, \boldsymbol{v}) = (p, v_1, \ldots, v_d) : \Omega \to \mathbb{R}^{d+1}$ ($\Omega \subset \mathbb{R}^d$ being an open bounded domain, $d \geq 2$) solving the equations:

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega\,, \tag{1}$$

$$\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div}[\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})] = -\nabla p + \boldsymbol{b} \quad \text{in } \Omega\,, \tag{2}$$

($\nabla$ denotes the Eulerian spatial gradient, $\mathbf{D}(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$ the symmetric part of the velocity gradient) completed by:

$$\int_\Omega p \, \mathrm{d}\boldsymbol{x} = 0 \tag{3}$$

and by the Dirichlet boundary condition

$$\boldsymbol{v} = \boldsymbol{\varphi} \qquad \text{on } \partial\Omega\,, \tag{4}$$

where $\boldsymbol{\varphi} : \partial\Omega \to \mathbb{R}^d$ and $\boldsymbol{b} : \Omega \to \mathbb{R}^d$ are given. We shall denote the system (1)-(4) by Problem (P). Standard notation[1] concerning function spaces is used.

For the viscosity $\nu(p, |\mathbf{D}|^2)$ the following assumptions are considered:

**A1** For a given $r \in (1, 2)$, there are positive constants $C_1$ and $C_2$ such that for all symmetric linear transformations $\mathbf{B}, \mathbf{D}$ and all $p \in \mathbb{R}$

$$C_1(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} |\mathbf{B}|^2 \leq \frac{\partial[\nu(p, |\mathbf{D}|^2)\mathbf{D}]}{\partial \mathbf{D}} \cdot (\mathbf{B} \otimes \mathbf{B}) \leq C_2(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} |\mathbf{B}|^2\,,$$

where $(\mathbf{B} \otimes \mathbf{B})_{ijkl} = \mathbf{B}_{ij}\mathbf{B}_{kl}$.

[1] For $1 \leq r \leq \infty$, the symbols $(\mathrm{L}^r(\Omega), ||\cdot||_r)$ and $(\mathrm{W}^{1,r}_{(0)}(\Omega), ||\cdot||_{1,r})$ denote the standard Lebesgue and Sobolev spaces (with zero trace on $\partial\Omega$). If $\mathrm{X}(\Omega)$ is a Banach space of functions defined on $\Omega$ then $(\mathrm{X}(\Omega))^*$ denotes its dual space. Also, $\mathbf{X}(\Omega) := \mathrm{X}(\Omega)^d = \{\boldsymbol{u} : \Omega \to \mathbb{R}^d; u_i \in \mathrm{X}(\Omega), i = 1, \ldots, d\}$. Further, $(\mathbf{W}^{-1,r'}(\Omega), ||\cdot||_{-1,r'}) := (\mathbf{W}^{1,r}_0)^*$, where $r' = \frac{r}{r-1}$. We use the Einstein summation convention in the text.

**A2** For all symmetric linear transformations $\mathbf{D}$ and for all $p \in \mathbb{R}$

$$\left|\frac{\partial[\nu(p,|\mathbf{D}|^2)\mathbf{D}]}{\partial p}\right| \leq \gamma_0(1+|\mathbf{D}|^2)^{\frac{r-2}{4}} \leq \gamma_0\,,$$

with

$$\gamma_0 < \frac{1}{C_{\mathrm{div},2}}\frac{C_1}{C_1+C_2} \leq \frac{1}{2C_{\mathrm{div},2}}\,.$$

The constant $C_{\mathrm{div},q}$ originates in the following problem, which is instrumental in the proof of the existence: For $g \in \mathrm{L}^q(\Omega)$ given, $\int_\Omega g\,\mathrm{d}\boldsymbol{x} = 0$, find $\boldsymbol{z}$ solving

$$\operatorname{div}\boldsymbol{z} = g \quad \text{in}\,\Omega, \quad \boldsymbol{z} = \mathbf{0} \quad \text{on}\,\partial\Omega\,. \tag{5}$$

For $q \in (1,\infty)$, the bounded linear Bogovskii operator $\mathcal{B}\,:\,\mathrm{L}^q(\Omega) \to \mathbf{W}_0^{1,q}(\Omega)$, assigning $\boldsymbol{z} := \mathcal{B}(g)$ the solution of (5), fulfills

$$||\boldsymbol{z}||_{1,q} = ||\mathcal{B}(g)||_{1,q} \leq C_{\mathrm{div},q}||g||_q\,. \tag{6}$$

Moreover, if $g = \operatorname{div}\boldsymbol{f}$, with $\boldsymbol{f} \in \mathbf{W}^{1,q}(\Omega)$ and $\boldsymbol{f}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, then

$$||\boldsymbol{z}||_q = ||\mathcal{B}(\operatorname{div}\boldsymbol{f})||_q \leq D_{\mathrm{div},q}||\boldsymbol{f}||_q\,. \tag{7}$$

Note that the assumptions **(A1)** and **(A2)** determine the fluid model to be shear-thinning and allow it to be pressure-thickening. Examples and more details can be found e.g. in [1]. Note also that the following inequalities result from **(A1)** and **(A2)**, see [1, 2] for their proofs. First,

$$\nu(p,|\mathbf{D}|^2)\mathbf{D}:\mathbf{D} \geq \frac{C_1}{2r}(|\mathbf{D}|^r - 1)\,, \tag{8}$$

$$|\nu(p,|\mathbf{D}|^2)\mathbf{D}| \leq \frac{C_2}{r-1}(1+|\mathbf{D}|)^{r-1} \tag{9}$$

holds for all symmetric $\mathbf{D}$ and all $p \in \mathbb{R}$. Then, defining

$$I^{1,2} := \int_0^1 \left(1+|\mathbf{D}^1+s(\mathbf{D}^2-\mathbf{D}^1)|^2\right)^{\frac{r-2}{2}} |\mathbf{D}^1-\mathbf{D}^2|^2\,\mathrm{d}s\,, \tag{10}$$

there hold

$$\frac{C_1}{2}I^{1,2} \leq \left(\nu(p^1,|\mathbf{D}^1|^2)\mathbf{D}^1 - \nu(p^2,|\mathbf{D}^2|^2)\mathbf{D}^2\right) : (\mathbf{D}^1-\mathbf{D}^2) + \frac{\gamma_0^2}{2C_1}\,|p^1-p^2|^2\,, \tag{11}$$

$$\left|\nu(p^1,|\mathbf{D}^1|^2)\mathbf{D}^1 - \nu(p^2,|\mathbf{D}^2|^2)\mathbf{D}^2\right| \leq C_2\left(I^{1,2}\right)^{\frac{1}{2}} + \gamma_0\,|p^1-p^2|\,. \tag{12}$$

### 1.2. Results

The model described above has been systematically studied in last decade or more; the reader is kindly asked to find references given in [1] and [2].

In [1], the existence of a weak solution to Problem (P) including the non-homogeneous Dirichlet boundary condition (4) was proved, either for small data or assuming the inner flows:

$$\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0 \quad \text{on}\,\partial\Omega\,. \tag{13}$$

The proof is given for $d = 2$ or 3 and for

$$\frac{3d}{d+2} \leq r < 2\,.$$

The lower bound relates to the fact, that with $r \geq \frac{3d}{d+2}$ the solution is a possible test function in the weak formulation and a standard monotone operator theory is applicable, supplied by proper estimates on the pressure. Within the proof, the following $\varepsilon$-approximate system is utilized, replacing equation (1) by

$$-\,\varepsilon\Delta p + \operatorname{div}\boldsymbol{v} = 0 \;\text{ in}\,\Omega, \quad \frac{\partial p}{\partial\boldsymbol{n}} = 0 \;\text{ on}\,\partial\Omega \tag{14}$$

for $\varepsilon > 0$. The solution to Problem (P) is obtained by the limit $\varepsilon \to 0$.

Recently in [2], the theory was extended to the case

$$\frac{2d}{d+2} < r \leq \frac{3d}{d+2}\,,$$

considering the homogeneous Dirichlet boundary condition

$$\boldsymbol{\varphi} = \mathbf{0} \quad \text{on}\,\partial\Omega\,.$$

The starting point is the following $\eta,\varepsilon$-approximate system, replacing (1) by (14) and replacing (2) by

$$\left.\begin{aligned} &\eta|\boldsymbol{v}|^{2r'-2}\boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\mathcal{P}\boldsymbol{v}) \\ &\quad -\operatorname{div}[\nu(p,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})] = -\nabla p + \boldsymbol{b} \end{aligned}\right\} \tag{15}$$

for $\eta > 0$, where $\mathcal{P}$ is a projection to divergence-free functions.

The goal of the presented paper is to follow these two results and to study the existence of a weak solution to Problem (P) with

$$r < \frac{3d}{d+2}$$

and subject to non-homogeneous Dirichlet boundary condition. Section 2 derives the energy estimates for the corresponding $\eta,\varepsilon$-approximate system, thereby showing the existence of its weak solution. In Section 3, the main existence theorem is merely stated, the remaining parts of the proof–the limit procedures $\varepsilon \to 0$ and $\eta \to 0$–being left to the reader, referring to [2]. The theorem assumes non-homogeneous Dirichlet b.c. with small data, its corollary then treats inner flows with large data. In the last section, some minor correction to [1] is mentioned.

## 2. Energy estimates

The main result of this paper is the following variation of Lemma 4.1, which is the starting point of the result established in [2].

**Lemma 1** *Let $\varepsilon, \eta > 0$ be arbitrary. Let $\Omega \in \mathcal{C}^{0,1}$, $d \geq 2$ and $\boldsymbol{b} \in \mathbf{W}^{-1,r'}(\Omega)$ be given. Let*

$$\frac{2d}{d+1} < r < \min\left\{2, \frac{3d}{d+2}\right\} \tag{16}$$

*and the assumptions* **A1** *and* **A2** *be satisfied. There are certain positive constants $H_1$, $H_2$ which depend on $r$, $\Omega$, $C_1$, $C_2$ and $\boldsymbol{b}$ and which are small enough such that they meet the inequality* (23). *Let there exist $\lambda \geq 1$ and $\boldsymbol{\Phi} \in \mathbf{W}^{1,r}(\Omega)$ such that, with $q := \frac{rd}{r(d+1)-2d}$ ,*

$$\operatorname{div} \boldsymbol{\Phi} = 0 \ \text{ in } \Omega, \quad \operatorname{tr} \boldsymbol{\Phi} = \boldsymbol{\varphi} \qquad \text{and} \qquad ||\boldsymbol{\Phi}||_q \leq H_1 \lambda^{r-2} \qquad \text{and} \qquad ||\nabla \boldsymbol{\Phi}||_r \leq ||\boldsymbol{\Phi}||_{1,r} \leq H_2 \lambda \,. \tag{17}$$

*Then there exists a couple $(p, \boldsymbol{v})$ satisfying*

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\Phi}, \qquad \boldsymbol{u} \in \mathbf{W}_0^{1,r}(\Omega) \cap \mathbf{L}^{2r'}(\Omega) \qquad \text{and} \qquad p \in \mathrm{W}^{1,2}(\Omega) \cap \mathrm{L}_0^2(\Omega)\,, \tag{18}$$

$$\varepsilon \int_\Omega \nabla p \cdot \nabla \xi \,\mathrm{d}\boldsymbol{x} + \int_\Omega \xi \operatorname{div} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = 0 \qquad \text{for all } \xi \in \mathrm{W}^{1,2}(\Omega)\,, \tag{19}$$

$$\left.\begin{aligned} &\eta \int_\Omega |\boldsymbol{u}|^{2r'-2} \boldsymbol{u} \cdot \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \nu(p, |\mathbf{D}(\boldsymbol{v})|^2) \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\psi}) \,\mathrm{d}\boldsymbol{x} - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} \\ &\quad - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{u})\, \boldsymbol{u} \cdot \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} = \int_\Omega p \operatorname{div} \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}, \boldsymbol{\psi} \rangle \qquad \text{for all } \boldsymbol{\psi} \in \mathbf{W}_0^{1,r}(\Omega) \cap \mathbf{L}^{2r'}(\Omega)\,. \end{aligned}\right\} \tag{20}$$

*Moreover, the following estimates hold:*

$$\varepsilon\, ||p||_{1,2}^2 + \eta\, ||\boldsymbol{v}||_{2r'}^{2r'} + ||\mathbf{D}(\boldsymbol{v})||_r^r \leq C < +\infty\,, \tag{21}$$

$$||\nu(p, |\mathbf{D}(\boldsymbol{v})|^2) \mathbf{D}(\boldsymbol{v})||_{r'} \leq C < +\infty \qquad \text{and} \qquad ||p||_{\frac{2dr}{r(d-2)+d}} \leq C(\eta) < +\infty\,. \tag{22}$$

**Proof:** Note that all integrals make sense:

$$\begin{aligned} &\boldsymbol{v} \in \mathbf{W}^{1,r}(\Omega) \cap \mathbf{L}^{2r'}(\Omega) \quad \Leftarrow \quad \boldsymbol{\Phi} \in \mathbf{W}^{1,r}(\Omega) \cap \mathbf{L}^q(\Omega), \ \text{ where } q > 2r' \text{ since } r < \frac{3d}{d+2}\,, \\ &\xi \operatorname{div} \boldsymbol{v} \in \mathrm{L}^1(\Omega) \quad \Leftarrow \quad \xi \in \mathrm{W}^{1,2}(\Omega) \hookrightarrow \mathrm{L}^{r'}(\Omega) \ \text{ since } r > \frac{2d}{d+2}\,, \\ &\nu(p, |\mathbf{D}(\boldsymbol{v})|^2) \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\psi}) \in \mathrm{L}^1(\Omega) \quad \Leftarrow \quad \boldsymbol{v}, \boldsymbol{\psi} \in \mathbf{W}^{1,r}(\Omega) \ \text{ and since (9).} \end{aligned}$$

The pair $(p, \boldsymbol{v})$ fulfilling (18)-(20) can be found as a limit of Galerkin approximations. The proof uses Brouwer's fixed point theorem, the compact embedding argument, the monotonicity conditions (11), (12) and Vitali's theorem. Here the first steps are provided in detail and, in time, the remainings are referred to [1].

Take $\{\alpha^k\}_{k=1}^\infty$ and $\{\boldsymbol{a}^k\}_{k=1}^\infty$ any bases of $\mathrm{W}^{1,2}(\Omega)$ and $\mathbf{W}_0^{1,2}(\Omega)$, respectively. Define the Galerkin approximations as follows:

$$\left.\begin{aligned} p^N &:= \textstyle\sum_{k=1}^N c_k^N (\alpha^k - \frac{1}{|\Omega|} \int_\Omega \alpha^k \,\mathrm{d}\boldsymbol{x}) \\ \boldsymbol{v}^N &:= \boldsymbol{\Phi} + \textstyle\sum_{k=1}^N d_k^N \boldsymbol{a}^k =: \boldsymbol{\Phi} + \boldsymbol{u}^N \end{aligned}\right\} \quad \text{for } N = 1, 2, \dots\,,$$

where $\boldsymbol{c}^N = (c_1^N, \dots, c_N^N)$ and $\boldsymbol{d}^N = (d_1^N, \dots, d_N^N)$ solve the algebraic system

$$\mathcal{M}([\boldsymbol{c}^N, \boldsymbol{d}^N]) = \mathbf{0}\,,$$

with $\mathcal{M} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ being a continuous mapping:

$$\begin{aligned} \mathcal{M}_k([\boldsymbol{c}^N, \boldsymbol{d}^N]) \;&:=\; \varepsilon \int_\Omega \nabla p^N \cdot \nabla \alpha^k \,\mathrm{d}\boldsymbol{x} + \int_\Omega \alpha^k \operatorname{div} \boldsymbol{v}^N \,\mathrm{d}\boldsymbol{x}\,, \quad k = 1, 2, \dots, N \\ \mathcal{M}_{N+l}([\boldsymbol{c}^N, \boldsymbol{d}^N]) \;&:=\; \eta \int_\Omega |\boldsymbol{u}^N|^{2r'-2} \boldsymbol{u}^N \cdot \boldsymbol{a}^l \,\mathrm{d}\boldsymbol{x} - \int_\Omega (\boldsymbol{v}^N \otimes \boldsymbol{v}^N) : \nabla \boldsymbol{a}^l \,\mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{u}^N) \boldsymbol{u}^N \cdot \boldsymbol{a}^l \,\mathrm{d}\boldsymbol{x} \\ &\quad + \int_\Omega \nu(p^N, |\mathbf{D}(\boldsymbol{v}^N)|^2) \mathbf{D}(\boldsymbol{v}^N) : \mathbf{D}(\boldsymbol{a}^l) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p^N \operatorname{div} \boldsymbol{a}^l \,\mathrm{d}\boldsymbol{x} - \langle \boldsymbol{b}, \boldsymbol{a}^l \rangle\,, \quad l = 1, 2, \dots, N\,. \end{aligned}$$

The basic estimate is obtained by testing the equation by $(p^N, \boldsymbol{u}^N)$ as follows. First, realize that (recall $\operatorname{div} \boldsymbol{v}^N = \operatorname{div} \boldsymbol{u}^N$)

$$\mathcal{M}([\boldsymbol{c}^N, \boldsymbol{d}^N]) \cdot ([\boldsymbol{c}^N, \boldsymbol{d}^N]) = \varepsilon \, ||\nabla p^N||_2^2 + \eta \, ||\boldsymbol{u}^N||_{2r'}^{2r'} \overbrace{- \int_\Omega (\boldsymbol{v}^N \otimes \boldsymbol{v}^N) : \nabla \boldsymbol{u}^N \, \mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{u}^N) \, |\boldsymbol{u}^N|^2 \, \mathrm{d}\boldsymbol{x}}^{=:I_{\mathrm{conv}}}$$
$$+ \int_\Omega \nu(p^N, |\mathbf{D}(\boldsymbol{v}^N)|^2) \mathbf{D}(\boldsymbol{v}^N) : \mathbf{D}(\boldsymbol{u}^N) \, \mathrm{d}\boldsymbol{x} - \langle \boldsymbol{b}, \boldsymbol{u}^N \rangle \, .$$

Since $\frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{u}^N) \, |\boldsymbol{u}^N|^2 \, \mathrm{d}\boldsymbol{x} = - \int_\Omega (\boldsymbol{u}^N \otimes \boldsymbol{u}^N) : \nabla \boldsymbol{u}^N \, \mathrm{d}\boldsymbol{x}$, it follows that

$$I_{\mathrm{conv}} = - \int_\Omega (\boldsymbol{\Phi} \otimes \boldsymbol{\Phi} + \boldsymbol{\Phi} \otimes \boldsymbol{u}^N + \boldsymbol{u}^N \otimes \boldsymbol{\Phi}) : \nabla \boldsymbol{u}^N \, \mathrm{d}\boldsymbol{x} \, ,$$

which implies (using Hölder's, Korn's and embeddings inequalities and using $r > \frac{2d}{d+1}$)

$$|I_{\mathrm{conv}}| \leq ||\nabla \boldsymbol{u}^N||_r \left( 2 \, ||\boldsymbol{u}^N||_{\frac{rd}{d-r}} \, ||\boldsymbol{\Phi}||_q + ||\boldsymbol{\Phi}||_{2r'}^2 \right) \leq C \, ||\mathbf{D}(\boldsymbol{u}^N)||_r^2 \, ||\boldsymbol{\Phi}||_q + C \, ||\mathbf{D}(\boldsymbol{u}^N)||_r \, ||\boldsymbol{\Phi}||_q^2 \, ,$$

where $q = \frac{dr}{r(d+1)-2d} > 2r'$. Throughout this text, the symbols $C$ denote positive, generally different constants. Further,

$$\int_\Omega \nu(p^N, |\mathbf{D}(\boldsymbol{v}^N)|^2) \mathbf{D}(\boldsymbol{v}^N) : \mathbf{D}(\boldsymbol{u}^N) \, \mathrm{d}\boldsymbol{x} = \int_\Omega \nu(p^N, |\mathbf{D}(\boldsymbol{v}^N)|^2) \mathbf{D}(\boldsymbol{v}^N) : (\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{\Phi})) \, \mathrm{d}\boldsymbol{x}$$
$$\geq \frac{C_1}{2r} \int_\Omega |\mathbf{D}(\boldsymbol{v}^N)|^r \, \mathrm{d}\boldsymbol{x} - \frac{C_1}{2r} |\Omega| - \frac{C_2}{r-1} \int_\Omega (1 + |\mathbf{D}(\boldsymbol{v}^N)|)^{r-1} |\mathbf{D}(\boldsymbol{\Phi})| \, \mathrm{d}\boldsymbol{x}$$
$$\geq C \, ||\mathbf{D}(\boldsymbol{u}^N) + \mathbf{D}(\boldsymbol{\Phi})||_r^r - C - C \, ||\mathbf{D}(\boldsymbol{\Phi})||_r \, ||1 + |\mathbf{D}(\boldsymbol{u}^N) + \mathbf{D}(\boldsymbol{\Phi})| \, ||_r^{r-1} \, .$$

Using $|a+b|^{r-1} \leq |a|^{r-1} + |b|^{r-1}$ due to $r - 1 < 1$, it follows

$$\int_\Omega \nu(p^N, |\mathbf{D}(\boldsymbol{v}^N)|^2) \mathbf{D}(\boldsymbol{v}^N) : \mathbf{D}(\boldsymbol{u}^N) \, \mathrm{d}\boldsymbol{x} \geq C \, ||\mathbf{D}(\boldsymbol{u}^N) + \mathbf{D}(\boldsymbol{\Phi})||_r \left( ||\mathbf{D}(\boldsymbol{u}^N)||_r^{r-1} - ||\mathbf{D}(\boldsymbol{\Phi})||_r^{r-1} \right)$$
$$- C - C \, ||\mathbf{D}(\boldsymbol{\Phi})||_r \left( 1 + ||\mathbf{D}(\boldsymbol{u}^N)||_r^{r-1} + ||\mathbf{D}(\boldsymbol{\Phi})||_r^{r-1} \right)$$
$$\geq D \, ||\mathbf{D}(\boldsymbol{u}^N)||_r^r - C \, ||\mathbf{D}(\boldsymbol{\Phi})||_r \, ||\mathbf{D}(\boldsymbol{u}^N)||_r^{r-1} - C \, ||\mathbf{D}(\boldsymbol{\Phi})||_r^{r-1} \, ||\mathbf{D}(\boldsymbol{u}^N)||_r - C \, ||\mathbf{D}(\boldsymbol{\Phi})||_r^r - C \, .$$

Finally, since $|\langle \boldsymbol{b}, \boldsymbol{u}^N \rangle| \leq C \, ||\boldsymbol{b}||_{-1,r'} \, ||\mathbf{D}(\boldsymbol{u}^N)||_r$ and noticing that there holds $||\nabla p^N||_2 \geq C \, ||p^N||_{1,2}$, we arrive at

$$\mathcal{M}([\boldsymbol{c}^N, \boldsymbol{d}^N]) : ([\boldsymbol{c}^N, \boldsymbol{d}^N]) \geq \varepsilon \, C \, ||p^N||_{1,2}^2 + \eta \, ||\boldsymbol{u}^N||_{2r'}^{2r'} + D \, ||\mathbf{D}(\boldsymbol{u}^N)||_r^r$$
$$- C \, ||\mathbf{D}(\boldsymbol{u}^N)||_r^2 \, ||\boldsymbol{\Phi}||_q - C \, ||\mathbf{D}(\boldsymbol{u}^N)||_r \, ||\boldsymbol{\Phi}||_q^2 - C \, ||\mathbf{D}(\boldsymbol{u}^N)||_r^{r-1} \, ||\nabla \boldsymbol{\Phi}||_r$$
$$- C \, ||\mathbf{D}(\boldsymbol{u}^N)||_r \, ||\nabla \boldsymbol{\Phi}||_r^{r-1} - C \, ||\nabla \boldsymbol{\Phi}||_r^r - C - C \, ||\mathbf{D}(\boldsymbol{u}^N)||_r \, .$$

At this point the assumption (17) is recalled and, denoting $\rho := ||\mathbf{D}(\boldsymbol{u}^N)||_r / \lambda$, the following is observed:

$$\mathcal{M}([\boldsymbol{c}^N, \boldsymbol{d}^N]) : ([\boldsymbol{c}^N, \boldsymbol{d}^N]) \geq \varepsilon \, C \, ||p^N||_{1,2}^2 + \eta \, ||\boldsymbol{u}^N||_{2r'}^{2r'} + D \rho^r \lambda^r$$
$$- C \rho^2 \lambda^2 H_1 \lambda^{r-2} - C \rho \lambda H_1 \lambda^{2r-4} - C \rho^{r-1} \lambda^{r-1} H_2 \lambda - C \rho \lambda H_2^{r-1} \lambda^{r-1} - C H_2^r \lambda^r - C \rho \lambda - C$$
$$\geq \varepsilon \, C \, ||p^N||_{1,2}^2 + \eta \, ||\boldsymbol{u}^N||_{2r'}^{2r'} + D \rho^r \lambda^r$$
$$- C H_1 \rho^2 \lambda^r - C H_1 \rho \lambda^{2r-3} - C H_2 \rho^{r-1} \lambda^r - C H_2^{r-1} \rho \lambda^r - C H_2^r \lambda^r - C \rho \lambda - C \, .$$

Since $1 \leq \lambda \leq \lambda^r$ and $\lambda^{2r-3} \leq \lambda^r$, this can be rewritten as

$$\mathcal{M}([\boldsymbol{c}^N, \boldsymbol{d}^N]) : ([\boldsymbol{c}^N, \boldsymbol{d}^N]) \geq \varepsilon \, C \, ||p^N||_{1,2}^2 + \eta \, ||\boldsymbol{u}^N||_{2r'}^{2r'}$$
$$+ \lambda^r \left[ \left( \frac{D}{2} \rho^r - C \rho - C \right) + \left( \frac{D}{2} \rho^r - C H_1 \rho^2 - C H_1 \rho - C H_2 \rho^{r-1} - C H_2^{r-1} \rho - C H_2^r \right) \right] .$$

Define $E > 0$ such that $\frac{D}{2}E^r - CE - C \geq 0$. The values of $C$, $D$ and $E$ define the following constraint, which is assumed to be fulfilled by the constants $H_1$ and $H_2$:

$$\frac{D}{2}E^r - (CE^2 + CE)H_1 - CE^{r-1}H_2 - CEH_2^{r-1} - CH_2^r \geq 0\,. \tag{23}$$

Note that, since $\frac{D}{2}E^r > 0$, some $H_1$, $H_2$ small enough to meet (23) can be found. Note that the values of $C$, $D$, $E$ and consequently $H_1$ and $H_2$ depend only on $C_1$, $C_2$, $r$, $\Omega$ and $\boldsymbol{b}$.

It follows that the inequality

$$\mathcal{M}([\boldsymbol{c}^N, \boldsymbol{d}^N]) : ([\boldsymbol{c}^N, \boldsymbol{d}^N]) \geq \mathbf{0} \tag{24}$$

holds for any $[\boldsymbol{c}^N, \boldsymbol{d}^N]$, provided that $||\mathbf{D}(\boldsymbol{u}^N)||_r = E$. Moreover, there exists some $C > 0$ independent of $\varepsilon$ and $\eta$, such that (24) holds also for any $[\boldsymbol{c}^N, \boldsymbol{d}^N]$, provided that $\varepsilon\, ||p^N||_{1,2}^2 \geq C$ or provided that $\eta\, ||\boldsymbol{u}^N||_{2r'}^{2r'} \geq C$. Applying Brouwer's fixed point theorem, a solution $(p^N, \boldsymbol{v}^N)$ of the Galerkin approximate system is obtained, fulfilling the estimate (21)

$$\varepsilon\, ||p^N||_{1,2}^2 + \eta\, ||\boldsymbol{v}^N||_{2r'}^{2r'} + ||\mathbf{D}(\boldsymbol{v}^N)||_r \leq C < \infty\,, \tag{25}$$

where $C$ does not depend on $\varepsilon$ neither on $\eta$. The estimate $(22)_1$

$$||\nu(p^N, |\mathbf{D}(\boldsymbol{v}^N)|^2)\mathbf{D}(\boldsymbol{v}^N)||_{r'} \leq C < \infty \tag{26}$$

then follows from (9).

With the estimates (25)-(26) in hand, the limit passage $N \to \infty$ follows exactly the steps given e.g. in [1]; the compact embedding, the monotonicity (11) and Vitali's theorem are used and a couple $(p, \boldsymbol{v})$ is found, which solves (18)-(20) and fulfills the estimates (21), $(22)_1$.

In order to obtain an estimate for pressure uniform with respect to $\varepsilon$, test the equation (20) with $\boldsymbol{\psi} := \mathcal{B}(|p|^{s-2}p - \frac{1}{|\Omega|}\int_\Omega |p|^{s-2}p \,\mathrm{d}\boldsymbol{x})$, denoting $s := \frac{2rd}{r(d-2)+d}$. Note that

$$\begin{aligned} &||\boldsymbol{\psi}||_{1,s'} \leq 2C_{\mathrm{div},s'}\, ||p||_s^{s-1} \\ &||\boldsymbol{\psi}||_{2r'} = ||\boldsymbol{\psi}||_{\frac{ds'}{d-s'}} \leq C\, ||\boldsymbol{\psi}||_{1,s'}\,, \qquad r \leq s' \quad \text{and} \quad s \leq r'\,. \end{aligned}$$

Since $\int_\Omega p \operatorname{div} \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} = ||p||_s^s$, this yields

$$\begin{aligned} ||p||_s^s &= \eta \int_\Omega |\boldsymbol{u}|^{2r'-2}\boldsymbol{u}\cdot \boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\boldsymbol{v}\otimes \boldsymbol{v}) : \nabla\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{u})\boldsymbol{u}\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\psi})\,\mathrm{d}\boldsymbol{x} - \langle \boldsymbol{b}, \boldsymbol{\psi}\rangle \\ &\leq \eta\, ||\boldsymbol{\psi}||_{2r'}||\boldsymbol{u}||_{2r'}^{2r'-1} + C\, ||\boldsymbol{\psi}||_{1,s'}\, ||\boldsymbol{v}\otimes\boldsymbol{v}||_s + C\, ||\mathbf{D}(\boldsymbol{u})||_r\, ||\boldsymbol{\psi}||_{2r'}\, ||\boldsymbol{u}||_{2r'} + C\, ||\boldsymbol{\psi}||_{1,r}(1 + ||\mathrm{D}(\boldsymbol{v})||_r)^{r-1} \\ &\quad + ||\boldsymbol{b}||_{-1,r'}\, ||\boldsymbol{\psi}||_{1,r} \leq C(\eta)\, ||\boldsymbol{\psi}||_{1,s'} \quad \leq \quad C(\eta)\, ||p||_s^{s-1}\,, \end{aligned}$$

which finally implies $(22)_2$

$$||p||_{\frac{2dr}{r(d-2)+d}} \leq C(\eta) < \infty\,. \tag{27}$$

□

## 3. Existence theorem

Lemma 1 allows to establish the following results. First, the generalization of Theorem 1 stated in [1] and of Theorem 2.1 stated in [2] can be formulated:

**Theorem 2** *Let $\Omega \in \mathcal{C}^{0,1}$, $d \geq 2$ and $\boldsymbol{b} \in \mathbf{W}^{-1,r'}(\Omega)$ be given. Let*

$$\frac{2d}{d+1} < r < \min\left\{2, \frac{3d}{d+2}\right\}$$

*and the assumptions* **A1** *and* **A2** *be satisfied. Let there exist $\lambda \geq 1$ and $\boldsymbol{\Phi} \in \mathbf{W}^{1,r}(\Omega)$ fulfilling* (17)*, with $H_1$ and $H_2$ meeting the inequality* (23).

*Then there exists at least one weak solution* $(p, \boldsymbol{v})$ *to Problem (P) such that*

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\Phi}, \qquad (p, \boldsymbol{u}) \in \mathrm{L}_0^{\frac{dr}{2(d-r)}}(\Omega) \times \mathbf{W}^{1,r}_{\mathrm{div},0}(\Omega)\,,$$

*and such that, for all* $\boldsymbol{\psi} \in \mathcal{C}_0^\infty(\Omega)^d$,

$$\int_\Omega \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\psi})\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega p \operatorname{div} \boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}, \boldsymbol{\psi}\rangle\,.$$

For the proof, the reader is asked to follow the complete procedure given in [2], starting with the above established Lemma 1 and using the method of Lipschitz approximations of Sobolev functions, developed in [3, 4].

The assumptions (17) on the non-homogeneous Dirichlet boundary condition contains, deliberately, the "free" parameter $\lambda \geq 1$. This allows, due to Lemma 3 in [1], to proceed to the following analogy of Corollary 4 in [1] concerned with the inner flows:

**Corollary 3** *Let* $\Omega$ *and* $\boldsymbol{b}$ *be the same as in Theorem 2. Let the assumptions* **(A1)** *and* **(A2)** *be satisfied with*

$$d = 3$$

*and with*

$$2 - \frac{1}{d} = \frac{5}{3} < r < \frac{9}{5} = \frac{3d}{d+2}\,. \tag{28}$$

*Let* $\boldsymbol{\varphi} = \operatorname{tr}\boldsymbol{\Phi}$ *for some* $\boldsymbol{\Phi} \in \mathbf{W}^{1,q}(\Omega) \cap \mathbf{L}^\infty(\Omega)$, $q = \frac{rd}{r(d+1)-2d}$, *where* $\boldsymbol{\varphi}$ *satisfies* (13)

$$\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega\,.$$

*Then there is at least one weak solution to Problem (P).*

A short proof given in [1] is reproduced here. The goal is to find $\boldsymbol{\Phi}^\eta$, $\eta \in (0, 1\rangle$ and $\lambda \geq 1$ such that the condition (17) is fulfilled, i. e.

$$||\boldsymbol{\Phi}^\eta||_{\frac{rd}{r(d+1)-2d}} \leq H_1 \lambda^{r-2}\,, \tag{29}$$

$$||\boldsymbol{\Phi}^\eta||_{1,r} \leq H_2 \lambda\,. \tag{30}$$

Then the assertion follows from Theorem 2.

For any $\eta \in (0, 1\rangle$, Lemma 3 in [1] gives a suitable extension $\boldsymbol{\Phi}^\eta$ of the boundary data $\boldsymbol{\varphi}$ and the estimate

$$||\boldsymbol{\Phi}^\eta||_q < H\eta^{\frac{1}{q}}\,, \tag{31}$$

$$||\boldsymbol{\Phi}^\eta||_{1,q} < H\eta^{\frac{1}{q}-1}\,, \tag{32}$$

where $q \in (0, \infty)$ and where $H$ depends only on $\Omega$ and $\boldsymbol{\Phi}$. Since $r > 2 - \frac{1}{d}$, an $s$ can be found such that

$$\frac{r-1}{r} < s < \frac{r(d+1)-2d}{rd(2-r)}\,.$$

Setting $\lambda := \eta^{-s}$ this means that for any positive constants $H$, $H_1$ and $H_2$, suitable $\eta \in (0, 1\rangle$ can be found such that

$$H_1\lambda^{r-2} = H_1\eta^{s(2-r)} > H\eta^{\frac{r(d+1)-2d}{rd}}\,,$$

$$H_2\lambda = H_2\eta^{-s} > H\eta^{\frac{1-r}{r}}\,.$$

For such $\eta$, the assertions (29)-(30) follow from (31) and (32).
□

## 4. Further notes

Note that in comparison to Theorem 1 in [1], its assumption (15) is not of any use here and is simply missing in Lemma 1 and Theorem 2. This is, however, not a generalization of the previous result but merely a correction of a mistake. The energy estimates procedure provided in [1] is formulated in terms of $\boldsymbol{v}^N$ instead of $\boldsymbol{u}^N$, which is (in the context of applying Brouwer's fixed point theorem) not correct. The author apologizes for this inconvenience.

Note that the constraint $r > 2 - \frac{1}{d}$ does not allow to extend the result for inner flows in case of two dimensions, because $2 - \frac{1}{d} = \frac{3}{2} = \frac{3d}{d+2}$. In three dimensions, while the "homogeneous Dirichlet" Theorem 2.1 in [2] holds for $r$ down to $\frac{2d}{d+2} = \frac{6}{5}$, the "small data" Theorem 2 requires $\frac{2d}{d+1} = \frac{3}{2} < r$ and the "inner flows" Corollary 3 assumes $2 - \frac{1}{d} = \frac{5}{3} < r$.

## References


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