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On the Independence of Axioms in BL and MTL

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Abstract

We show by standard automated theorem proving methods and freely available automated theorem prover software that axiom (A2), stating that multiplicative conjunction implies its first member, is provable from other axioms in fuzzy logics BL and MTL without using axiom (A3), which is known to be provable from other axioms [1]. We also use freely available automated model generation software to show that all other axioms in BL and MTL are independent.

1. Introduction

Among propositional fuzzy logics Hájek's basic logic BL [3] and Esteva and Godo's monoidal t-norm based logic MTL [2] play prominent roles. BL, which was introduced as a common fragment of Łukasiewicz, Gödel and product logics, is the logic of continuous t-norms[1] and their residua[2]. However, in [2] was shown that the minimal condition for a t-norm to have a residuum is left-continuity and authors proposed logic MTL, which was later proved to be the logic of left-continuous t-norms and their residua.

Standard Hilbert style calculus for BL comes from Hájek. Esteva and Godo slightly addapted this system for MTL by replacing one axiom by three other axioms. Generally, both systems are almost identical. In a short note by Cintula [1], it was shown that axiom (A3), stating commutativity of multiplicative conjunction, is provable from other axioms and thus redundant. Lehmke proved that also axiom (A2), stating that multiplicative conjunction implies its first member, is provable from other axioms by using his own Hilbert style proof generation software [4]. However, the proof used axiom (A3) and thus was not a proof of independence of both axioms (A2) and (A3).

We use a well known technique of automated theorem proving to encode the Hilbert style calculus of a fuzzy propositional logic into classical first order logic, and standard automated theorem proving software to prove axiom (A2), without using axiom (A3), in BL and MTL. Moreover, by an easy application of similar technique and standard automated model generation software we show that none of the other axioms is redundant in BL and MTL, independently of presence of axioms (A2) and (A3).

The interest of this paper is solely in above stated properties of Hilbert style calculus of BL and MTL. The technique used to obtain them can be in our case used completely naive.

The paper is organised as follows. In Section 2 we set up notation and terminology. In Section 3 we give a brief exposition of techniques used to obtain presented results. Section 4.1 contains the proof of derivability of axiom (A2) for MTL and Section 4.2 for BL. In Section 5 the semantic proofs of independence of other axioms are presented.

2. Preliminaries

We will touch only a few aspects of the theory. For simplicity of notation, we use fuzzy logic for fuzzy propositional logic and first order logic (FOL) for classical first order logic. First order fuzzy logics and classical propositional logic are not discussed in this paper.

We define standard Hilbert style calculus for the *Basic*

[1] A t-norm is a binary function $\star$ on linearly ordered real interval $[0, 1]$ which satisfies commutativity, monotonicity, associativity and 1 acts as identity element.

[2] The operation $x \Rightarrow y$ is the residuum of the t-norm $\star$ if $x \Rightarrow y = \max\{z \mid x \star z \leq y\}$.

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Logic (BL) and the *Monoidal T-norm based Logic* (MTL), which consist of axioms and modus ponens as the only deduction rule. The language of BL and MTL consists of implication ($\rightarrow$), multiplicative ($\&$) and additive ($\wedge$) conjunctions and a constant for falsity ($\overline{0}$).

Definition 2.1 *We define the basic logic BL as a Hilbert style calculus with following formulae as axioms*

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$,

(A2) $(\varphi \,\&\, \psi) \rightarrow \varphi$,

(A3) $(\varphi \,\&\, \psi) \rightarrow (\psi \,\&\, \varphi)$,

(A4) $(\varphi \,\&\, (\varphi \rightarrow \psi)) \rightarrow (\psi \,\&\, (\psi \rightarrow \varphi))$,

(A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \,\&\, \psi) \rightarrow \chi)$,

(A5b) $((\varphi \,\&\, \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$,

(A6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$,

(A7) $\overline{0} \rightarrow \varphi$.

The only deduction rule of BL is modus ponens

(MP) If φ is derivable and $\varphi \rightarrow \psi$ is derivable then ψ is derivable.

Let us note properties stated by each axiom, following [3]. Axiom (A1) is transitivity of implication. Axiom (A2) states that multiplicative conjunction implies its first member. Axiom (A3) is commutativity of multiplicative conjunction. In BL, additive conjunction $\varphi \wedge \psi$ is definable as $\varphi \,\&\, (\varphi \rightarrow \psi)$. The equivalence of these two formulae is the divisibility axiom. Axiom (A4) is commutativity of additive conjunction. Axioms (A5a) and (A5b) represent residuation. Axiom (A6) is a variant of proof by cases, and states that if both $\varphi \rightarrow \psi$ and $\psi \rightarrow \varphi$ implies χ, then χ. Axiom (A7) states that false implies everything.

Definition 2.2 *Hilbert style calculus BL$^-$ is obtained by dropping axioms* (A2) *and* (A3) *from BL.*

We obtain a Hilbert style calculus of the monoidal t-norm based logic MTL by weakening properties on additive conjunction. In BL, we define $\varphi \wedge \psi$ as an abbreviation for $\varphi\&(\varphi\rightarrow\psi)$. In MTL, we define additive conjunction directly by three new axioms which state that additive conjunction is commutative, implies its first member and one implication of divisibility property.

Definition 2.3 *We obtain the monoidal t-norm based logic MTL by replacing axiom* (A4) *in BL by following three axioms*

(A4a) $(\varphi \,\&\, (\varphi \rightarrow \psi)) \rightarrow (\varphi \wedge \psi)$,

(A4b) $(\varphi \wedge \psi) \rightarrow \varphi$,

(A4c) $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$.

Definition 2.4 *Hilbert style calculus MTL$^-$ is obtained by dropping axioms* (A2) *and* (A3) *from MTL.*

2.1. First order logic and automated theorem proving

A FOL *model* is a pair $\langle D, I \rangle$, where domain D is a set of elements and I is an interpretation of symbols of a language.

In FOL, *terms* are defined inductively as the smallest set of all variables and constants closed under function symbols in given language. We will have only one predicate symbol Pr and thus all our *atomic formulae* have a form $Pr(t)$, where t is a term. A *literal* l is an atomic formula (*positive literal*) or a negative atomic formula (*negative literal*). A *clause* C is a finite disjunction of literals. Specifically, a *Horn clause* is a clause with at most one positive literal. All clauses will be for our purposes implicitly universally quantified. *Unification of literals* l *and* l' is a substitution σ which gives $l\sigma = l'\sigma$. So called *most general unifier of* l *and* l', denoted $\mathrm{mgu}(l, l')$, is a unification σ such that for every unification θ of l and l' exists a unification η satisfying $\theta = (\sigma)\eta$.

The standard FOL automated theorem proving strategy is resolution [5]. We can transform a problem of $\Gamma \vdash \varphi$ to the problem of deciding whether set $\{\Gamma, \neg\varphi\}$ is contradictionary. Let $\sigma = \mathrm{mgu}(l, l')$, then resolution calculus with (binary) resolution rule

$$\frac{C \vee l \qquad D \vee \neg l'}{(C \vee D)\sigma}$$

and factoring rule

$$\frac{C \vee l \vee l'}{(C \vee l)\sigma}$$

is refutational complete [5], which means that for every contradictionary set eventually find a derivation of empty clause which represents a contradiction.

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3. Usage of ATP methods

There is a well known technique for encoding propositional Hilbert style calculus into classical FOL through terms. The key idea is that formula variables in axioms and rules are encoded as universally quantified first order variables and propositional connectives as first order function symbols. Moreover, we use one unary predicate which says which terms are provable (encoding of axioms) and how another provable term can be obtained from provable terms (encoding of rules). It is evident that our axioms and modus ponens rule can be encoded easily. However, for more complicated axioms and rules problems may arise.

For simplicity of notation, we write Fle_L instead of the set of all formulae in language L.

Definition 3.1 *Let L be BL or MTL or their fragment. We define term encoding $f : Fle_L \to Fle_{FOL}$.*

First, a function $f' : Fle_L \to Fle_{FOL}$ is defined recursively as follows

$$f'(\varphi) = \begin{cases} 0_f & \varphi \text{ is } \overline{0}, \\ f'(\psi) \to_f f'(\chi) & \varphi \text{ is } \psi \to \chi, \\ f'(\psi) \mathbin{\&}_f f'(\chi) & \varphi \text{ is } \psi \mathbin{\&} \chi, \\ f'(\psi) \wedge_f f'(\chi) & \varphi \text{ is } \psi \wedge \chi, \\ X_\psi & \varphi \text{ is a formula variable } \psi, \end{cases}$$

where $\&_f$, $\wedge_f$ and $\to_f$ are new binary function symbols, written for better readability in infix notation, 0_f is a new FOL constant and X_ψ is a new FOL variable for every formula variable ψ, but the same for every occurrence of ψ in the encoded formula.

Second, formula $f(\varphi)$ is the universal closure of formula $Pr(f'(\varphi))$, where Pr is a common new unary predicate saying which terms are provable.

Finally, let $\varphi_1, \ldots, \varphi_n \vdash \psi$ be a propositional rule (in our case just (MP)*), we define term encoding f into classical FOL as the universal closure of formula $(f'(\varphi_1) \wedge \ldots \wedge f'(\varphi_n)) \Rightarrow f'(\psi)$, where $\wedge$ and $\Rightarrow$ are standard logical connectives for conjunction and implication in classical FOL and function f' is defined as above.*

Example Let us have a system with axioms (A2), (A3) and the only rule (MP). This propositional system will be formalised, for better readability with X and Y instead of X_φ and X_ψ, in FOL as follows

(A1$_f$) $(\forall X, Y) Pr((X \mathbin{\&}_f Y) \to_f X)$,

(A2$_f$) $(\forall X, Y) Pr((X \mathbin{\&}_f Y) \to_f (Y \mathbin{\&}_f X))$,

(MP$_f$) $(\forall X, Y)(Pr(X) \wedge Pr(X \to_f Y) \Rightarrow Pr(Y))$.

Before stating a crucial lemma we make some remarks. For a set of formulae Γ, we define $f(\Gamma)$ as a set of all f-translated formulae from Γ. We write $f(MP)$ for the term encoding f of modus ponens rule.

By an easy observation we realize that all translated axioms and modus ponens translation, written in form of disjunction, are Horn clauses.

Lemma 3.2 *Let L be BL or MTL or their fragment with the set of axioms Δ, Γ arbitrary set of formulae, and φ arbitrary formula, both in language of L. Then $\Gamma \vdash_L \varphi$, if and only if $f(\Delta), f(\Gamma), f(MP) \vdash_{FOL} f(\varphi)$.*

Proof: A Hilbert style proof of φ from Γ can be easily translated into a Hilbert style proof of $f(\varphi)$ from $f(\Delta), f(\Gamma)$ and $f(MP)$ in classical FOL using generalisation rule, if $\vdash_{FOL} \psi$ then $\vdash_{FOL} \forall x\psi$, and $\vdash_{FOL} \forall x\psi \to \psi$.

The opposite direction can be shown by using a resolution refutation. It is an easy observation that only Horn clauses occur in such a resolution refutation. And this fragment has a property that given resolution refutation can be reordered in such a way that a backward translation gives a proof of φ in Γ. ∎

Demonstrating the independence of some axiom, we are also interested in unprovability. There is a standard model theoretical technique for proving that some formula φ is unprovable from a set of formulae Γ. From soundness theorem in FOL it is enough to show a FOL model in which all formulae from Γ are true and formula φ is false. By previous lemma we can easily transform a problem of unprovability φ from Γ in a Hilbert style calculus to a problem of finding classical FOL model in which $f(\Delta), f(\Gamma), f(MP)$ and $\neg f(\varphi)$ are true.

We have thus transformed the problem of provability of formula in propositional fuzzy logic Hilbert style calculus into FOL and we can try to solve it by standard automated theorem proving software. We can use a theorem prover for showing that some formula (in an encoded form) is provable from other formulae using given rules, or a model generator software to find a model which demonstrates its unprovability. Traditionally, both computations are executed in parallel.

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Generally speaking, because of undecidability of FOL, this technique cannot be fully satisfiable. Moreover, abilities of automated theorem provers and automated model generators are very limited and highly dependent on software configuration. However, several results were obtained by this or similar techniques, which proved its usability, see for instance Wos's papers [6].

We are not going to describe technique used by automated theorem provers and model generators, because these systems are rather complicated. For our experiments we used freely available E prover in version 0.999-001[3], which is based on superposition (restricted paramodulation) calculus. For building models we used freely available Paradox 2.3[4] finite model finder which iteratively tries to find finite models by transforming a given problem into SAT problems.

Tuning software for obtaining results can be highly complicated. Nevertheless, for all our results standard configuration is sufficient as well as almost any state of the art prover or model generator. However, the presented form of results was obtained by experimenting with software configuration and some configurations are better suited for direct extraction of proofs in Hilbert style calculus.

4. Provability of axiom (A2)

We present a proof of axiom (A2) separetely for MTL$^-$ and BL$^-$. Both proofs are obtained by proving weakening formula $\varphi \to (\psi \to \varphi)$ which immediately gives a proof of axiom (A2). We note that the original prover proofs were slightly adapted.

4.1. MTL$^-$

First, we present proof for MTL$^-$which is shorter. It may look surprising, because MTL$^-$ is weaker than BL$^-$. However, for the proof of axiom (A2), axioms (A4a)–(A4c) are evidently better suited than axiom (A4).

Lemma 4.1 *The following formulae are provable in MTL$^-$:*

(a) $(\varphi \,\&\, (\varphi \to \psi)) \to \varphi$,

(b) $(((\varphi \to \psi) \to \varphi) \to \chi) \to (\varphi \to \chi)$,

(c) $\varphi \to (\varphi \to \varphi)$,

(d) $\varphi \to (\psi \to \varphi)$.

Proof:

a)

1:	$((\varphi \,\&\, (\varphi \to \psi)) \to (\varphi \wedge \psi)) \to (((\varphi \wedge \psi) \to \chi) \to ((\varphi \,\&\, (\varphi \to \psi)) \to \chi))$	(A1)
2:	$((\varphi \wedge \psi) \to \chi) \to ((\varphi \,\&\, (\varphi \to \psi)) \to \chi)$	by (A4a), 1
3:	$(\varphi \,\&\, (\varphi \to \psi)) \to \varphi$	by (A4b), 2

b)

4:	$\varphi \to ((\varphi \to \psi) \to \varphi)$	by (a), (A5b)
5:	$(((\varphi \to \psi) \to \varphi) \to \chi) \to (\varphi \to \chi)$	by 4, (A1)

c)

6:	$((\varphi \to ((\varphi \to \psi) \to \varphi)) \to \varphi) \to (\varphi \to ((\varphi \to \psi) \to \varphi))$	by 4, 4
7:	$\varphi \to (\varphi \to ((\varphi \to \psi) \to \varphi))$	by 6, (b)
8:	$((\varphi \to ((\varphi \to \psi) \to \varphi)) \to \chi) \to (\varphi \to \chi)$	by (7), (A1)
9:	$\varphi \to (\varphi \to \varphi)$	by 8, (b)

[3] `http://www.eprover.org`
[4] `http://www.cs.chalmers.se/~koen/folkung/`

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d)

10:	$((\varphi \to \varphi) \to \psi) \to (\varphi \to \psi)$	by (c), (A1)
11:	$((\varphi \to (\psi \to (\varphi \,\&\, \psi))) \to \varphi) \to (((\varphi \,\&\, \psi) \to (\varphi \,\&\, \psi)) \to \varphi)$	by (A5b), (A1)
12:	$\varphi \to (((\varphi \,\&\, \psi) \to (\varphi \,\&\, \psi)) \to \varphi)$	by 11, (b)
13:	$(\varphi \to (\varphi \to \varphi)) \to ((((\varphi \to (\varphi \to \varphi)) \,\&\, \psi) \to ((\varphi \to (\varphi \to \varphi)) \,\&\, \psi)) \to (\varphi \to (\varphi \to \varphi)))$	12
14:	$(((\varphi \to (\varphi \to \varphi)) \,\&\, \psi) \to ((\varphi \to (\varphi \to \varphi)) \,\&\, \psi)) \to (\varphi \to (\varphi \to \varphi))$	by (c),13
15:	$((\varphi \to (\varphi \to \varphi)) \,\&\, \psi) \to (\varphi \to (\varphi \to \varphi))$	by 14, 10
16:	$(\varphi \to (\varphi \to \varphi)) \to (\psi \to (\varphi \to (\varphi \to \varphi)))$	by 15, (A5b)
17:	$\psi \to (\varphi \to (\varphi \to \varphi))$	by (c), 16
18:	$((\varphi \to (\varphi \to \varphi)) \to \varphi) \to (\psi \to \varphi)$	by 17, (A1)
19:	$\varphi \to (\psi \to \varphi)$	by 18, (b)

■

Now by application of (A5a) we immediately obtain

Corollary 4.2 *Axiom* (A2) *is derivable in MTL$^-$.*

Let us note that we do not use axioms (A4c), (A6) and (A7). On the contrary, all other axioms are necessary, which can be demonstrated by Section 5 methods.

Corollary 4.3 (see Cintula [1]) *Axiom* (A3) *is derivable in MTL$^-$.*

It is worth pointing out that axiom (A3) can be proved by similar technique used to prove axiom (A2).

4.2. BL$^-$

We are going to prove a similar lemma for logic BL$^-$. Let us note that we will use axiom (A6) and axiom (A7), which are not necessary, but shorten the proof, whereas all other axioms are necessary.

Lemma 4.4 *The following formulae are provable in BL$^-$:*

(a) $\varphi \to \varphi$,

(b) $(\varphi \,\&\, (\varphi \to \overline{0})) \to \psi$,

(c) $(\varphi \,\&\, \psi) \to \psi$,

(d) $\varphi \to (\psi \to \varphi)$.

Proof:

a)

1:	$((\varphi \to \varphi) \to (\varphi \to \varphi)) \to (((\varphi \to \varphi) \,\&\, \varphi) \to \varphi)$	(A5a)
2:	$(((\varphi \to \varphi) \to (\varphi \to \varphi)) \to (((\varphi \to \varphi) \,\&\, \varphi) \to \varphi)) \to ((((\varphi \to \varphi) \to (\varphi \to \varphi)) \to (((\varphi \to \varphi) \,\&\, \varphi) \to \varphi)) \to (((\varphi \to \varphi) \,\&\, \varphi) \to \varphi))$	(A6)
3:	$(((\varphi \to \varphi) \to (\varphi \to \varphi)) \to (((\varphi \to \varphi) \,\&\, \varphi) \to \varphi)) \to (((\varphi \to \varphi) \,\&\, \varphi) \to \varphi)$	by 1, 2
4:	$((\varphi \to \varphi) \,\&\, \varphi) \to \varphi$	by 1, 3
5:	$(((\varphi \to \varphi) \,\&\, \varphi) \to \varphi) \to ((\varphi \to \varphi) \to (\varphi \to \varphi))$	by (A5b)
6:	$(\varphi \to \varphi) \to (\varphi \to \varphi)$	by 4, 5

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7:	$((\varphi \to \varphi) \to (\varphi \to \varphi)) \to (((\varphi \to \varphi) \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$	(A6)
8:	$((\varphi \to \varphi) \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$	by 6, 7
9:	$\varphi \to \varphi$	by 6, 8

b)

10:	$((\varphi \to \psi) \to (\varphi \to \psi)) \to (((\varphi \to \psi) \,\&\, \varphi) \to \psi)$	(A5a)
11:	$((\varphi \to \psi) \,\&\, \varphi) \to \psi$	by (a), 10
12:	$(((\varphi \to \psi) \,\&\, \varphi) \to \psi) \to ((\psi \to \chi) \to (((\varphi \to \psi) \,\&\, \varphi) \to \chi))$	(A1)
13:	$(\psi \to \chi) \to (((\varphi \to \psi) \,\&\, \varphi) \to \chi)$	by 11, 12
14:	$(\overline{0} \to (\psi \to \varphi)) \to ((\overline{0} \,\&\, \psi) \to \varphi)$	(A5a)
15:	$(\overline{0} \,\&\, \psi) \to \varphi$	by (A7), 14
16:	$((\overline{0} \,\&\, \psi) \to \varphi) \to (((\chi \to (\overline{0} \,\&\, \psi)) \,\&\, \chi) \to \varphi)$	13
17:	$((\chi \to (\overline{0} \,\&\, \psi)) \,\&\, \chi) \to \varphi$	by 15, 16
18:	$(((\varphi \to (\overline{0} \,\&\, \psi)) \,\&\, \varphi) \to \chi) \to ((\varphi \to (\overline{0} \,\&\, \psi)) \to (\varphi \to \chi))$	(A5b)
19:	$(\varphi \to (\overline{0} \,\&\, \psi)) \to (\varphi \to \chi)$	by 17, 18
20:	$(\varphi \,\&\, (\varphi \to \overline{0})) \to (\overline{0} \,\&\, (\overline{0} \to \varphi))$	(A4)
21:	$((\varphi \,\&\, (\varphi \to \overline{0})) \to (\overline{0} \,\&\, (\overline{0} \to \varphi))) \to ((\varphi \,\&\, (\varphi \to \overline{0})) \to \psi)$	19
22:	$(\varphi \,\&\, (\varphi \to \overline{0}) \to \psi$	by 20, 21

c)

23:	$\varphi \to ((\varphi \to \overline{0}) \to \psi)$	by (b), (A5b)
24:	$(\varphi \to ((\varphi \to \overline{0}) \to \psi)) \to ((((\varphi \to \overline{0}) \to \psi) \to \chi) \to (\varphi \to \chi))$	(A1)
25:	$(((\varphi \to \overline{0}) \to \psi) \to \chi) \to (\varphi \to \chi)$	by 23, 24
26:	$(\varphi \to \psi) \to (\overline{0} \to \psi)$	by (A7), (A1)
27:	$(((\varphi \to \overline{0}) \to \psi) \to (\overline{0} \to \psi)) \to (\varphi \to (\overline{0} \to \psi))$	25
28:	$\varphi \to (\overline{0} \to \psi)$	by 26, 27
29:	$\varphi \to (\psi \to (\varphi \,\&\, \psi))$	by (a), (A5b)
30:	$(\varphi \to \varphi) \to (\psi \to ((\varphi \to \varphi) \,\&\, \psi))$	29
31:	$\psi \to ((\varphi \to \varphi) \,\&\, \psi)$	by (a), 30
32:	$(\varphi \to (\overline{0} \to \psi)) \to ((\chi \to \chi) \,\&\, (\varphi \to (\overline{0} \to \psi)))$	31
33:	$(\chi \to \chi) \,\&\, (\varphi \to (\overline{0} \to \psi))$	by 28, 32
34:	$((\varphi \to \varphi) \,\&\, ((\varphi \to \varphi) \to (\overline{0} \to \psi))) \to ((\overline{0} \to \psi) \,\&\, ((\overline{0} \to \psi) \to (\varphi \to \varphi)))$	(A4)
35:	$(\overline{0} \to \psi) \,\&\, ((\overline{0} \to \psi) \to (\varphi \to \varphi))$	by 33, 34

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36: $((\varphi \,\&\, \psi) \to (\overline{0} \to \chi)) \to (\varphi \to (\psi \to (\overline{0} \to \chi)))$ (A5b)

37: $\varphi \to (\psi \to (\overline{0} \to \chi))$ by 28, 36

38: $(\varphi \to (\psi \to (\overline{0} \to \chi))) \to (((\psi \to (\overline{0} \to \chi)) \to \xi) \to (\varphi \to \xi))$ (A1)

39: $((\psi \to (\overline{0} \to \chi)) \to \xi) \to (\varphi \to \xi)$ by 37, 38

40: $(\varphi \to (\overline{0} \to \psi)) \to (((\overline{0} \to \psi) \to \chi) \to (\varphi \to \chi))$ (A1)

41: $((\varphi \to (\overline{0} \to \psi)) \to (((\overline{0} \to \psi) \to \chi) \to (\varphi \to \chi))) \to (\xi \to (((\overline{0} \to \psi) \to \chi) \to (\varphi \to \chi)))$ 39

42: $\xi \to (((\overline{0} \to \psi) \to \chi) \to (\varphi \to \chi))$ by 40, 41

43: $(\varphi \to (((\overline{0} \to \psi) \to \chi) \to (\xi \to \chi))) \to ((\varphi \,\&\, ((\overline{0} \to \psi) \to \chi)) \to (\xi \to \chi))$ (A5a)

44: $(\varphi \,\&\, ((\overline{0} \to \psi) \to \chi)) \to (\xi \to \chi)$ by 42, 43

45: $((\overline{0} \to \varphi) \,\&\, ((\overline{0} \to \varphi) \to (\psi \to \psi))) \to (\varphi \to (\psi \to \psi))$ 44

46: $\varphi \to (\psi \to \psi)$ by 35, 45

47: $(\varphi \,\&\, \psi) \to \psi$ by 46, (A5a)

d)

48: $((\psi \,\&\, \chi) \to \chi) \to (((\varphi \to (\psi \,\&\, \chi)) \,\&\, \varphi) \to \chi)$ 13

49: $((\varphi \to (\psi \,\&\, \chi)) \,\&\, \varphi) \to \chi$ by (c), 48

50: $(\varphi \to (\psi \,\&\, \chi)) \to (\varphi \to \chi)$ by 49, (A5b)

51: $((\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi))) \to ((\varphi \,\&\, (\varphi \to \psi)) \to (\psi \to \varphi))$ 50

52: $(\varphi \,\&\, (\varphi \to \psi)) \to (\psi \to \varphi)$ by (A4), 51

53: $\varphi \to ((\varphi \to \psi) \to (\psi \to \varphi))$ by 52, (A5b)

54: $(((\varphi \to \psi) \to (\psi \to \varphi)) \to \chi) \to (\varphi \to \chi)$ by 53, (A1)

55: $((\psi \to \varphi) \to (\psi \to \varphi)) \to (((\varphi \to \psi) \to (\psi \to \varphi)) \to (\psi \to \varphi))$ (A6)

56: $((\varphi \to \psi) \to (\psi \to \varphi)) \to (\psi \to \varphi)$ by (a), 55

57: $\varphi \to (\psi \to \varphi)$ by 56, 54

■

Now again by application of (A5a) we immediately obtain

Corollary 4.5 *Axiom* (A2) *is derivable in BL$^-$.*

Corollary 4.6 (see Cintula [1]) *Axiom* (A3) *is derivable in BL$^-$.*

It is worth pointing out that axiom (A3) can be again proved by similar technique used to prove axiom (A2).

5. The independence of axioms

We know that axioms (A2) and (A3) are redundant in BL and MTL. Is any other axiom redundant in BL or MTL? We answer this question negatively for every remaining axiom by presenting a model and a valuation which make the axiom false, but all other axioms including (A2) and (A3) and modus ponens rule are true in the model. It means that none of the axioms but (A2) and (A3) is redundant in original systems BL and MTL. We obtain immediately that all axioms in BL$^-$ and MTL$^-$ are independent.

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All models are finitely valued structures with elements labeled by natural numbers, presented in form of truth tables. Let us note that in all models except for (A7) we interpret constant $\overline{0}$ as the minimal element 0 and truth as the maximal value in a model, e.g. in a four member model it has value 3.

The important point to note is that checking falsity of axiom in a given model under a given valuation is an easy task. On the other hand, to show that all other axioms are true in the model, exhausting checking is sometimes needed. Fortunately, for computer it is an easy task. We naturally do not present these proofs.

For shortening the presentation we present models for BL and MTL at once. Only models for logic specific axioms (A4) and (A4a)–(A4c) are presented separately. Moreover, we prefer the same definition for multiplicative and additive conjunction.

We start by a group of axioms common to BL and MTL.

5.1. Axiom (A1)

For showing the independence of axiom (A1) we need a model in which implication is not transitive. We present such a model which falsifies axiom (A1) for valuation $\varphi = 1, \psi = 0$ and $\chi = 2$.

$\&, \wedge$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	1	0	3

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	3	3	1	3
2	3	3	3	3
3	1	1	1	3

Table 1: Truth tables for (A1)

5.2. Axiom (A5a)

First of the residuation axioms (A5a) fails evidently for $\varphi = 2, \psi = 1$ and $\chi = 0$. Both conjunctions are defined separately.

$\&$	0	1	2	3
0	0	0	0	0
1	0	0	2	2
2	0	2	0	2
3	0	2	2	3

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	3

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	1	3	3	3
2	2	3	3	3
3	0	2	1	3

Table 2: Truth tables for (A5a)

5.3. Axiom (A5b)

To demonstrate the independence of axiom (A5b), much easier model than for axiom (A5a) is needed. A two valued model with classical implication and both conjunctions false for all values is sufficient. Axiom (A5b) fails for $\varphi = 1, \psi = 1$ and $\chi = 0$.

$\&, \wedge$	0	1
0	0	0
1	0	0

$\rightarrow$	0	1
0	1	1
1	0	1

Table 3: Truth tables for (A5b)

5.4. Axiom (A6)

The independence of axiom (A6) can be easily shown by an algebraic arguments. It represents prelinearity and logics without prelinearity have been already studied. Moreover, MTL without axiom (A6) represents Höhle Monoidal Logic ML. Nevertheless, we present our standard semantic argument. Axiom (A6) fails for φ, ψ and χ represented by $1, 2$ and 3.

$\&, \wedge$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	1	1
2	0	0	2	2	2
3	0	1	2	3	3
4	0	1	2	3	4

$\rightarrow$	0	1	2	3	4
0	4	4	4	4	4
1	2	4	2	4	4
2	1	1	4	4	4
3	0	1	2	4	4
4	0	1	2	3	4

Table 4: Truth tables for (A6)

5.5. Axiom (A7)

It is evident that axiom (A7) is independent of other axioms, because of new symbol $\overline{0}$. For demonstration it is enough to interpret $\overline{0}$ as truth and all connectives classically. In such model, axiom (A7) easily fails and all other axioms are evidently true.

$\&, \wedge$	0	1
0	0	0
1	0	1

$\rightarrow$	0	1
0	1	1
1	0	1

Table 5: Truth tables for (A7)

Now we present BL and MTL specific cases.

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5.6. Axiom (A4)

If we take $\varphi \wedge \psi$ as an abbreviation for $\varphi \,\&\, (\varphi \rightarrow \psi)$, axiom (A4) represents commutativity of additive conjunction in BL. For $\varphi = 1$ and $\psi = 2$, additive conjunction is not commutative.

&	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	0	2
3	0	1	2	3

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	2	3	3	3
2	2	2	3	3
3	0	1	2	3

Table 6: Truth tables for (A4)

We show the independence of axioms (A4a)–(A4c) by small models, in which axioms (A1)–(A3) and (A5a)–(A7) are evidently true, because of & and $\rightarrow$ definition. Therefore to complete the proof it is sufficient to show the (in)validity of axioms (A4a)–(A4c) in the corresponding truth tables only.

5.7. Axiom (A4a)

Axiom (A4a) fails for $\varphi = 1$ and $\psi = 1$, but axioms (A4b) and (A4c) are evidently true.

&	0	1
0	0	0
1	0	1

$\wedge$	0	1
0	0	0
1	0	0

$\rightarrow$	0	1
0	1	1
1	0	1

Table 7: Truth tables for (A4a)

5.8. Axiom (A4b)

Axiom (A4b) fails for $\varphi = 0$ and $\psi = 1$, but axioms (A4a) and (A4c) are evidently true.

&	0	1
0	0	0
1	0	1

$\wedge$	0	1
0	0	1
1	1	1

$\rightarrow$	0	1
0	1	1
1	0	1

Table 8: Truth tables for (A4b)

5.9. Axiom (A4c)

Axiom (A4c) fails for $\varphi = 1$ and $\psi = 0$, but axioms (A4a) and (A4b) are evidently true.

&	0	1
0	0	0
1	0	1

$\wedge$	0	1
0	0	0
1	1	1

$\rightarrow$	0	1
0	1	1
1	0	1

Table 9: Truth tables for (A4c)

Corollary 5.1 *All axioms but* (A2) *and* (A3) *are independent of each other in BL.*

Corollary 5.2 *All axioms but* (A2) *and* (A3) *are independent of each other in MTL.*

It is worth pointing out that the independence of axioms could be presented also by studying some known algebraic structures, which has several indisputable theoretical advantages. On the other hand, our approach seems to be easier for presentation.

6. Summary and conclusion

We presented the complete solution of dependence and independence of axioms in prominent fuzzy propositional logics BL and MTL by using simple technique from automated theorem proving. Also other similar problems can be solved using these methods and state of the art theorem provers and model generators.

Nevertheless, our approach has several drawbacks. First, abilities of current theorem provers are limited and in some situations even short proofs are inaccessible for them without special settings. Second, abilities of automated model generators are also very limited, e.g. infinite models are highly problematic.

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