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Abstract:

This report contains a description of subroutines which can be used for testing large-scale optimization codes. These subroutines can easily be obtained from the web page `http://www.cs.cas.cz/~luksan/test.html`. Furthermore, all test problems contained in these subroutines are presented in the analytic form.

Keywords:
large-scale optimization, least squares, test problems

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1 Introduction

This report describes subroutines TIUB26, TIUD26, TAFU26, TAGU26 which contain 60 sparse sum of squares problems for testing unconstrained optimization codes. We want to find a minimum of the objective function

$$F(x) = \frac{1}{2} \sum_{k=1}^{n_A} f_k^2(x), \quad x \in R^n,$$

where $f_k(x)$, $1 \le k \le n_A$, are twice continuously differentiable partial functions. All subroutines are written in the standard Fortran 77 language. Their names are derived from the following rule:

- The first letter is T - test subroutines.
- The second letter is either I - initiation, or F - objective function, or A - partial function, or C - constraint function.
- The third letter is either U - initiation of unconstrained problem, or F - computation of the function value, or G - computation of the function gradient.
- The fourth letter is either U - universal subroutine, or B - subroutine for partially separable problems, or D - subroutine for general dense problems, or S - subroutine for general sparse problems.

The last two digits determine a given collection (numbering corresponds to the UFO system [19], which contains similar collections).

Initiation subroutines use the following parameters (array dimensions are given in parentheses):

N	input	number of variables
NA	output	number of partial functions (or equations)
MA	output	number of nonzero elements in the sparse Jacobian matrix
IA(NA+1)	output	pointers of the Jacobian matrix nonzero elements beginning of each row
JA(MA)	output	column indices of the Jacobian matrix nonzero elements of each row
X(N)	output	vector of variables
FMIN	output	lower bound of the objective function value
XMAX	output	maximum stepsize
NEXT	input	number of the problem selected
IERR	output	error indicator (0 - correct data, 1 - N is too small)

Although N is an input parameter, it can be changed by the initiation subroutine when its value does not satisfy the required conditions. For example, most of the problems require N to be even or a multiple of a positive integer. If the Jacobian matrix has the following structure

$$J_A = \begin{pmatrix} *, & 0, & 0, & * \\ 0, & *, & 0, & * \\ 0, & 0, & *, & 0 \\ *, & *, & *, & 0 \\ 0, & 0, & *, & * \end{pmatrix},$$

then the subroutine TIUB26 returns

```
NA=5, MA=10,
IA(1)=1, IA(2)=3, IA(3)=5, IA(4)=6, IA(5)=9, IA(6)=11,
JA(1)=1, JA(2)=4, JA(3)=2, JA(4)=4, JA(5)=3, JA(6)=1, JA(7)=2,
JA(8)=3, JA(9)=3, JA(10)=4.
```

Evaluation subroutines use the following parameters (array dimensions are given in parentheses):

N	input	number of variables
X(N)	input	vector of variables
KA	input	index of the partial function (or equation) selected
FA	output	value of the partial function (or equation) selected
GA(N)	output	gradient of the partial function (or equation) selected
NEXT	input	number of the problem selected

2 Test problems for large-scale nonlinear least squares

Calling statements have the form

```
CALL TIUB26(N,NA,MA,X,IA,JA,FMIN,XMAX,NEXT,IERR)
CALL TIUD26(N,NA,X,FMIN,XMAX,NEXT,IERR)
CALL TAFU26(N,KA,X,FA,NEXT)
CALL TAGU26(N,KA,X,GA,NEXT)
```

with the following significance

- TIUB26 – initiation of vector of variables N, X and definition of the Jacobian matrix structure NA, MA, IA, JA
- TIUD26 – initiation of vector of variables N, X
- TAFU15 – evaluation of the KA-th partial function value FA at point X
- TAGU15 – evaluation of the KA-th partial function gradient GA at point X

The following table gives typical sizes of problems in case we choose N=1000 (M is the number of nonzero elements in the Hessian matrix $\nabla^2 F$).

NEXT	N	NA	MA	M	NEXT	N	NA	MA	M
1	1000	1998	2997	1999	31	1000	1000	2998	2997
2	1000	2994	4990	1999	32	1000	1000	2000	1500
3	1000	1996	3992	2498	33	1000	1000	2500	2250
4	1000	2495	3992	1999	34	1000	1000	2998	2997
5	1000	1000	2998	2997	35	1000	1000	2998	2997
6	1000	1000	6984	6979	36	1000	1000	2998	2997
7	1000	1998	3996	1999	37	1000	1000	2998	2997
8	1000	5000	10000	1500	38	1000	1000	4994	4990
9	1000	2994	11976	3496	39	1000	1000	6988	7475
10	1000	1999	4996	2997	40	961	961	4681	6419
11	1000	1998	2997	1999	41	961	961	4681	6419
12	998	1992	3652	2991	42	961	961	4681	6419
13	998	2324	5312	2991	43	961	961	4681	6419
14	1000	1996	7984	3496	44	961	961	4681	6419
15	1000	1996	7984	3496	45	961	961	11877	18351
16	1000	1996	7984	3496	46	961	961	11877	18351
17	1000	1000	3996	4492	47	1000	1000	2998	2997
18	1000	1000	2998	2997	48	1000	1000	2998	2997
19	1000	1000	7984	7972	49	1000	1000	1999	1999
20	1000	1000	2998	2997	50	1000	1000	2000	2000
21	998	2324	4316	2659	51	1000	1000	2998	2997
22	1000	1998	4993	2997	52	1000	1000	2998	2997
23	1000	1000	4993	5985	53	1000	1000	6988	6979
24	1000	1000	5000	3000	54	1000	1000	1999	1999
25	1000	1000	2998	2997	55	1000	1000	1999	1999
26	999	999	3993	4488	56	1000	1000	2998	2997
27	1000	1000	2998	2997	57	1000	1000	6984	6979
28	1000	1000	4994	4990	58	1000	1000	2998	2997
29	1000	1000	6988	6979	59	1000	1000	2998	2997
30	1000	1000	2000	1500	60	1000	1000	3996	4492

We seek a local minimum of the function

$$F(x) = \frac{1}{2} \sum_{k=1}^{n_A} f_k^2(x), \quad x \in R^n,$$

from the starting point $\bar{x}$. For positive integers k and l, we use the notation $\mathrm{div}(k,l)$ for integer division, i.e., maximum integer not greater than k/l, and $\mathrm{mod}(k,l)$ for the remainder after integer division, i.e., $\mathrm{mod}(k,l) = l(k/l - \mathrm{div}(k,l))$. The description of individual problems follows.

Problem 2.1. Chained Rosenbrock function [17].

$$\begin{aligned} f_k(x) &= 10(x_i^2 - x_{i+1}) \quad , \quad \mathrm{mod}(k,2) = 1, \\ f_k(x) &= x_i - 1 \quad , \quad \mathrm{mod}(k,2) = 0, \end{aligned}$$

$$
\begin{aligned}
n_A &= 2(n-1), \quad i = \mathrm{div}(k+1,2), \\
\bar{x}_l &= -1.2, \quad \mathrm{mod}(l,2)=1, \qquad \bar{x}_l = 1.0, \quad \mathrm{mod}(l,2)=0.
\end{aligned}
$$

Problem 2.2. Chained Wood function [29].

$$
\begin{aligned}
f_k(x) &= 10(x_i^2 - x_{i+1}) &&, \quad \mathrm{mod}(k,6)=1, \\
f_k(x) &= x_i - 1 &&, \quad \mathrm{mod}(k,6)=2, \\
f_k(x) &= \sqrt{90}\,(x_{i+2}^2 - x_{i+3}) &&, \quad \mathrm{mod}(k,6)=3, \\
f_k(x) &= x_{i+2} - 1 &&, \quad \mathrm{mod}(k,6)=4, \\
f_k(x) &= \sqrt{10}\,(x_{i+1} + x_{i+3} - 2) &&, \quad \mathrm{mod}(k,6)=5, \\
f_k(x) &= (x_{i+1} - x_{i+3})/\sqrt{10} &&, \quad \mathrm{mod}(k,6)=0, \\
n_A &= 3(n-2), \quad i = 2\,\mathrm{div}(k+5,6) - 1, \\
\bar{x}_l &= -3, \quad \mathrm{mod}(l,2)=1, \quad l \le 4, \qquad \bar{x}_l = -2, \quad \mathrm{mod}(l,2)=1, \quad l > 4, \\
\bar{x}_l &= -1, \quad \mathrm{mod}(l,2)=0, \quad l \le 4, \qquad \bar{x}_l = 0, \quad \mathrm{mod}(l,2)=0, \quad l > 4.
\end{aligned}
$$

Problem 2.3. Chained Powell singular function [17].

$$
\begin{aligned}
f_k(x) &= x_i + 10x_{i+1} &&, \quad \mathrm{mod}(k,4)=1, \\
f_k(x) &= \sqrt{5}\,(x_{i+2} - x_{i+3}) &&, \quad \mathrm{mod}(k,4)=2, \\
f_k(x) &= (x_{i+1} - 2x_{i+2})^2 &&, \quad \mathrm{mod}(k,4)=3, \\
f_k(x) &= \sqrt{10}\,(x_i - x_{i+3})^2 &&, \quad \mathrm{mod}(k,4)=0, \\
n_A &= 2(n-2), \quad i = 2\,\mathrm{div}(k+3,4) - 1, \\
\bar{x}_l &= 3, \quad \mathrm{mod}(l,4)=1, \qquad \bar{x}_l = -1, \quad \mathrm{mod}(l,4)=2, \\
\bar{x}_l &= 0, \quad \mathrm{mod}(l,4)=3, \qquad \bar{x}_l = 1, \quad \mathrm{mod}(l,4)=0.
\end{aligned}
$$

Problem 2.4. Chained Cragg and Levy function [29].

$$
\begin{aligned}
f_k(x) &= (\exp(x_i) - x_{i+1})^2 &&, \quad \mathrm{mod}(k,5)=1, \\
f_k(x) &= 10(x_{i+1} - x_{i+2})^3 &&, \quad \mathrm{mod}(k,5)=2, \\
f_k(x) &= \frac{\sin^2(x_{i+2} - x_{i+3})}{\cos^2(x_{i+2} - x_{i+3})} &&, \quad \mathrm{mod}(k,5)=3, \\
f_k(x) &= x_i^4 &&, \quad \mathrm{mod}(k,5)=4, \\
f_k(x) &= x_{i+3} - 1 &&, \quad \mathrm{mod}(k,5)=0, \\
n_A &= 5(n-2)/2, \quad i = 2\,\mathrm{div}(k+4,5) - 1, \\
\bar{x}_l &= 1 \quad, l = 1 \quad, \quad \bar{x}_l = 2 \quad, \quad 2 \le l \le n.
\end{aligned}
$$

Problem 2.5. Generalized Broyden tridiagonal function [17].

$$
\begin{aligned}
f_k(x) &= (3 - 2x_k)\,x_k + 1 - x_{k-1} - x_{k+1}, \\
n_A &= n, \quad x_0 = x_{n+1} = 0, \\
\bar{x}_l &= -1, \quad 1 \le l \le n.
\end{aligned}
$$

Problem 2.6. Generalized Broyden banded function [17].

$$\begin{aligned}
f_k(x) &= (2+5x_k^2)x_k + 1 + \sum_{j=k_1}^{k_2} x_j(1+x_j), \\
n_A &= n, \quad k_1 = \max(1, k-5), \quad k_2 = \min(n, k+1), \\
\bar{x}_l &= -1, \quad 1 \le l \le n.
\end{aligned}$$

Problem 2.7. Chained Freudenstein and Roth function [29].

$$\begin{aligned}
f_k(x) &= x_i + x_{i+1}((5-x_{i+1})x_{i+1} - 2) - 13 \ , \quad \mathrm{mod}(k,2) = 1, \\
f_k(x) &= x_i + x_{i+1}((1+x_{i+1})x_{i+1} - 14) - 29, \quad \mathrm{mod}(k,2) = 0, \\
n_A &= 2(n-1), \quad i = \mathrm{div}(k+1,2), \\
\bar{x}_l &= 0.5, \quad 1 \le l < n,
\end{aligned}$$

Problem 2.8. Wright and Holt zero residual problem [30].

$$\begin{aligned}
f_k(x) &= (x_i^a - x_j^b)^c, \\
a &= 1, \qquad k \le m/2, \qquad a = 2, \quad k > m/2, \\
b &= 5 - \mathrm{div}(k, m/4), \qquad c = \mathrm{mod}(k,5) + 1, \\
n_A &= 5n, \quad i = \mathrm{mod}(k, n/2) + 1, \quad j = i + n/2, \\
\bar{x}_l &= \sin^2(l), \quad 1 \le l \le n.
\end{aligned}$$

Problem 2.9. Toint quadratic merging problem [29].

$$\begin{aligned}
f_k(x) &= x_i + 3x_{i+1}(x_{i+2} - 1) + x_{i+3}^2 - 1 \quad , \quad \mathrm{mod}(k,6) = 1, \\
f_k(x) &= (x_i + x_{i+1})^2 + (x_{i+2} - 1)^2 - x_{i+3} - 3 \quad , \quad \mathrm{mod}(k,6) = 2, \\
f_k(x) &= x_i x_{i+1} - x_{i+2} x_{i+3} \quad , \quad \mathrm{mod}(k,6) = 3, \\
f_k(x) &= 2x_i x_{i+2} + x_{i+1} x_{i+3} - 3 \quad , \quad \mathrm{mod}(k,6) = 4, \\
f_k(x) &= (x_i + x_{i+1} + x_{i+2} + x_{i+3})^2 + (x_i - 1)^2 \quad , \quad \mathrm{mod}(k,6) = 5, \\
f_k(x) &= x_i x_{i+1} x_{i+2} x_{i+3} + (x_{i+3} - 1)^2 - 1 \quad , \quad \mathrm{mod}(k,6) = 0, \\
n_A &= 3(n-2), \quad i = 2\,\mathrm{div}(k+5,6) - 1, \\
\bar{x}_l &= 5, \quad 1 \le l \le n.
\end{aligned}$$

Problem 2.10. Chained exponential problem [17].

$$\begin{aligned}
f_k(x) &= 4 - \exp(x_i) - \exp(x_{i+1}) \quad , \quad \mathrm{mod}(k,2) = 1, \ i = 1, \\
f_k(x) &= 8 - \exp(3x_{i-1}) - \exp(3x_i) \\
&+ 4 - \exp(x_i) - \exp(x_{i+1}) \quad , \quad \mathrm{mod}(k,2) = 1, \ 1 < i < n, \\
f_k(x) &= 8 - \exp(3x_{i-1}) - \exp(3x_i) \quad , \quad \mathrm{mod}(k,2) = 1, \ i = n, \\
f_k(x) &= 6 - \exp(2x_i) - \exp(2x_{i+1}) \quad , \quad \mathrm{mod}(k,2) = 0, \\
n_A &= 2(n-1), \quad i = \mathrm{div}(k+1,2), \\
\bar{x}_l &= 0.2, \quad 1 \le l \le n.
\end{aligned}$$

Problem 2.11. Chained serpentine function [18].

$$\begin{aligned} f_k(x) &= 10(2x_i/(1+x_i^2) - x_{i+1}) \quad , \quad \text{mod}(k,2) = 1, \\ f_k(x) &= x_i - 1 \quad , \quad \text{mod}(k,2) = 0, \\ n_A &= 2(n-1), \quad i \;=\; \text{div}(k+1,2), \\ \bar{x}_l &= -0.8, \quad 1 \le l \le n. \end{aligned}$$

Problem 2.12. Chained and modified problem HS47 [18].

$$\begin{aligned} f_k(x) &= 10(x_i^2 - x_{i+1}) \quad , \quad \text{mod}(k,6) = 1, \\ f_k(x) &= x_{i+2} - 1 \quad , \quad \text{mod}(k,6) = 2, \\ f_k(x) &= (x_{i+3} - 1)^2 \quad , \quad \text{mod}(k,6) = 3, \\ f_k(x) &= (x_{i+4} - 1)^3 \quad , \quad \text{mod}(k,6) = 4, \\ f_k(x) &= x_i^2 x_{i+3} + \sin(x_{i+3} - x_{i+4}) - 10 \quad , \quad \text{mod}(k,6) = 5, \\ f_k(x) &= x_{i+1} + x_{i+2}^4 x_{i+3}^2 - 20 \quad , \quad \text{mod}(k,6) = 0, \\ n_A &= 6(\text{div}(n-5,3)+1), \quad i \;=\; 3\,\text{div}(k+5,6) - 2, \\ \bar{x}_l &= -1, \quad 1 \le l \le n. \end{aligned}$$

Problem 2.13. Chained and modified problem HS48 [18].

$$\begin{aligned} f_k(x) &= 10(x_i^2 - x_{i+1}) \quad , \quad \text{mod}(k,7) = 1, \\ f_k(x) &= 10(x_{i+1}^2 - x_{i+2}) \quad , \quad \text{mod}(k,7) = 2, \\ f_k(x) &= (x_{i+2} - x_{i+3})^2 \quad , \quad \text{mod}(k,7) = 3, \\ f_k(x) &= (x_{i+3} - x_{i+4})^2 \quad , \quad \text{mod}(k,7) = 4, \\ f_k(x) &= x_i + x_{i+1}^2 + x_{i+2} - 30 \quad , \quad \text{mod}(k,7) = 5, \\ f_k(x) &= x_{i+1} - x_{i+2}^2 + x_{i+3} - 10 \quad , \quad \text{mod}(k,7) = 6, \\ f_k(x) &= x_i x_{i+4} - 10 \quad , \quad \text{mod}(k,7) = 0, \\ n_A &= 7(\text{div}(n-5,3)+1), \quad i \;=\; 3\,\text{div}(k+6,7) - 2, \\ \bar{x}_l &= -1, \quad 1 \le l \le n. \end{aligned}$$

Problem 2.14. Sparse signomial function [18].

$$\begin{aligned} f_k(x) &= y_j - \sum_{p=1}^{3} (p^2/j) \prod_{q=1}^{4} \text{sign}(x_{i+q}) |x_{i+q}|^{q/(pj)}, \\ n_A &= 4(\text{div}(n-4,2)+1), \quad i \;=\; 2\,\text{div}(k+3,4) - 2, \quad j \;=\; \text{mod}(k-1,4)+1, \\ \bar{x}_l &= -0.8 \quad , \quad \text{mod}(l,4) = 1, \quad y_1 = 14.4, \\ \bar{x}_l &= 1.2 \quad , \quad \text{mod}(l,4) = 2, \quad y_2 = 6.8, \\ \bar{x}_l &= -1.2 \quad , \quad \text{mod}(l,4) = 3, \quad y_3 = 4.2, \\ \bar{x}_l &= 0.8 \quad , \quad \text{mod}(l,4) = 0, \quad y_4 = 3.2. \end{aligned}$$

Problem 2.15. Sparse exponential function [18].

$$f_k(x) = y_j - \sum_{p=1}^{3} (p^2/j) \exp\left(\sum_{q=1}^{4} x_{i+q} q/(pj) \right),$$

$$
\begin{aligned}
n_A &= 4(\mathrm{div}(n-4,2)+1), \quad i = 2\,\mathrm{div}(k+3,4)-2, \quad j = \mathrm{mod}(k-1,4)+1,\\
\bar{x}_l &= -0.8 \quad , \quad \mathrm{mod}(l,4)=1, \quad y_1 = 35.8,\\
\bar{x}_l &= 1.2 \quad , \quad \mathrm{mod}(l,4)=2, \quad y_2 = 11.2,\\
\bar{x}_l &= -1.2 \quad , \quad \mathrm{mod}(l,4)=3, \quad y_3 = 6.2,\\
\bar{x}_l &= 0.8 \quad , \quad \mathrm{mod}(l,4)=0, \quad y_4 = 4.4.
\end{aligned}
$$

Problem 2.16. Sparse trigonometric function [18].

$$
\begin{aligned}
f_k(x) &= y_j - \sum_{q=1}^{4}[(-1)^q j q^2 \sin(x_{i+q}) + j^2 q \cos(x_{i+q})],\\
n_A &= 4(\mathrm{div}(n-4,2)+1), \quad i = 2\,\mathrm{div}(k+3,4)-2, \quad j = \mathrm{mod}(k-1,4)+1,\\
\bar{x}_i &= -0.8 \quad , \quad \mathrm{mod}(l,4)=1, \quad y_1 = 30.6,\\
\bar{x}_i &= 1.2 \quad , \quad \mathrm{mod}(l,4)=2, \quad y_2 = 72.2,\\
\bar{x}_i &= -1.2 \quad , \quad \mathrm{mod}(l,4)=3, \quad y_3 = 124.4,\\
\bar{x}_i &= 0.8 \quad , \quad \mathrm{mod}(l,4)=0, \quad y_4 = 187.4.
\end{aligned}
$$

Problem 2.17. Countercurrent reactors problem 1 [7] (modified).

$$
\begin{aligned}
f_k(x) &= \alpha - (1-\alpha)x_{k+2} - x_k(1+4x_{k+1}) \quad &, \ k=1,\\
f_k(x) &= -(2-\alpha)x_{k+2} - x_k(1+4x_{k-1}) \quad &, \ k=2,\\
f_k(x) &= \alpha x_{k-2} - (1-\alpha)x_{k+2} - x_k(1+4x_{k+1}) \quad &, \ \mathrm{mod}\ (k,2)=1\ , \ 2<k<n-1,\\
f_k(x) &= \alpha x_{k-2} - (2-\alpha)x_{k+2} - x_k(1+4x_{k-1}) \quad &, \ \mathrm{mod}\ (k,2)=0\ , \ 2<k<n-1,\\
f_k(x) &= \alpha x_{k-2} - x_k(1+4x_{k+1}) \quad &, \ k=n-1,\\
f_k(x) &= \alpha x_{k-2} - (2-\alpha) - x_k(1+4x_{k-1}) \quad &, \ k=n,\\
n_A &= n, \quad \alpha = 1/2,\\
\overline{x}_l &= 0.1 \ , \ \mathrm{mod}\ (l,8)=1, \quad \overline{x}_l = 0.2 \ , \ \mathrm{mod}\ (l,8)=2,\\
\overline{x}_l &= 0.3 \ , \ \mathrm{mod}\ (l,8)=3, \quad \overline{x}_l = 0.4 \ , \ \mathrm{mod}\ (l,8)=4,\\
\overline{x}_l &= 0.5 \ , \ \mathrm{mod}\ (l,8)=5, \quad \overline{x}_l = 0.4 \ , \ \mathrm{mod}\ (l,8)=6,\\
\overline{x}_l &= 0.3 \ , \ \mathrm{mod}\ (l,8)=7, \quad \overline{x}_l = 0.2 \ , \ \mathrm{mod}\ (l,8)=0.
\end{aligned}
$$

Problem 2.18. Tridiagonal system [16].

$$
\begin{aligned}
f_k(x) &= 4(x_k - x_{k+1}^2) \quad &, \ k=1,\\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) + 4(x_k - x_{k+1}^2) \quad &, \ 1<k<n,\\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) \quad &, \ k=n,\\
n_A &= n, \quad \overline{x}_l = 12, \quad 1 \le l \le n.
\end{aligned}
$$

Problem 2.19. Structured Jacobian problem [12].

$$
\begin{aligned}
f_k(x) &= -2x_k^2 + 3x_k - 2x_{k+1} + 3x_{n-4} - x_{n-3}\\
&- \ x_{n-2} + 0.5x_{n-1} - x_n + 1 \quad , \ k=1,
\end{aligned}
$$

$$
\begin{aligned}
f_k(x) &= -2x_k^2 + 3x_k - x_{k-1} - 2x_{k+1} + 3x_{n-4} - x_{n-3} \\
&- \ x_{n-2} + 0.5x_{n-1} - x_n + 1 \qquad , \ 1 < k < n, \\
f_k(x) &= -2x_k^2 + 3x_k - x_{k-1} + 3x_{n-4} - x_{n-3} \\
&- \ x_{n-2} + 0.5x_{n-1} - x_n + 1 \qquad , \ k = n, \\
n_A &= n, \quad \overline{x}_l = -1, \quad 1 \le l \le n.
\end{aligned}
$$

Problem 2.20. Modified discrete boundary value problem [18].

$$
\begin{aligned}
f_k(x) &= 2x_k + (1/2)h^2(x_k + hk + 1)^3 - x_{k-1} - x_{k+1} + 1, \\
n_A &= n, \quad h = 1/(n+1), \quad x_0 = x_{n+1} = 0, \\
\bar{x}_l &= l(l-1)h, \quad 1 \le l \le n.
\end{aligned}
$$

Problem 2.21. Chained and modified problem HS53 [18].

$$
\begin{aligned}
f_k(x) &= 10(x_i^2 - x_{i+1}) \qquad , \quad \mathrm{mod}(k,7) = 1, \\
f_k(x) &= x_{i+1} + x_{i+2} - 2 \qquad , \quad \mathrm{mod}(k,7) = 2, \\
f_k(x) &= x_{i+3} - 1 \qquad , \quad \mathrm{mod}(k,7) = 3, \\
f_k(x) &= x_{i+4} - 1 \qquad , \quad \mathrm{mod}(k,7) = 4, \\
f_k(x) &= x_i + 3x_{i+1} \qquad , \quad \mathrm{mod}(k,7) = 5, \\
f_k(x) &= x_{i+2} + x_{i+3} - 2x_{i+4} \ , \quad \mathrm{mod}(k,7) = 6, \\
f_k(x) &= 10(x_{i+1}^2 - x_{i+4}) \qquad , \quad \mathrm{mod}(k,7) = 0, \\
n_A &= 7(\mathrm{div}(n-5,3) + 1) \ , \quad i = 3\ \mathrm{div}(k+6,7) - 2, \\
\bar{x}_l &= -1, \quad 1 \le l \le n.
\end{aligned}
$$

Problem 2.22. Attracting-Repelling problem [18].

$$
\begin{aligned}
f_k(x) &= x_1 - 1 \qquad , \quad k = 1, \\
f_k(x) &= 10(x_i^2 - x_{i+1}), \qquad , \quad k > 1, \quad \mathrm{mod}(k,2) = 0, \\
f_k(x) &= 2\exp(-(x_i - x_{i+1})^2) + \exp(-2(x_{i+1} - x_{i+2})^2) \ , \quad k > 1, \quad \mathrm{mod}(k,2) = 1, \\
n_A &= 2(n-1), \quad i = \mathrm{div}(k,2), \\
\bar{x}_l &= -1.2 \ , \ \mathrm{mod}(l,2) = 1, \qquad \bar{x}_l = 1.0 \ , \ \mathrm{mod}(l,2) = 0.
\end{aligned}
$$

Problem 2.23. Countercurrent reactors problem 2 [7] (modified).

$$
\begin{aligned}
f_k(x) &= x_1 - (1 - x_1)x_{k+2} - \alpha(1 + 4x_{k+1}) \qquad , \ k = 1, \\
f_k(x) &= -(1 - x_1)x_{k+2} - \alpha(1 + 4x_k) \qquad , \ k = 2, \\
f_k(x) &= \alpha x_1 - (1 - x_1)x_{k+2} - x_k(1 + 4x_{k-1}) \qquad , \ k = 3, \\
f_k(x) &= x_1 x_{k-2} + (1 - x_1)x_{k+2} - x_k(1 + 4x_{k-1}) \quad , \ 3 < k < n-1, \\
f_k(x) &= x_1 x_{k-2} - x_k(1 + 4x_{k-1}) \qquad , \ k = n-1, \\
f_k(x) &= x_1 x_{k-2} - (1 - x_1) - x_k(1 + 4x_{k-1}) \qquad , \ k = n, \\
n_A &= n, \quad \alpha = 0.414214, \\
\overline{x}_i &= 0.1 \ , \ \mathrm{mod}\ (i,8) = 1, \qquad \overline{x}_i = 0.2 \ , \ \mathrm{mod}\ (i,8) = 2,
\end{aligned}
$$

$$
\begin{aligned}
\overline{x}_i &= 0.3 \ , \ \mathrm{mod}\,(i,8)=3, & \overline{x}_i &= 0.4 \ , \ \mathrm{mod}\,(i,8)=4,\\
\overline{x}_i &= 0.5 \ , \ \mathrm{mod}\,(i,8)=5, & \overline{x}_i &= 0.4 \ , \ \mathrm{mod}\,(i,8)=6,\\
\overline{x}_i &= 0.3 \ , \ \mathrm{mod}\,(i,8)=7, & \overline{x}_i &= 0.2 \ , \ \mathrm{mod}\,(i,8)=0.
\end{aligned}
$$

Problem 2.24. Trigonometric system [28].

$$
\begin{aligned}
f_k(x) &= 5-(l+1)(1-\cos x_k)-\sin x_k-\sum_{j=5l+1}^{5l+5}\cos x_j,\\
n_A &= n, \quad l=\mathrm{div}(k-1,5),\\
\overline{x_i} &= 1/n, \quad 1\le i\le n.
\end{aligned}
$$

Problem 2.25. Trigonometric - exponential system (trigexp 1) [28].

$$
\begin{aligned}
f_k(x) &= 3x_k^3+2x_{k+1}-5+\sin(x_k-x_{k+1})\sin(x_k+x_{k+1}) \ , & k&=1,\\
f_k(x) &= 3x_k^3+2x_{k+1}-5+\sin(x_k-x_{k+1})\sin(x_k+x_{k+1}) & &\\
&+ 4x_k-x_{k-1}\exp(x_{k-1}-x_k)-3 \ , & 1&<k<n,\\
f_k(x) &= 4x_k-x_{k-1}\exp(x_{k-1}-x_k)-3 \ , & k&=n,\\
n_A &= n, \quad \overline{x}_i=0, \quad 1\le i\le n.
\end{aligned}
$$

Problem 2.26. Trigonometric - exponential system (trigexp 2) [28].

$$
\begin{aligned}
f_k(x) &= 3(x_k-x_{k+2})^3-5+2x_{k+1},\\
&+ \sin(x_k-x_{k+1}-x_{k+2})\sin(x_k+x_{k+1}-x_{k+2}) \ , \ \mathrm{mod}\,(k,2)=1, \ k=1,\\
f_k(x) &= -6(x_{k-2}-x_k)^3+10-4x_{k-1}\\
&- 2\sin(x_{k-2}-x_{k-1}-x_k)\sin(x_{k-2}+x_{k-1}-x_k)\\
&+ 3(x_k-x_{k+2})^3-5+2x_{k+1}\\
&+ \sin(x_k-x_{k+1}-x_{k+2})\sin(x_k+x_{k+1}-x_{k+2}) \ , \ \mathrm{mod}\,(k,2)=1, \ 1<k<n,\\
f_k(x) &= -6(x_{k-2}-x_k)^3+10-4x_{k-1}\\
&- 2\sin(x_{k-2}-x_{k-1}-x_k)\sin(x_{k-2}+x_{k-1}-x_k) \ , \ \mathrm{mod}\,(k,2)=1, \ k=n,\\
f_k(x) &= 4x_k-(x_{k-1}-x_{k+1})\exp(x_{k-1}-x_k-x_{k+1})-3 \ , \ \mathrm{mod}\,(k,2)=0,\\
n_A &= n, \quad \overline{x}_i=1, \quad 1\le i\le n.
\end{aligned}
$$

Problem 2.27. Singular Broyden problem [12].

$$
\begin{aligned}
f_k(x) &= ((3-2x_k)x_k-2x_{k+1}+1)^2 \ , & k&=1,\\
f_k(x) &= ((3-2x_k)x_k-x_{k-1}-2x_{k+1}+1)^2 \ , & 1&<k<n,\\
f_k(x) &= ((3-2x_k)x_k-x_{k-1}+1)^2 \ , & k&=n,\\
n_A &= n, \quad \overline{x}_i=-1, \quad 1\le i\le n.
\end{aligned}
$$

Problem 2.28. Five-diagonal system [16].

$$
f_k(x) = 4(x_k-x_{k+1}^2)+x_{k+1}-x_{k+2}^2 \ , \quad k=1,
$$

$$
\begin{aligned}
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) \\
&+ 4(x_k - x_{k+1}^2) + x_{k+1} - x_{k+2}^2 & , \; k = 2, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) \\
&+ 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} - x_{k+2}^2 & , \; 2 < k < n-1, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) \\
&+ 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} & , \; k = n-1, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) + x_{k-1}^2 - x_{k-2} & , \; k = n, \\
n_A &= n, \quad \overline{x}_i = -2, \quad 1 \le i \le n.
\end{aligned}
$$

Problem 2.29. Seven-diagonal system [16].

$$
\begin{aligned}
f_k(x) &= 4(x_k - x_{k+1}^2) + x_{k+1} - x_{k+2}^2 + x_{k+2} - x_{k+3}^2 & , \; k = 1, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) \\
&+ 4(x_k - x_{k+1}^2) + x_{k-1}^2 + x_{k+1} - x_{k+2}^2 + x_{k+2} - x_{k+3}^2 & , \; k = 2, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) \\
&+ 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} - x_{k+2}^2 \\
&+ x_{k-2}^2 + x_{k+2} - x_{k+3}^2 & , \; k = 3, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) \\
&+ 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} - x_{k+2}^2 \\
&+ x_{k-2}^2 + x_{k+2} - x_{k-3} - x_{k+3}^2 & , \; 3 < k < n-2, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) \\
&+ 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} - x_{k+2}^2 \\
&+ x_{k-2}^2 + x_{k+2} - x_{k-3} & , \; k = n-2, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) \\
&+ 4(x_k - x_{k+1}^2) + x_{k-1}^2 - x_{k-2} + x_{k+1} \\
&+ x_{k-2}^2 - x_{k-3} & , \; k = n-1, \\
f_k(x) &= 8x_k(x_k^2 - x_{k-1}) - 2(1-x_k) + x_{k-1}^2 - x_{k-2} \\
&+ x_{k-2}^2 - x_{k-3} & , \; k = n, \\
n_A &= n, \quad \overline{x}_i = -3, \quad 1 \le i \le n.
\end{aligned}
$$

Problem 2.30. Extended Freudenstein and Roth function [6].

$$
\begin{aligned}
f_k &= x_k + ((5 - x_{k+1})x_{k+1} - 2)x_{k+1} - 13 & , \; \mathrm{mod}(k,2) = 1, \\
f_k &= x_{k-1} + ((x_k + 1)x_k - 14)x_k - 29 & , \; \mathrm{mod}(k,2) = 0, \\
n_A &= n, \\
\overline{x}_i &= 90 \;, \; \mathrm{mod}(i,2) = 1, \quad \overline{x}_i = 60 \;, \; \mathrm{mod}(i,2) = 0.
\end{aligned}
$$

Problem 2.31. Broyden tridiagonal problem [23].

$$
\begin{aligned}
f_k(x) &= x_k(0.5x_k - 3) + 2x_{k+1} - 1 & , \; k = 1, \\
f_k(x) &= x_k(0.5x_k - 3) + x_{k-1} + 2x_{k+1} - 1 & , \; 1 < k < n, \\
f_k(x) &= x_k(0.5x_k - 3) - 1 + x_{k-1} & , \; k = n, \\
n_A &= n, \quad \overline{x}_i = -1, \quad 1 \le i \le n.
\end{aligned}
$$

Problem 2.32. Extended Powell badly scaled function [23].

$$
\begin{aligned}
f_k(x) &= 10000\, x_k\, x_{k+1} - 1 &&, \quad \mathrm{mod}(k,2)=1,\\
f_k(x) &= \exp(-x_{k-1}) + \exp(-x_k) - 1.0001 &&, \quad \mathrm{mod}(k,2)=0,\\
n_A &= n,\\
\overline{x}_i &= 0 \;,\;\; \mathrm{mod}(i,2)=1, \qquad \overline{x}_i = 1 \;,\;\; \mathrm{mod}(i,2)=0.
\end{aligned}
$$

Problem 2.33. Extended Wood problem [13].

$$
\begin{aligned}
f_k(x) &= -200x_k(x_{k+1} - x_k^2) - (1 - x_k) &&, \quad \mathrm{mod}(k,4)=1,\\
f_k(x) &= 200(x_k - x_{k-1}^2) + 20.2(x_k - 1) + 19.8(x_{k+2} - 1) &&, \quad \mathrm{mod}(k,4)=2,\\
f_k(x) &= -180x_k(x_{k+1} - x_k^2) - (1 - x_k) &&, \quad \mathrm{mod}(k,4)=3,\\
f_k(x) &= 180(x_k - x_{k-1}^2) + 20.2(x_k - 1) + 19.8(x_{k-2} - 1) &&, \quad \mathrm{mod}(k,4)=0,\\
n_A &= n,\\
\overline{x}_i &= -3, \;\; \mathrm{mod}(i,2)=1, \qquad \overline{x}_i = -1, \;\; \mathrm{mod}(i,2)=0.
\end{aligned}
$$

Problem 2.34. Tridiagonal exponential problem [6].

$$
\begin{aligned}
f_k(x) &= x_k - \exp(\cos(k(x_k + x_{k+1}))) &&, \; k=1,\\
f_k(x) &= x_k - \exp(\cos(k(x_{k-1} + x_k + x_{k+1}))) &&, \; 1<k<n,\\
f_k(x) &= x_k - \exp(\cos(k(x_{k-1} + x_k))) &&, \; k=n,\\
n_A &= n, \quad \overline{x}_i = 1.5, \quad 1 \le i \le n.
\end{aligned}
$$

Problem 2.35. Discrete boundary value problem [23].

$$
\begin{aligned}
f_k(x) &= 2x_k + 0.5h^2(x_k + hk + 1)^3 - x_{k+1} &&, \; k=1,\\
f_k(x) &= 2x_k + 0.5h^2(x_k + hk + 1)^3 - x_{k-1} - x_{k+1} &&, \; 1<k<n,\\
f_k(x) &= 2x_k + 0.5h^2(x_k + hk + 1)^3 - x_{k-1} &&, \; k=n,\\
n_A &= n, \quad h = 1/(n+1),\\
\overline{x}_i &= ih(ih-1), \quad 1 \le i \le n.
\end{aligned}
$$

Problem 2.36. Brent problem [5].

$$
\begin{aligned}
f_k(x) &= 3x_k(x_{k+1} - 2x_k) + x_{k+1}^2/4 &&, \; k=1,\\
f_k(x) &= 3x_k(x_{k+1} - 2x_k + x_{k-1}) + (x_{k+1} - x_{k-1})^2/4 &&, \; 1<k<n,\\
f_k(x) &= 3x_k(20 - 2x_k + x_{k-1}) + (20 - x_{k-1})^2/4 &&, \; k=n,\\
n_A &= n, \quad \overline{x}_i = 10, \quad 1 \le i \le n.
\end{aligned}
$$

Problem 2.37. Troesch problem [26].

$$
\begin{aligned}
f_k(x) &= 2x_k + \rho h^2 \sinh(\rho x_k) - x_{k+1} &&, \; k=1,\\
f_k(x) &= 2x_k + \rho h^2 \sinh(\rho x_k) - x_{k-1} - x_{k+1} &&, \; 1<k<n,\\
f_k(x) &= 2x_k + \rho h^2 \sinh(\rho x_k) - x_{k-1} - 1 &&, \; k=n,\\
n_A &= n, \quad \rho = 10, \quad h = 1/(n+1),\\
\overline{x}_i &= 1, \quad 1 \le i \le n.
\end{aligned}
$$

Problem 2.38. Flow in a channel [4].

This is a finite difference analogue of the following nonlinear ordinary differential equation

$$u'''' = R(u'u'' - uu'''), \quad R = 500$$

over a unit interval Ω with boundary conditions $u(0) = 0$, $u'(0) = 0$, $u(1) = 1$, $u'(1) = 0$. We use standard 5-point finite differences on a uniform grid having 5000 internal nodes. The initial approximate solution is a discretization of $u_0(x) = (x - 1/2)^2$.

Problem 2.39. Swirling flow [4].

This is a finite difference analogue of the following system of two nonlinear ordinary differential equations

$$\begin{aligned} u'''' + R(uu''' + vv') &= 0 \\ v'' + R(uv' + u'v) &= 0, \quad R = 500 \end{aligned}$$

over a unit interval Ω with boundary conditions $u(0) = u'(0) = u(1) = u'(1) = 0$, $v(0) = -1$, $v(1) = 1$. We use standard 5-point finite differences on a uniform grid having 2500 internal nodes. The initial approximate solution is a discretization of $u_0(x) = (x - 1/2)^2$ and $v_0(x) = x - 1/2$.

Problem 2.40. Bratu problem [14].

This is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u + R \exp(u) = 0, \quad R = 6.8$$

over a unit square Ω with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 2.41. Poisson problem 1 [12].

This is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u = \frac{u^3}{1 + x^2 + y^2}$$

over a unit square Ω with Dirichlet boundary conditions $u(0, y) = 1$, $u(1, y) = 2 - \exp(y)$, $u(x, 0) = 1$, $u(x, 1) = 2 - \exp(x)$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = -1$.

Problem 2.42. Poisson problem 2 [21].

This is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u + \sin(2\pi u) + \sin\left(2\pi \frac{\partial u}{\partial x}\right) + \sin\left(2\pi \frac{\partial u}{\partial y}\right) + f(x, y) = 0,$$

where $f(x, y) = 1000((x - 1/4)^2 + (y - 3/4)^2)$, over a unit square Ω with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 2.43. Porous medium problem [10].

This is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u^2 + R\left(\frac{\partial u^3}{\partial x} + f(x, y)\right) = 0, \quad R = 50,$$

where $f(1/71, 1/71) = 1$ and $f(x, y) = 0$ for $(x, y) \neq (1/71, 1/71)$, over a unit square Ω with Dirichlet boundary conditions $u(0, y) = 1$, $u(1, y) = 0$, $u(x, 0) = 1$, $u(x, 1) = 0$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 1 - xy$.

Problem 2.44. Convection-difussion problem [15].

This is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta u - Ru\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + f(x, y) = 0, \quad R = 20,$$

where $f(x, y) = 2000x(1 - x)y(1 - y)$, over a unit square Ω with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We use standard 5-point finite differences on a uniform grid having 70×70 internal nodes. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 2.45. Nonlinear biharmonic problem [20].

This is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta\Delta u + R\left(\max(0, u) + \operatorname{sign}(x - \frac{1}{2})\right) = 0, \quad R = 500$$

over a unit square Ω with the boundary conditions $u = 0$ on $\partial\Omega$ and $\partial u(0, y)/\partial x = 0$, $\partial u(1, y)/\partial x = 0$, $\partial u(x, 0)/\partial y = 0$, $\partial u(x, 1)/\partial y = 0$. We use standard 13-point finite differences on a shifted uniform grid having 50×50 internal nodes [14]. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 2.46. Driven cavity problem [14].

This is a finite difference analogue of the following nonlinear partial differential equation

$$\Delta\Delta u + R\left(\frac{\partial u}{\partial y}\frac{\partial \Delta u}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial \Delta u}{\partial y}\right) = 0, \quad R = 500$$

over a unit square Ω with the boundary conditions $u = 0$ on $\partial\Omega$ and $\partial u(0, y)/\partial x = 0$, $\partial u(1, y)/\partial x = 0$, $\partial u(x, 0)/\partial y = 0$, $\partial u(x, 1)/\partial y = 1$. We use standard 13-point finite differences on a shifted uniform grid having 50×50 internal nodes [14]. The initial approximate solution is a discretization of $u_0(x, y) = 0$.

Problem 2.47.

$$f_k(x) = 2x_k - x_{k+1} - x_{k-1}$$

$$
\begin{aligned}
&\quad +h^2\left(x_k^3 + 2.10^{-4}(2.10^{-4}a_2 - 1)x_k - 10^9\exp(-3.10^4 a_2)\right) , \\
n_A &= n, \quad h = 1/(n+1), \quad a_1 = h\,k, \quad a_2 = (a_1 - 1/2)^2 , \\
x_0 &= x_{n+1} = 0, \quad \overline{x}_l = 5\min(lh, 1-lh), \quad 1 \le l \le n .
\end{aligned}
$$

Problem 2.48.

$$
\begin{aligned}
f_k(x) &= 2x_k - x_{k+1} - x_{k-1} \\
&\quad +h^2\left(x_k^3\exp(x_k) + 5.10^8\exp(-10^4 a_2)\sqrt{|a_1 - 1/2|}(x_{k+1} - x_{k-1}) + a_3\right) , \\
n_A &= n, \quad h = 1/(n+1), \quad a_1 = h\,k, \quad a_2 = (a_1 - 1/2)^2, \quad a_3 = 10^6\mathrm{sign}(a_1 - 1/2) , \\
x_0 &= x_{n+1} = 0, \quad \overline{x}_l = 5\min(lh, 1-lh), \quad 1 \le l \le n .
\end{aligned}
$$

Problem 2.49. [25], Problem 202 in [27].

$$
\begin{aligned}
f_k(x) &= x_k - \frac{x_{k+1}^2}{10}, \quad 1 \le k < n , \\
f_k(x) &= x_k - \frac{x_1^2}{10}, \quad k = n , \\
n_A &= n, \quad \overline{x}_l = 2, \quad 1 \le l \le n .
\end{aligned}
$$

Problem 2.50. [9], Problem 206 in [27].

$$
\begin{aligned}
f_k(x) &= x_{k-1} - 2x_k + x_{k+1} - h^2\exp(x_k), \quad 1 \le k \le n , \\
n_A &= n, \quad h = 1/(n+1), \quad x_0 = x_{n+1} = 0 , \\
\overline{x}_l &= 1, \quad 1 \le l \le n .
\end{aligned}
$$

Problem 2.51. [8], Problem 207 in [27].

$$
\begin{aligned}
f_k(x) &= (3 - x_k/10)x_k + 1 - x_{k-1} - 2x_{k+1}, \quad 1 \le k \le n , \\
x_0 &= x_{n+1} = 0 , \\
n_A &= n, \quad \overline{x}_l = -1, \quad 1 \le l \le n .
\end{aligned}
$$

Problem 2.52. [8], Problem 208 in [27].

$$
\begin{aligned}
f_k(x) &= (1 + x_k^2)x_k + 1 - \sum_{i \in I_k}(x_i + x_i^2), \quad 1 \le k \le n , \\
I_k &= \{i : i \ne k, \max(1, k-3) \le i \le \min(n, k+3)\} , \\
n_A &= n, \quad \overline{x}_l = -1, \quad 1 \le l \le n .
\end{aligned}
$$

Problem 2.53. [22], Problem 212 in [27].

$$
\begin{aligned}
f_k(x) &= x_k \qquad\qquad\qquad\qquad , \quad k = 1 , \\
f_k(x) &= \cos(x_{k-1}) + x_k - 1), \quad 1 < k \le n , \\
n_A &= n, \quad \overline{x}_l = 1/2, \quad 1 \le l \le n .
\end{aligned}
$$

Problem 2.54. [1], Problem 213 in [27].

$$\begin{aligned} f_k(x) &= 2x_k + h^2(x_k + \sin(x_k)) - x_{k-1} - x_{k+1}, \quad 1 \le k \le n\,, \\ n_A &= n, \quad h = 1/(n+1), \quad x_0 = 0, \quad x_{n+1} = 1\,, \\ \overline{x}_l &= 1, \quad 1 \le l \le n\,. \end{aligned}$$

Problem 2.55. Broyden banded function [23].

$$\begin{aligned} f_k(x) &= x_k(2 + 5x_k^2) + 1 - \sum_{i \in I_k} x_i(1 + x_i), \quad 1 \le k \le n\,, \\ I_k &= \{i : i \ne k, \max(1, k-5) \le i \le \min(n, k+1)\}\,, \\ n_A &= n, \quad \overline{x}_l = -1, \quad 1 \le l \le n\,. \end{aligned}$$

Problem 2.56. Ascher and Russel boundary value problem [3].

$$\begin{aligned} f_k(x) &= 2x_k - 2h^2\left(x_k^2 + \frac{x_{k+1} - x_{k-1}}{2h}\right) - x_{k-1} - x_{k+1}, \quad 1 \le k \le n\,, \\ n_A &= n, \quad h = 1/(n+1), \quad x_0 = 0, \quad x_{n+1} = 1/2\,, \\ \overline{x}_l &= 1, \quad 1 \le l \le n\,. \end{aligned}$$

Problem 2.57. Allgower and Georg boundary value problem [2] (modified).

$$\begin{aligned} f_k(x) &= 2x_k + 0.3h^2\left[\exp(20(x_k + 25(kh-1))) - \exp(-20(x_k + 25kh)) - t_k\right] \\ &\quad -x_{k-1} - x_{k+1}\,, \\ t_k &= \operatorname{sign}(kh - 0.5), \quad k \ge 1\,, \\ n_A &= n, \quad h = 1/(n+1), \quad x_0 = 0, \quad x_{n+1} = 25\,, \\ \overline{x}_l &= 1, \quad 1 \le l \le n\,. \end{aligned}$$

Problem 2.58. Potra and Rheinboldt boundary value problem [24].

$$\begin{aligned} f_k(x) &= 2x_k - x_{k-1} - x_{k+1} + h^2(x_k^2 + x_k + 0.1x_{k+n/2}^2 - 1.2)\;, \quad 1 \le k < n/2\,, \\ f_k(x) &= 2x_k - x_{k-1} + h^2(x_k^2 + x_k + 0.1x_{k+n/2}^2 - 1.2)\;, \quad k = n/2\,, \\ f_k(x) &= 2x_k - x_{k+1} + h^2(0.2x_{k-n/2}^2 + x_k^2 + 2x_k - 0.6), \quad k = n/2+1\,, \\ f_k(x) &= 2x_k - x_{k-1} - x_{k+1} + h^2(0.2x_{k-n/2}^2 + x_k^2 + 2x_k - 0.6), \quad n/2+1 < k \le n\,, \\ n_A &= n, \quad h = 1/(n/2+1), \quad x_0 = x_{n+1} = 0\,, \\ \overline{x}_l &= lh(1 - lh), \quad \overline{x}_{l+n/2} = \overline{x}_l, \quad 1 \le l \le n/2\,. \end{aligned}$$

Problem 2.59. Modified Bratu problem [11].

$$\begin{aligned} f_k(x) &= 4x_k - x_{k-1} - x_{k+1} - x_{k-\sqrt{n}} - x_{k+\sqrt{n}} + h^2\exp(x_k), \quad 1 \le k \le n\,, \\ n_A &= n, \quad h = 1/(\sqrt{n}+1), \quad x_l = 0 \quad \text{for} \quad l < 1 \quad \text{or} \quad l > n, \\ \overline{x}_l &= 1, \quad 1 \le l \le n\,. \end{aligned}$$

Problem 2.60. Nonlinear Dirichlet problem [11].

$$\begin{aligned} f_k(x) &= 4x_k - x_{k-1} - x_{k+1} - x_{k-\sqrt{n}} - x_{k+\sqrt{n}} + h^2x_k^2 - y_k, \quad 1 \le k \le n\,, \\ n_A &= n, \quad h = 1/(\sqrt{n}+1), \quad x_l = 0 \quad \text{for} \quad l < 1 \quad \text{or} \quad l > n, \\ \overline{x}_l &= 1, \quad 1 \le l \le n\,. \end{aligned}$$

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