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Technical report No. 1015

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Abstract:

Johnson mean is a new characteristic of the central tendency of continuous distributions. Johnson variance was introduced as a new characteristics of the variability of distributions. In this paper we introduce a Johnson difference in the sample space, which is used for a construction of confidence intervals for the Johnson mean, and replace the expression for the Johnson variance by a more suitable one.

Keywords:
measure of central tendency, measure of variability, interval estimates

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1 INTRODUCTION: THE JOHNSON MEAN

It has been shown by (Fabián, 2008) that a continuous probability distribution F with interval support $\mathcal{X} \in \mathbb{R}$ can be characterized, besides the distribution function $F(x)$ and density $f(x)$, by its Johnson score, defined as follows. Mapping $\eta : \mathcal{X} \to \mathbb{R}$, where

$$\eta(x) = \begin{cases} x & \text{if} \quad (a,b) = \mathbb{R} \\ \log(x-a) & \text{if} \quad -\infty < a < b = \infty \\ \log \dfrac{(x-a)}{(b-x)} & \text{if} \quad -\infty < a < b < \infty \\ -\log(b-x) & \text{if} \quad -\infty = a < b < \infty, \end{cases} \tag{1.1}$$

is the *Johnson transformation* (Johnson, 1949) adapted for arbitrary interval support $\mathcal{X} = (a,b) \in \mathbb{R}$. The *Johnson score* of distribution F with continuously differentiable density is

$$T(x) = \frac{1}{f(x)} \frac{d}{dx} \left(-\frac{1}{\eta'(x)} f(x) \right) \tag{1.2}$$

where η is given by (1.1).

The philosophy behind this concept is the following. Any distribution F with interval support $\mathcal{X} \neq \mathbb{R}$ is viewed as a transformed '*prototype*' G with support $\mathbb{R}$, that is, its distribution function is $F(x) = G(\eta(x))$. Denoting by g the density of G, the density of F is

$$f(x) = g(\eta(x))\eta'(x), \qquad x \in \mathcal{X}, \tag{1.3}$$

where $\eta'(x) = d\eta(x)/dx$ is the Jacobian of the transformation. Let us denote the score function of G by Q so that

$$Q(y) = -\frac{g'(y)}{g(y)}.$$

By setting $y = \eta(x)$ we obtain from (1.2) and (1.3)

$$T(x) = \frac{1}{g(y)\eta'(x)} \frac{d}{dy}(-g(y)) \frac{dy}{dx} = Q(\eta(x)). \tag{1.4}$$

By (1.1), the Johnson score of a prototype (a distribution with support $\mathbb{R}$) is its score function. By (1.4), the Johnson score of a distribution with interval support $\mathcal{X} \neq \mathbb{R}$ is the transformed score function of its prototype. By (1.4) is defined a unique and useful scalar inference function for arbitrary continuous distribution F satisfying the usual regularity conditions. We note that the Johnson transformation was chosen not only due to its mathematical convenience for many distributions used in statistics, but also on the base of other reasons discussed in (Fabián, 2008).

A unique solution x^* of equation

$$T(x) = 0 \tag{1.5}$$

we call a *Johnson mean* of distribution F. Due to (1.4), the solution of (1.5) is unique if G is unimodal. Confining ourselves to distributions with unimodal prototypes, the Johnson mean was shown to characterize the typical value of distributions including the heavy-tailed distributions without mean, being a value near the mean of the light-tailed ones.

From the point of view of the structure of the parameters, there are two different types of distributions with support $\mathcal{X} \neq \mathbb{R}$:

i/ Distributions of the first type are the transformed distributions, prototypes of which have the location parameter μ. These distributions have a parameter

$$t = \eta^{-1}(\mu),$$

called a *Johnson parameter*, the value of which is the Johnson mean of the distribution. Denoting by $f(x,t)$ the density and $T(x;t)$ the Johnson score of a distribution of this type, it was shown that function

$$S(x;t) = \eta'(t)T(x;t)$$

equals to the likelihood score $l(x;t) = (\partial/\partial t)\log f(x;t)$ for parameter t. The value

$$I(t) = ES^2 = [\eta'(t)]^2 ET^2 \tag{1.6}$$

thus appears to be the Fisher information for the Johnson parameter. An example is the exponential distribution with density $f(x;\lambda) = \lambda^{-1}e^{-x/\lambda}$ and Johnson score $T(x;\lambda) = x/\lambda - 1$ with Johnson parameter λ and Fisher information $I(\lambda) = 1/\lambda^2$.

ii/ Distributions of the second type are the transformed distributions with prototypes without location parameter. The Johnson score of them appears to be a new function and the Johnson mean, a function $x^* = x^*(\theta)$ of the parameters, is a new characteristic of their central tendency. For some two-parameter distributions, x^* is the ratio of the parameters. For example, the gamma distribution with density $f(x;\alpha,\gamma) = \frac{\gamma^\alpha}{x\Gamma(\alpha)}x^\alpha e^{-\gamma x}$ and Johnson score $T(x;\alpha,\gamma) = \gamma x - \alpha$ has Johnson mean $x^* = \alpha/\gamma$.

2 JOHNSON DISTANCE

Consider distribution F_t of the first type mentioned above. Its Johnson mean is $x^* = t$ so that

$$T(t;t) = 0. \tag{2.1}$$

Let us measure the difference (the oriented distance) between $x_1 \in \mathcal{X}$ and $x_2 \in \mathcal{X}$ as a difference of values of the Johnson score

$$d_t(x_1,x_2) = T(x_1;t) - T(x_2;t). \tag{2.2}$$

Let $\mathbb{X}_n = (X_1, ..., X_n)$ be random sample drawn from F_t. Due to (2.1),

$$\frac{1}{n}\sum_{i=1}^{n} d_t(x_i, t) = \frac{1}{n}\sum_{i=1}^{n} T(x_i; t)$$

so that the mean difference of the observed values from the maximum likelihood estimate $\hat{t}$ of t is zero. Thus (2.2) can be considered as a 'maximum likelihood distance' in the sample space of distribution F_t.

We suggest generalization of (2.2) for distributions of the second type (and, hence, for arbitrary F).

Definition 1 *Let T be the Johnson score of distribution F with support $\mathcal{X}$ and Johnson mean x^*. Define a Johnson difference of points $x_1, x_2 \in \mathcal{X}$ by*

$$d^*(x_1, x_2) = T(x_2) - T(x_1). \tag{2.3}$$

By (2.3) we define a distance of statistical individuals $x_1, x_2 \in \mathcal{X}$ in the sample space of distribution F, a simple distance compatible in particular cases with the maximum likelihood estimator. Johnson distances $|d^*(x_1, x_2)|$ for a few distributions are given in Table 1.

Table 1. Johnson distance in the sample space of some distributions.

Distribution	$f(x)$	$T(x)$	$D(x_1, x_2)$
normal	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\frac{x-\mu}{\sigma^2}$	$\frac{\lvert x_2-x_1\rvert}{\sigma^2}$
lognormal	$\frac{c}{\sqrt{2\pi}x}e^{-\frac{1}{2}\log^2(\frac{x}{t})^c}$	$c\log(\frac{x}{t})^c$	$c^2\lvert\log\frac{x_2}{x_1}\rvert$
Weibull	$\frac{c}{x}(\frac{x}{t})^c e^{-(\frac{x}{t})^c}$	$c((\frac{x}{t})^c - 1)$	$\frac{c}{t}\lvert x_2^c - x_1^c\rvert$
beta-prime	$\frac{1}{xB(p,q)}\frac{x^p}{(x+1)^{p+q}}$	$\frac{qx-p}{x+1}$	$\frac{q\lvert x_2-x_1\rvert}{(x_1+1)(x_2+1)}$

3 ESTIMATES OF JOHNSON MEAN

Let $\Theta \subset \mathbb{R}^m$ and $\{f(x,\theta), \theta \in \Theta\}$ be a family of distributions with support $\mathcal{X} \subseteq \mathbb{R}$. In what follows we use a notation $T(x; x^*, \theta)$ instead of $T(x; \theta)$, accenting superfluously a real parameter $x^* = t$ for distributions of the first type, and showing a virtual parameter x^* for distributions of the second type. For the gamma distribution, for instance, $T(x; x^*, \theta) = \gamma(x - x^*)$.

Having random sample $\mathbb{X}_n$ from F_θ with unknown θ, the estimate $\hat{x}^*$ of the Johnson mean x^* is either the maximum likelihood estimate of the Johnson parameter for distributions of the first type, or, in cases of distributions of the second type, it can be constructed from the maximum likelihood estimate $\hat{\theta}$ of vector θ by setting $\hat{x}^* = x^*(\hat{\theta})$. However, Fabián (2008) has shown that, in some important particular cases, x^* can be estimated by a direct way. Denote by

$$T'_{x^*} = \frac{\partial T(x; x^*, \theta)}{\partial x^*}$$

the derivative of the Johnson score according to virtual parameter x^*. By Proposition 2, Fabián (2008), the estimate of $\hat{x}^*$ of x^* from the equation

$$\sum_{i=1}^{n} T(x_i; x^*, \theta) = 0 \tag{3.1}$$

is $AN(x^*, \omega^2)$, where

$$\omega^2 = \frac{ET^2}{(ET'_{x^*})^2} \tag{3.2}$$

and where AN means 'asymptotically normal'.

4 CONFIDENCE INTERVALS FOR JOHNSON MEAN

The Johnson distance can be used for establishing confidence intervals for Johnson mean estimated from (3.1).

THEOREM 1 *Let assumptions of Definition 1 hold and let $\hat{x}^*$ be the estimate of x^* from (3.1) based on sample $\mathbb{X}_n$ drawn from F. Let d^* be given by (2.3), $T'(x^*) = \partial T(x; x^*, \theta)/\partial x|_{x=x^*} \neq 0$ and ω^2 be given by (3.2). Random variable $\sqrt{n}d^*(x^*, \hat{x}^*)$ is $AN(0, [T'(x^*)]^2\omega^2)$.*

Proof. By Proposition 2 (Fabián, 2008), $\sqrt{n}(\hat{x}^* - x^*)$ is $AN(0, \omega^2)$ where ω^2 is given by (3.2). As $T'(x^*) \neq 0$, $\sqrt{n}[T(x^*; \hat{x}^*, \theta) - T(\hat{x}^*; \hat{x}^*, \theta)]$ is $AN(0, [T'(x^*)]^2\omega^2)$ in distribution according Theorem A, Chap.3.1 (Serfling, 1980). □

Corollary. The normalized Johnson difference

$$\sqrt{n}D(x^*, \hat{x}^*) = \sqrt{n}\frac{d^*(x^*, \hat{x}^*|\hat{x}^*)}{|T'(x^*)|\omega} = \frac{T(x^*; \hat{x}^*, \theta)}{|T'(x^*)|\omega/\sqrt{n}} \tag{4.1}$$

is $AN(0, 1)$.

After replacing x^* and ω in the denominator of (4.1) by their estimates $\hat{x}^*$ and $\hat{\omega}$, respectively, we obtain the approximate $(100 - \alpha)\%$ confidence interval for x^* from equation

$$|T(x^*; \hat{x}^*, \theta)| \leq \lambda_n |T'(\hat{x}^*)|\hat{\omega}, \tag{4.2}$$

where $\lambda_n = u_{\alpha/2}/\sqrt{n}$ and $u_{\alpha/2}$ is the $(\alpha/2)$-th quantile of the normal distribution.

The densities and Johnson scores of some commonly used two-parameter distributions, together with their Johnson mean x^* and variance ω^2 of its estimate $\hat{x}^*$ computed in (Fabián, 2008) are given in Table 2.

We find the $(100-\alpha)\%$ confidence intervals for the Johnson mean of them. Since they all are two-parameter distributions, the estimate of ω can be constructed not only from the maximum likelihood estimates of the parameters, but in some cases directly using the second equation of the system of the Johnson score moment equations (Fabián, 2008) in the form

$$\frac{1}{n}\sum_{i=1}^{n} T^2(x_i; x^*, \theta) = ET^2. \tag{4.3}$$

The first two distributions in Table 2 are the distributions with support $\mathbb{R}$.

Normal distribution: The maximum likelihood equations and equations (3.1) and (4.3) are identical. $\hat{x}^*$ is the arithmetic mean with the usual confidence bounds.

Logistic distribution: By Table 2, $\hat{x}^* = \hat{\mu}$ and $\hat{\omega}^2 = 3\hat{\sigma}^2$ where $\hat{\mu}$ and $\hat{\sigma}$ are either the maximum likelihood or Johnson score moment estimates. Since $T'_\mu(\hat{\mu}) = 1/2\hat{\sigma}$, condition (4.2) appears to be $|e^{\frac{\hat{\mu}-\mu}{\hat{\sigma}}} - 1| \le \lambda_n \rho (e^{\frac{\hat{\mu}-\mu}{\hat{\sigma}}} + 1)$ where $\rho = \sqrt{3}/2$ so that

$$\mu \in \left(\hat{\mu} - \hat{\sigma} \log \frac{1-\lambda_n \rho}{1+\lambda_n \rho}, \hat{\mu} + \hat{\sigma} \log \frac{1+\lambda_n \rho}{1-\lambda_n \rho}\right).$$

Table 2. Density, Johnson score, Johnson mean and variance of the estimate of the Johnson mean of some distributions.

Distribution	$f(x)$	$T(x)$	x^*	ω^2
normal ($\mathbb{R}$)	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\frac{x-\mu}{\sigma^2}$	μ	σ^2
logistic ($\mathbb{R}$)	$\frac{1}{\sigma} \frac{e^{\frac{x-\mu}{\sigma}}}{(1+e^{\frac{x-\mu}{\sigma}})^2}$	$\frac{1}{\sigma} \frac{e^{\frac{x-\mu}{\sigma}}-1}{e^{\frac{x-\mu}{\sigma}}+1}$	μ	$3\sigma^2$
lognormal	$\frac{c}{\sqrt{2\pi}x} e^{-\frac{1}{2}\log^2(\frac{x}{t})^c}$	$c\log(\frac{x}{t})^c$	t	$\frac{t^2}{c^2}$
Weibull	$\frac{c}{x}(\frac{x}{t})^c e^{-(\frac{x}{t})^c}$	$c((\frac{x}{t})^c - 1)$	t	$\frac{t^2}{c^2}$
hyperbolic	$\frac{1}{2K_0(c)x} e^{-\frac{c}{2}(\frac{x}{t}+\frac{t}{x})}$	$\frac{c}{2}(\frac{x}{t} - \frac{t}{x})$	t	$\frac{2}{K(c)}\frac{t^2}{c^2}$
gamma	$\frac{\gamma^\alpha}{x\Gamma(\alpha)} x^\alpha e^{-\gamma x}$	$\gamma x - \alpha$	α/γ	α/γ^2
inverse gamma	$\frac{\gamma^\alpha}{x\Gamma(\alpha)} x^{-\alpha} e^{-\gamma/x}$	$\alpha - \gamma/x$	γ/α	γ^2/α^3
beta-prime	$\frac{1}{xB(p,q)} \frac{x^p}{(x+1)^{p+q}}$	$\frac{qx-p}{x+1}$	p/q	$\frac{p(p+q)^2}{q^3(p+q+1)}$
Pareto (a, ∞)	ca^c/x^{c+1}	$\frac{(c+1)(x-a)}{x} - 1$	$\frac{c+1}{c}a$	$\frac{a^2(c+1)^2}{c^3(c+2)}$
beta $(0,1)$	$\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}$	$(p+q)x - p$	$\frac{p}{p+q}$	$\frac{pq}{(p+q+1)(p+q)^2}$

Here Γ is the gamma function, B the beta function, $K(\alpha) = (\frac{K_2(\alpha)}{K_0(\alpha)} - 1)$ where K_ν is the McDonald function. If not indicated, $\mathcal{X} = (0, \infty)$.

Next three distributions in Table 2 are the distributions with support $(0, \infty)$ which are of the first type, i.e., with Johnson parameter t. The Johnson mean is the value of the Johnson parameter.

Lognormal distribution: By Table 2, $T(x; t, c) = c^2 \log(x/t)$, $T'(t) = c^2/t$. By (3.1), $\hat{t}$ is the geometric mean. By (4.2), $D(t, \hat{t}) = \hat{c} \log(t/\hat{t})$ so that

$$t \in (\hat{t}e^{-\lambda_n/\hat{c}}, \hat{t}e^{\lambda_n/\hat{c}}).$$

Since $ET^2 = 1$, it follows from (4.3) that $\hat{c}^2 = n/\sum_{i=1}^{n} \log^2(x_i/\hat{t})$.

Weibull distribution: Here $T(x; t, c) = (x/t)^c - 1$, $T'(t) = c/t$ and $D(t; \hat{t}) = (t/\hat{t})^{\hat{c}} - 1$ so that

$$t \in (\hat{t}(1-\lambda_n)^{1/\hat{c}}, \hat{t}(1+\lambda_n)^{1/\hat{c}})$$

where $\hat{t}$ and $\hat{c}$ are the maximum likelihood estimates or solutions of (3.1) and (4.3) with $ET^2 = 1$.

Hyperbolic distribution: Here $T(x;t,c) = \frac{c}{2}(\frac{x}{t} - \frac{t}{x})$ and $T'(t) = c/t$. From (3.1), $\hat{t} = \sqrt{\bar{x}\bar{x}_H}$ where $\bar{x}$ is the mean and $\bar{x}_H = n/\sum_{i=1}^n 1/x_i$ is the harmonic mean. Using ω from Table 2, we obtain condition (4.2) in the form

$$\left|t/\hat{t} - \hat{t}/t\right| \leq \lambda_n \sqrt{2/K(\hat{c})}.$$

The estimate $\hat{c}$ can be obtained from (4.3), which appears to be

$$\frac{\sum_{i=1}^n x_i^2}{n\hat{t}^2} + \frac{n\hat{t}^2}{\sum_{i=1}^n 1/x_i^2} = 2\frac{K_2(c)}{K_0(c)}.$$

The remaining distributions in Table 2 are the distributions of the second type without Johnson parameter.

Gamma distribution: Here $T(x;x^*,\gamma,\alpha) = \gamma(x - x^*)$ so that from (3.1) $\hat{x}^* = \bar{x}$. Since $D(x^*;\bar{x}) = \sqrt{\hat{\alpha}}|x^*/\bar{x} - 1|$ and $ET^2 = \alpha$, we obtain

$$x^* \in (\bar{x} - \lambda_n\hat{\rho}, \bar{x} + \lambda_n\hat{\rho}) \tag{4.4}$$

where $\hat{\rho}^2 = 1/\hat{\alpha} = \frac{1}{n}\sum_{i=1}^n x_i^2/\bar{x}^2 - 1$ using (4.3). (4.4) is the usual confidence interval for the mean (the reason is the linear Johnson score of the distribution).

Inverse gamma distribution: $T(x;x^*,\gamma,\alpha) = \alpha(1 - x^*/x)$ so that from (3.1) $\hat{x}^* = \bar{x}_H$. Since $D(x^*;\hat{x}^*) = \sqrt{\hat{\alpha}}(1 - \hat{x}_H/x^*)$ and $ET^2 = \alpha$, we obtain

$$x^* \in \left(\frac{\bar{x}_H}{1 + \lambda_n\hat{\rho}}, \frac{\bar{x}_H}{1 - \lambda_n\hat{\rho}}\right)$$

where $\hat{\rho}^2 = 1/\hat{\alpha} = (\bar{x}_H^2/\frac{1}{n}\sum_{i=1}^n 1/x_i^2 - 1)$ by (4.3).

Beta-prime distribution: $T(x;x^*,p,q) = q(x - x^*)/(x+1)$ and $T'(x^*) = q/(1+\hat{x}^*)$ so that condition (4.2) appears to be $|x^* - \hat{x}^*| \leq \lambda_n\hat{\omega}(x^* + 1)/(\hat{x}^* + 1)$ and the confidence interval is

$$x^* \in \left(\frac{\hat{x} - \lambda_n\hat{\rho}}{1 + \lambda_n\hat{\rho}}, \frac{\hat{x} + \lambda_n\hat{\rho}}{1 - \lambda_n\hat{\rho}}\right)$$

where $\hat{\rho} = \hat{\omega}/(\hat{x}^* + 1)$. Explicit formulas for $\hat{x}^*$ and $\hat{\omega}$ see Fabián (2008).

Pareto distribution: Using Table 2, the Johnson score of the Pareto distribution with support (a,∞) is $T(x;x^*,c) = (c+1)(1 - a/x) - 1 = c(1 - x^*/x)$ where $x^* = \frac{c+1}{c}a$. Making use of (3.1), $\hat{x}^* = \bar{x}_H$. Since $T'(x^*) = c/x^*$ and $ET^2 = c/(c+2)$ so that $\omega = \hat{x}^*/\sqrt{c(c+2)}$, the confidence interval for Johnson mean is

$$x^* \in \left(\frac{\bar{x}_H}{1 + \lambda_n\hat{\rho}}, \frac{\bar{x}_H}{1 - \lambda_n\hat{\rho}}\right)$$

where $\hat{\rho} = 1/\sqrt{\hat{c}(\hat{c}+2)} = \frac{1}{n}\sum_{i=1}^n(1 - x^*/x_i)^2 = \bar{x}_{2H}/(\bar{x}_H - \bar{x}_{2H})^{1/2}$ and where $\bar{x}_{2H} = n/\sum_{i=1}^n 1/x_i^2$.

Beta distribution: The Johnson mean $\hat{x}^* = p/(p+q) = \bar{x}$ from (3.1). $T(x;x^*,p,q) = (p+q)(x - x^*)$, $T'(x^*) = p + q$, $ET^2 = pq/(p+q+1)$ so that $\omega^2 = pq/(p+q+1)(p+q)^2 = \sigma^2$ and

$$x^* \in (\bar{x} - \lambda_n\hat{\sigma}, \bar{x} + \lambda_n\hat{\sigma}),$$

where $\hat{\sigma}^2$.

5 VARIANCE OF JOHNSON MEAN AND JOHNSON VARIANCE

This paper is a continuation of the paper by (Fabián, 2008) where value (3.2) was proposed to be considered a measure of variability of distributions. However, we later noticed that ω^2 of some distributions of the second type is finite. Since there is no reason for such behavior of the measure of the variability of any distribution, it seems to us now that the measure proposed by (Fabián, 2006, 2007) is the more suitable one. It is the reciprocal value of generalization of (1.6) for arbitrary distribution.

Definition 2 *Let T be the Johnson score of distribution F with support $\mathcal{X}$ and Johnson mean x^*. The value*

$$\omega_*{}^2 = \frac{1}{[\eta'(x^*)]^2 ET^2} \tag{5.1}$$

will be called a Johnson variance.

According to Proposition 1 (Fabián, 2008), for distributions of the first type it holds that $\omega_* = \omega$. This relation holds true also for some distributions of the second type, such as the gamma, inverse gamma and Burr XII distributions. However, in a general case are ω_*^2 and ω^2 different. Let us give a simple example.

EXAMPLE 5.1 Johnson variance and variance of the estimate of Johnson mean of the power distribution.

Consider the power distribution with support (0,1) and density $f(x;\alpha) = \alpha x^{\alpha-1}$. According to (1.2), its Johnson score is

$$T(x) = \frac{1}{\alpha x^{\alpha-1}}\frac{d}{dx}(-x(1-x)\alpha x^{\alpha-1}) = (\alpha+1)x - \alpha$$

so that the Johnson mean $x^* = \frac{\alpha}{\alpha+1}h$ equals to the mean. Since $\mathrm{ET}^2 = \alpha/(\alpha+2)$ and $1/\eta'(x^*) = x^*(1-x^*) = \alpha/(\alpha+1)^2$,

$$\omega_*^2 = \frac{1}{[\eta'(x^*)]^2\mathrm{ET}^2} = \frac{\alpha(\alpha+2)}{(\alpha+1)^4}.$$

On the other hand, $T(x) = (\alpha+1)(x-x^*)$ and $T'_{x^*} = -(\alpha+1)$ so that

$$\omega^2 = \frac{\mathrm{ET}^2}{(ET'_{x^*})^2} = \frac{\alpha}{(\alpha+1)^2(\alpha+2)}.$$

The variance of the estimate of the Johnson mean equals to the usual variance [cf. (Balakrishnan, Nevzorov, 2003)]. Both measures ω_*^2 and ω^2 as functions of $1/\alpha$ are compared in Fig. 1. Apart from the absolute values, the course of both curves is very similar, maxima are at $\alpha = \sqrt{2}-1$ for ω_*^2 and at $\alpha = (\sqrt{5}-1)/2$ for ω^2. Power distributions corresponding these maxima are plotted in Fig. 2.

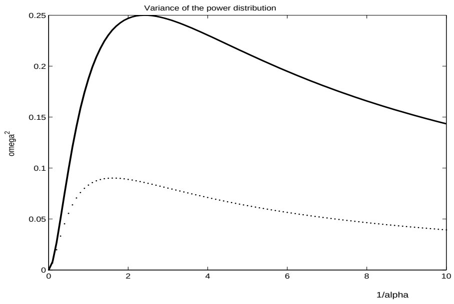

Figure 1. Johnson variance (full line) and variance (dotted line) of power distribution as functions of $1/\alpha$

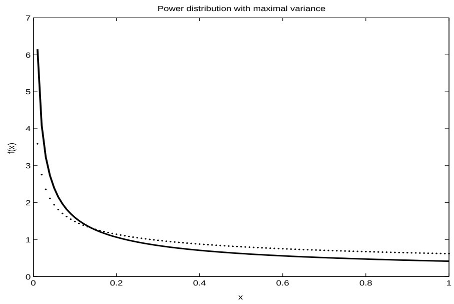

Figure 2. Power distributions with maximal Johnson variance (full line) and maximal variance (dotted line).

Other distributions of the second type with $\omega_* \neq \omega$ discussed in Fabián (2008) or in the present paper are given in Table 3.

Table 3. Johnson variance and variance of the estimate of the Johnson mean of some distributions of the second type.

Distribution	$\mathcal{X}$	$f(x)$	${\omega_*}^2$	ω^2
beta-prime	$(0,\infty)$	$\frac{1}{xB(p,q)}\frac{x^p}{(x+1)^{p+q}}$	$\frac{p(p+q+1)}{q^3}$	$\frac{p(p+q)^2}{q^3(p+q+1)}$
Fisher-Snedecor	$(0,\infty)$	$\frac{(p/q)^p}{B(p,q)}\frac{x^{p-1}}{(\frac{p}{q}x+1)^{p+q}}$	$\frac{p+q+1}{pq}$	$\frac{(p+q)^2}{pq(p+q+1)}$
Pareto	(a,∞)	ca^c/x^{c+1}	$\frac{a^2(c+1)^2(c+2)}{c^3}$	$\frac{a^2(c+1)^2}{c^3(c+2)}$
beta	$(0,1)$	$\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}$	$\frac{p}{q}\frac{p+q+1}{(p+q)^2}$	$\frac{pq}{(p+q+1)(p+q)^2}$

EXAMPLE 5.2 Confidence intervals for distributions with equal Johnson variances.

The theoretical confidence intervals (b^-, b^+) for the Johnson mean $x^* = 2$ of some distributions from Table 2 with the values of parameters chosen such that all distributions have the Johnson variance $\omega_* = 1$ are given for sample length $n = 50$ in Table 4. Except the gamma and beta distributions with usual confidence interval, Johnson confidence intervals are non-symmetric, the heavier-tailed distribution the more skewed to the right.

Table 4. Theoretical confidence intervals for $n = 50$, $\alpha = 0.05$ for some distributions with $x^* = 2, \omega_* = 1$.

Distribution	b^-	b^+
Weibull	1.7004	2.2603
gamma	1.7238	2.2772
lognormal	1.7412	2.2973
beta-prime	1.7462	2.3054
inverse gamma	1.7565	2.3218

The results of the simulation experiments are in an excellent agreement with the predicted intervals. As an example, for the beta-prime distribution ($x^* = 2, \omega_* = 1$) we obtained after the 5000 simulation experiments with sample length $n = 50$ the average value of the sample Johnson mean $\bar{x}^* = 2.0090$ with standard deviation $\hat{\sigma}_{x^*} = 0.1343$, which is equal to the theoretical value $\omega/\sqrt{50}$ (according to the formula in Table 2, $\omega = 0.95$). The average confidence interval was $(b^-, b^+) = (1.7552, 2.3143)$, shifted with respect to the predicted interval on $\hat{x}^* - x^* = 0.009$.

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