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2007

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Technical report No. 1014

December 2007

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Abstract:

In this report we define Johnson score moments and a parametric estimation method based on them. After showing basic properties of Johnson score moments we describe concept of related parametric families. The main and the last part of the report contains the overview of related two-parameter families and the corresponding explicit estimation equations.

Keywords:
Johnson score, parametric estimation, continuous distributions

[1]The research reported in this paper has been supported by the Institutional Research Plan AVOZ10300504 and partially supported by the ASO Project Nr. SK-0607-BA-018.

1 JOHNSON SCORE AND JOHNSON MEAN

It has been shown by (Fabián, 2008) that a continuous probability distribution F with interval support $\mathcal{X} \in \mathbb{R}$ can be characterized, besides the distribution function $F(x)$ and density $f(x)$, by its Johnson score, defined as follows. Mapping $\eta : \mathcal{X} \to \mathbb{R}$, where

$$\eta(x) = \begin{cases} x & \text{if} \quad (a,b) = \mathbb{R} \\ \log(x-a) & \text{if} \quad -\infty < a < b = \infty \\ \log\dfrac{(x-a)}{(b-x)} & \text{if} \quad -\infty < a < b < \infty \\ -\log(b-x) & \text{if} \quad -\infty = a < b < \infty, \end{cases} \tag{1.1}$$

is the *Johnson transformation* (Johnson, 1949) adapted for arbitrary interval support $\mathcal{X} = (a,b) \in \mathbb{R}$.

Definition 1 (Fabián 2001). The *Johnson score* of distribution F with interval support $\mathcal{X}$ and continuously differentiable density is

$$T(x) = \frac{1}{f(x)}\frac{d}{dx}\left(-\frac{1}{\eta'(x)}f(x)\right) \tag{1.2}$$

where η is given by (1.1).

The philosophy behind this concept is the following. Any distribution F with interval support $\mathcal{X} \neq \mathbb{R}$ is viewed as a transformed '*prototype*' G with support $\mathbb{R}$, that is, its distribution function is $F(x) = G(\eta(x))$. Denoting by g the density of G, the density of F is

$$f(x) = g(\eta(x))\eta'(x), \qquad x \in \mathcal{X}, \tag{1.3}$$

where $\eta'(x) = d\eta(x)/dx$ is the Jacobian of the transformation. Let us denote the score function of G by Q so that

$$Q(y) = -\frac{g'(y)}{g(y)}.$$

By setting $y = \eta(x)$ we obtain from (1.2) and (1.3)

$$T(x) = \frac{1}{g(y)\eta'(x)}\frac{d}{dy}(-g(y))\frac{dy}{dx} = Q(\eta(x)). \tag{1.4}$$

By (1.1), the Johnson score of a prototype (a distribution with support $\mathbb{R}$) is its score function. By (1.4), the Johnson score of a distribution with interval support $\mathcal{X} \neq \mathbb{R}$ is the transformed score function of its prototype. By (1.4) is defined a unique and useful scalar inference function for arbitrary continuous distribution F satisfying the usual regularity conditions. We note that the Johnson transformation was chosen not only due to its mathematical convenience for many distributions used in statistics, but also on the base of other reasons discussed in (Fabián, 2008).

A unique solution x_* of equation

$$T(x) = 0 \tag{1.5}$$

we call a *Johnson mean* of distribution F. Due to (1.4), the solution of (1.5) is unique if G is unimodal. Confining ourselves to distributions with unimodal prototypes, the Johnson mean was shown to characterize the typical value of distributions including the heavy-tailed distributions without mean, being a value near the mean of the light-tailed ones.

From the point of view of the structure of the parameters, there are two different types of distributions with support $\mathcal{X} \neq \mathbb{R}$:

i/ Distributions of the first type are the transformed distributions, prototypes of which have the location parameter μ. These distributions have a parameter

$$t = \eta^{-1}(\mu),$$

called a *Johnson parameter*, the value of which is the Johnson mean of the distribution. Denoting by $f(x,t)$ the density and $T(x;t)$ the Johnson score of a distribution of this type, it was shown that function

$$S(x;t) = \eta'(t)T(x;t)$$

equals to the likelihood score $l(x;t) = (\partial/\partial t)\log f(x;t)$ for parameter t. The value

$$I(t) = ES^2 = [\eta'(t)]^2 ET^2 \tag{1.6}$$

thus appears to be the Fisher information for the Johnson parameter. An example is the exponential distribution with density $f(x;\lambda) = \lambda^{-1}e^{-x/\lambda}$ and Johnson score $T(x;\lambda) = x/\lambda - 1$ with Johnson parameter λ and Fisher information $I(\lambda) = 1/\lambda^2$.

ii/ Distributions of the second type are the transformed distributions with prototypes without location parameter. The Johnson score of them appears to be a new function and the Johnson mean, a function $x_* = x_*(\theta)$ of the parameters, is a new characteristic of their central tendency. For some two-parameter distributions, x_* is the ratio of the parameters. For example, the gamma distribution with density $f(x;\alpha,\gamma) = \frac{\gamma^\alpha}{x\Gamma(\alpha)}x^\alpha e^{-\gamma x}$ and Johnson score $T(x;\alpha,\gamma) = \gamma x - \alpha$ has Johnson mean $x_* = \alpha/\gamma$.

2 JOHNSON SCORE MOMENTS AND JOHNSON VARIANCE

Definition. (Fabián 2001). Moments of the Johnson score T of distribution F with support $\mathcal{X}$

$$M_k = ET^k = \int_{\mathcal{X}} T(x)^k \, dF(x), \qquad k = 1, 2, ...$$

will be called *Johnson score moments.*

Proposition. *The $k-$th order Johnson score moment of transformed distribution*

$$F(x) = G(\eta(x)) \tag{2.1}$$

coincides with the $k-$th order score moment of its prototype G.

Proof. Making use of (2.1) and (1.4),

$$\int_{\mathcal{X}} T^k(x)\, dF(x) = \int_{\mathcal{X}} Q^k(\eta(x))\, dG(\eta(x)) = \int_{-\infty}^{\infty} Q^k(y)\, dG(y).$$

□

It follows from (1.4) that if $g(y) = O(e^{-y})$ then $Q(y) = O(1)$. The Johnson scores of heavy-tailed distributions are bounded. Although the usual moments $m_k = \int x^k dF(x)$ of many heavy-tailed distributions do not exist or do exist only within a certain range of parameters, the existence of the corresponding Johnson score moments is obvious.

Definition 2 (Fabián 2006, 2008). Let T be the Johnson score of distribution F with support $\mathcal{X}$ and Johnson mean x_*. The reciprocal value of generalization of (1.6) for arbitrary distribution,

$$\omega_*{}^2 = \frac{1}{[\eta'(x_*)]^2 ET^2} \tag{2.2}$$

is called *Johnson variance.*

It was shown that in Fabián (2006, 2007, 2008) that (2.2) appears to be a suitable description of the variability of distributions both heavy-tailed (the usual variance of which do not exist) and light-tailed ones.

In cases of the three most encountered supports, Johnson variance is

$$\omega_*^2 = \frac{1}{ET^2} \cdot \begin{cases} 1 & \text{if} \quad \mathcal{X} = \mathbb{R} \\ x_*^2 & \text{if} \quad \mathcal{X} = (0,\infty) \\ x_*^2(1-x_*^2) & \text{if} \quad (0,1). \end{cases} \tag{2.3}$$

3 JOHNSON SCORE MOMENT ESTIMATES

Let $\Theta \subset \mathbb{R}$ and $\mathbb{X}_n = (X_1, ..., X_n)$ be a random sample of size n with components i.i.d. by F_{θ_0} from a general parametric family $\mathcal{F} = \{F_\theta(x) : \theta \in \Theta\}$. We are interested in estimators of the true parameter $\theta_0 \in \Theta$, i.e. in sequences of R^m-valued measurable mappings $\hat{\theta}_n = \hat{\theta}_n(X_1, ..., X_n)$. The maximum likelihood method provides m likelihood scores, whereas the Johnson score is a unique function. Nevertheless, it can be used for the estimation of m-dimensional parameter θ using a modification of the moment method, utilizing the natural estimators of the corresponding Johnson score moments of the empirical distribution function $F_n(x) = n^{-1} \sum_{i=1}^n \delta(x_i \leq x)$.

Definition. *Let T_{F_θ} be the Johnson score of a distribution F_θ regular in the usual sense* (e.g. Serfling (1980, pp. 144) *on support $\mathcal{X}$ and $M_k(\theta)$, $k = 1, ..., m$ be the corresponding moments. Define the Johnson score moment estimate as a $(X_1, ..., X_n)$-measurable solution of systems of equations*

$$\hat{\theta}_n : \qquad \frac{1}{n}\sum_{i=1}^n T_{F_\theta}^k(X_i) = M_k(\theta), \qquad 1 \leq k \leq m. \tag{3.1}$$

Obviously, the system (3.1) can be written in the equivalent form

$$\int_{\mathcal{X}} \Pi_{F_\theta}(x)\, dF_n(x) = 0,$$

where, denoting by comma the matrix transpose,

$$\Pi_{F_\theta}(x)' = [\Pi_{k,\theta}(x)]' = [T_{F_\theta}(x)^k - M_k(\theta)]' : 1 \leq k \leq m).$$

Estimate (3.1) is an M-estimate so that we can immediately state the following theorem:

Theorem 1 (Fabián, 2001). *Let T_{F_θ} and $M_k(\theta)$ be continuously differentiable according to θ_j. Let for each $\theta \in \Theta$ exist finite $D_{j,k}(\theta), j, k = 1, ..., m$, where*

$$D_{j,k}(\theta) = \int_{\mathcal{X}} \frac{\partial \Pi_{k,\theta}(x)}{\partial \theta_j}\, dF_\theta(x).$$

Set $\mathbf{D}(\theta) = [D_{jk}(\theta)]_{j,k=1,...,m}$. Let Det $\mathbf{D}(\theta) \neq 0$. Then $\hat{\theta} = (\hat{\theta}_n)$ exists, is strongly consistent and asymptotically normal with the asymptotic variance-covariance matrix

$$\mathbf{B}(\hat{\theta}_n) = \mathbf{D}^{-1}(\theta_0)\mathbf{Q}(\mathbf{D}^{-1}(\theta_0))' \tag{3.2}$$

where $\mathbf{Q} = \int_{\mathcal{X}} \Pi_{\theta_0}(x)\Pi_{\theta_0}(x)'\, dF_{\theta_0}(x)$.

In the case of the member of the family with bounded Johnson score the Johnson score moment estimates are robust (B-robust in the sense of Hampel et al. (1986), whereas the maximum likelihood estimates are often sensitive to outliers.

4 ESTIMATES OF THE JOHNSON MEAN AND JOHNSON VARIANCE

In this report we study estimates of Johnson mean and Johnson variance of two-parameter distributions. The estimates of x_* and ω_*^2 can be constructed from either the maximum likelihood estimate or the Johnson moment estimate $\hat{\theta}$ of vector θ by setting $\hat{x}^* = x_*(\hat{\theta})$ and $\hat{\omega}^* = \omega(\hat{\theta})$. However, we showed (Fabián, 2008) that in some particular cases x_* can be estimated directly from the first or first two equations of the system (3.1)

$$\sum_{i=1}^n T(x_i;\theta) = 0 \tag{4.1}$$

$$\frac{1}{n}\sum_{i=1}^n T^2(x_i;\theta) = ET^2. \tag{4.2}$$

In some particular cases, the equation (4.1) can be written in a form

$$\sum_{i=1}^{n} T(x_i; x_*) = 0. \tag{4.3}$$

Let us denote by

$$\omega^2 = \frac{ET^2}{(ET'_{x_*})^2} \tag{4.4}$$

with derivative according to the *virtual parameter* x^*. Let AN means, as usually, 'asymptotically normal'. By Proposition 2, Fabián (2008), the estimate of $\hat{x}^*$ of x_* from (4.3) is $AN(x_*, \omega^2)$.

5 RELATED PARAMETRIC FAMILIES

In this section we give an overview of related parametric families, consisting of the prototype G with support $\mathbb{R}$ and all transformed families $F = G\eta^{-1}$ on arbitrary interval supports (a, b). The densities and Johnson scores of the families are given in general forms $\lambda f(u)$ and $T(u)$, where λ and u are specific particular supports. By the use of the proper λ and u, described in every subsection, one can easily obtain the explicit formulas for the concrete family with arbitrary support.

5.1 Related families with the prototype with location parameter

The explicit expressions for distributions with supports $\mathbb{R}, (a, \infty)$ and (a, b) can be obtained from the general forms described in this subsection by setting

$$u = \begin{cases} e^{c(y-\mu)} & \text{if} \quad \mathcal{X} = \mathbb{R} \\ \left(\frac{x-a}{\tau-a}\right)^c & \text{if} \quad \mathcal{X} = (a, \infty) \\ \left(\frac{(z-a)(b-\nu)}{(\nu-a)(b-z)}\right)^c & \text{if} \quad \mathcal{X} = (a, b) \end{cases}$$

and

$$\lambda = \begin{cases} c & \text{if} \quad \mathcal{X} = \mathbb{R} \\ \frac{c}{x-a} & \text{if} \quad \mathcal{X} = (a, \infty) \\ \frac{c(b-a)}{(z-a)(b-z)} & \text{if} \quad \mathcal{X} = (a, b) \end{cases} \tag{5.1}$$

where μ is the location parameter and

$$c = 1/\sigma$$

is the reciprocal scale. From now on we will consider for simplicity the partial supports $\mathbb{R}$, $(0, \infty)$ and $(0, 1)$ only. The Johnson mean equals to the Johnson parameter

$$\eta^{-1}(\mu) = \begin{cases} \tau = e^{\mu} & \text{if} \quad \mathcal{X} = (0, \infty) \\ \nu = \frac{e^{\mu}}{1+e^{\mu}} & \text{if} \quad \mathcal{X} = (0, 1) \end{cases} \tag{5.2}$$

and the Fisher information of distributions is $I^* = [\eta'(x_*)]^2 ET^2$ where $ET^2 = c^2\delta$. The Johnson variance is

$$\omega_*^2 = \begin{cases} \frac{1}{\delta c^2} & \text{if} \quad \mathcal{X} = \mathbb{R} \\ \frac{1}{\delta c^2}\tau^2 & \text{if} \quad \mathcal{X} = (0, \infty) \\ \frac{1}{\delta c^2}\frac{\nu^2}{(1-\nu)^2} & \text{if} \quad \mathcal{X} = (0, 1) \end{cases}$$

Normal family

$$f(x) = \frac{\lambda}{\sqrt{2\pi}} e^{-\frac{1}{2}\log^2 u} \qquad T(x) = c\log u$$

The distributions of the triplet with supports $\mathcal{X} = \mathbb{R}, (0, \infty)$ and $(0, 1)$ are the normal, lognormal and Johnson U_B distributions. $\delta = 1$. Johnson mean and Johnson variance of the normal distribution are identical with the usual mean and variance.

Gumbel family

$$f(u) = \lambda u e^{-u} \qquad T(u) = c(u-1)$$

This is a non-symmetric family skewed to the right with unbounded Johnson scores in the right part and bounded in the left part of the support. The members with supports $\mathcal{X} = \mathbb{R}$ and $(0,\infty)$ are the Gumbel and Weibull distributions, the latter with special forms Rayleigh and Maxwell distributions. The member on $(0,1)$ has no name. $\delta = 1$.

Extreme value family

$$f(u) = \lambda \frac{1}{u} e^{-1/u} \qquad T(u) = c(1-1/u)$$

This is a non-symmetric family of distributions skewed to the left with unbounded Johnson scores in the left part and bounded in the right part of the support. The prototype of the family is symmetric according to the axis $x = 0$ with the prototype of the Gumbel family. The members with supports $\mathcal{X} = \mathbb{R}$ and $(0,\infty)$ are the extreme value and Fréchet distributions, the member on $(0,1)$ has no name. $\delta = 1$.

Logistic family

$$f(u) = \frac{u}{(1+u)^2} \qquad T(u) = c\frac{u-1}{u+1}$$

Distributions of the logistic family have the heavy-tailed densities and bounded Johnson scores. The members with supports $\mathcal{X} = \mathbb{R}$ and $(0,\infty)$ are the logistic and log-logistic distributions, the member on $(0,1)$ has no name. $\delta = 1/3$.

Cauchy family

$$f(u) = \frac{\lambda}{\pi}\frac{1}{1+u^2} \qquad T(u) = c\frac{2u}{1+u^2}$$

Distributions of the Cauchy family have heavy-tailed densities and redescending Johnson scores. The members with supports $\mathcal{X} = \mathbb{R}$ and $(0,\infty)$ are the Cauchy and log-Cauchy distributions, the member on $(0,1)$ has no name. $\delta = 1/2$.

Hyperbolic family

$$f(u) = \frac{\lambda}{2K_0(1)} e^{-\frac{1}{2}(u+1/u)} \qquad T(u) = \frac{c}{2}(u-1/u)$$

Distributions of the hyperbolic family have very light-tailed densities and unbounded Johnson scores. From this extended family, only the hyperbolic distribution with support $(0,\infty)$ is used. $\delta = 2K_2(1)/K_0(1) - 1 = K$, say, where K_β is the McDonald function.

5.2 Related families with prototypes without location

In this subsection we give an overview of general forms $\lambda f(u)$ and $T(u)$ of densities and Johnson scores of parametric families with prototypes given by parametric forms without location parameter. The explicit expressions for the triplet of distributions with supports $\mathbb{R}, (0,\infty)$ and $(0,1)$ can be obtained from the general forms by setting

$$u = \begin{cases} e^y & \text{if} \quad \mathcal{X} = \mathbb{R} \\ x & \text{if} \quad \mathcal{X} = (0,\infty) \\ \frac{z}{1-z} & \text{if} \quad \mathcal{X} = (0,1) \end{cases}$$

and

$$\lambda = \begin{cases} 1 & \text{if} \quad \mathcal{X} = \mathbb{R} \\ \frac{1}{x} & \text{if} \quad \mathcal{X} = (0,\infty) \\ \frac{1}{z(1-z)} & \text{if} \quad \mathcal{X} = (0,1) \end{cases}$$

We give explicit expressions for the Johnson mean and Johnson variances of triplets of distributions.

Gamma family

$$f(u) = \frac{\gamma^\alpha}{\lambda\Gamma(\alpha)} u^\alpha e^{-\gamma u} \qquad T(u) = \gamma u - \alpha$$

This is a non-symmetric family skewed to the right with unbounded Johnson scores in the right part and bounded in the left part of the support. The member with support $\mathcal{X} = (0, \infty)$ is the gamma distribution with a special form the chi-squared distribution, $ET^2 = \alpha$. The Johnson mean and Johnson variance of the triplet are

prototype gamma	$y_* = \log(\alpha/\gamma)$	$\omega_*^2 = 1/\alpha$
gamma	$x_* = \alpha/\gamma$	$\omega_*^2 = \alpha/\gamma^2$
gamma on $(0,1)$	$z_* = \alpha/(\alpha+\gamma)$	$\omega_*^2 = \alpha/(\gamma+\alpha)^2$

Inverse gamma family

$$f(u) = \frac{\lambda\gamma^\alpha}{\Gamma(\alpha)} u^{-\alpha} e^{-\gamma/u} \qquad T(u) = \alpha - \gamma/u$$

This is a non-symmetric family skewed to the right, heavy-tailed in the right part of the support. The prototype of the family is symmetric according to the axis $x = 0$ with the prototype of the gamma family. However, only the inverse gamma distribution with support $(0, \infty)$ is used. The Johnson mean and Johnson variance are

prototype inv. gamma	$y_* = \log\gamma/\alpha$	$\omega_*^2 = 1/\alpha$
inverse gamma	$x_* = \gamma/\alpha$	$\omega_*^2 = \gamma^2/\alpha^3$
inv. gamma on $(0,1)$	$z_* = \gamma/(\gamma+\alpha)$	$\omega_*^2 = \gamma^2/\alpha(\gamma+\alpha)^2$

Beta family

$$f(u) = \frac{\lambda}{B(p,q)} \frac{u^p}{(u+1)^{p+q}} \qquad T(u) = \frac{qu-p}{u+1}$$

is a heavy-tailed family. Its prototype is known as the generalized logistic of the third kind (Balakrishnan, Nevzorov, 2003) and distribution on $(0, \infty)$ as the beta-prime or the beta of the second kind. By setting u and λ into above expressions we obtain the density of the beta distribution in the form $f(x) = B(p,q)^{-1} z^{p-1}(1-z)^{q-1}$ and a linear Johnson score $T(x) = (p+q)x - p$. $ET^2 = pq/(p+q+1)$. The Johnson mean and Johnson variance of distributions of the beta triplet are

prototype beta	$y_* = \log p/q$	$\omega_*^2 = \frac{p+q+1}{pq}$
beta-prime	$x_* = p/q$	$\omega_*^2 = \frac{p(p+q+1)}{q^3}$
beta	$z_* = \frac{p}{p+q}$	$\omega_*^2 = \frac{p(p+q+1)}{q(p+q)^2}$

For $p = q = 1$ we have the it uniform distribution with $z_* = 1/2$ and $\omega_*^2 = 3/4$.

Fisher-Snedecor family

$$f(u) = \frac{\lambda(p/q)^p}{B(p,q)} \frac{u^{p-1}}{(\frac{p}{q}u+1)^{p+q}} \qquad T(u) = \frac{p(u-1)}{pu/q+1}$$

is a variant of the beta family. $ET^2 = pq/(p+q+1)$. Johnson means and Johnson variances of the Fisher-Snedecor triplet are

prototype	$y_* = 0$	$\omega_*^2 = \frac{p+q+1}{pq}$
Fisher-Snedecor	$x_* = 1$	$\omega_*^2 = \frac{(p+q+1)}{pq}$
Fisher-Snedecor on $(0,1)$	$z_* = 1/2$	$\omega_*^2 = 4\frac{(p+q+1)}{pq}$

Burr family

$$f(u) = \frac{\lambda kcu^c}{(u^c+1)^{k+1}} \qquad T(u) = c\frac{ku^c-1}{u^c+1}$$

heavy-tailed family with the used member on support $(0, \infty)$. $ET^2 = k/(k+2)$. Johnson means and Johnson variances of the Burr triplet are

prototype Burr	$y_* = c^{-1}\log(1/k)$	$\omega_*^2 = \frac{k+2}{c^2 k}$
Burr	$x_* = (1/k)^{1/c}$	$\omega_*^2 = \frac{k+2}{c^2 k^{2/c+1}}$
Burr on $(0,1)$	$z_* = \frac{1}{k^{1/c}+1}$	$\omega_*^2 = \frac{k+2}{c^2 k(k^{1/c}+1)^2}$

5.3 Other families

In this subsection we present some families with specific support without attempts to generalize them to obtain the whole related family. The families are thus described by the full forms of the densities and Johnson scores.

Pareto family has support (a, ∞) and density

$$f(x) = \frac{ca^c}{x^{c+1}}.$$

By (1.2),

$$T(x) = -1 - (x-a)f'(x)/f(x) = c(1 - x_*/x)$$

where $x_* = a(c+1)/c$ is the Johnson mean. Since $ET^2 = c/(c+2)$ and $\eta'(x^*) = (\log(x-a))'|_{x=x^*} = 1/(x^* - a)$,

$$\omega_*^2 = \frac{a^2(c+2)^2}{c^2}\frac{1}{c^2}\frac{(c+2)^2}{c^2} = \frac{a^2(c+2)}{c^3}.$$

American version of the Pareto family with support $(0, \infty)$ has densities and Johnson scores is

$$f(x) = \frac{\alpha\lambda^\alpha}{(x+\lambda)^{\alpha+1}} \qquad T(x) = \frac{\alpha x - \lambda}{x + \lambda}$$

with Johnson mean $x_* = \lambda/\alpha$. It in fact belongs to distributions described in previous subsection. $ET^2 = \alpha/(\alpha+2)$ so that by (4.4) $\omega_*^2 = \lambda^2(\alpha+2)/\alpha^3$.

Log-gamma family with support $\mathcal{X} = (1, \infty)$ is given by

$$f(x) = \frac{c^\alpha}{\Gamma(\alpha)}(\log x)^{\alpha-1}x^{-(c+1)}. \tag{5.3}$$

In this case is better to use mapping $\eta(x) : (1, \infty) \to \mathbb{R}$

$$\eta(x) = \log(\log x).$$

Since $\eta'(x) = 1/(x \log x)$, by (1.2)

$$T(x) = \frac{1}{f(x)}\frac{d}{dx}\left(-(\log x)^c x^{-\alpha}\right) = c\log x - \alpha$$

so that the 'loglog' mean is $x_* = e^{\alpha/c}$. As the 'second log-log moment' $ET^2 = E[c^2\log^2(x/x^*)] = \alpha$, $\omega^2 = x_*^2(\log x_*)^2/ET^2 = \frac{\alpha}{c^2}e^{2\alpha/c}$.

6 ESTIMATION EQUATIONS

In the present section we give explicit Johnson score equations for estimation of the parameters. Usually, they are to be solved, similarly as the maximum likelihood equations, by an iterative procedure. However, in some cases are the Johnson mean and Johnson variance expressed in closed formulas.

The equations are given for one particular member of the related family. Parameters of other members can be estimated after the transformation of the data to the considered case. For instance, the data $(y_1, ..., y_n)$ assumed to be taken from the logistic distribution is to transform to $(x_1, ..., x_n)$ where $x_j = \exp(y_j)$ and $\hat{y}_* = \log \hat{x}_*$ where $\hat{x}_*$ is the estimated Johnson mean of the log-logistic distribution, and $\hat{\omega} = 1/\hat{c}$.

6.1 Distributions with prototype with location parameter

Normal distribution

$$\begin{aligned}\hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} y_i \\ \hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{\mu})^2\end{aligned}$$

Weibull distribution

$$\begin{aligned}\hat{\tau}^c &= \frac{1}{n}\sum_{i=1}^{n} x_i^c \\ \hat{\tau}^{2c} &= \frac{1}{2n}\sum_{i=1}^{n} x_i^{2c}\end{aligned}$$

Fréchet distribution

$$\begin{aligned}1/\hat{\tau}^c &= \frac{1}{n}\sum_{i=1}^{n} 1/x_i^c \\ 1/\hat{\tau}^{2c} &= \frac{1}{2n}\sum_{i=1}^{n} 1/x_i^{2c},\end{aligned}$$

Log-logistic distribution

$$\begin{aligned}\sum_{i=1}^{n}\frac{(x_i/\tau)^c - 1}{(x_i/\tau)^c + 1} &= 0 \\ \frac{1}{n}\sum_{i=1}^{n}\left(\frac{(x_i/\tau)^c - 1}{(x_i/\tau)^c + 1}\right)^2 &= \frac{1}{3}\end{aligned}$$

Cauchy distribution

$$\begin{aligned}\sum_{i=1}^{n}\frac{y_i - \mu}{1 + ((y_i - \mu)/\sigma)^2} &= 0 \\ \frac{1}{n}\sum_{i=1}^{n}\left(\frac{2(y_i - \mu)/\sigma)}{(1 + (y_i - \mu)/\sigma)^2)}\right)^2 &= \frac{1}{2}\end{aligned}$$

Hyperbolic distribution

$$\begin{aligned}\tau^{2c} &= \sum_{i=1}^{n} x_i^c / \sum_{i=1}^{n} 1/x_i^c \\ \tau^{4c}\sum_{i=1}^{n} 1/x_i^{2c} &= \sum_{i=1}^{n} x_i^{2c} + (\delta + 2)\sum_{i=1}^{n} x_i^c / \sum_{i=1}^{n} 1/x_i^c\end{aligned}$$

where δ is given in Section 5.1.

6.2 Distributions with prototype without location parameter

Gamma distribution

$$\begin{aligned}\sum_{i=1}^{n}(\gamma x_i - \alpha) &= 0 \\ \frac{1}{n}\sum_{i=1}^{n}(\gamma x_i - \alpha)^2 &= \alpha\end{aligned}$$

from which $\hat{x}^* = \bar{x}$ and $\hat{\omega}_*^2 = \frac{1}{n}\sum_{i=1}^n x_i^2 - \bar{x}^2$.

Inverse gamma distribution

$$\begin{aligned}\sum_{i=1}^n (\alpha - \gamma/x_i) &= 0\\ \frac{1}{n}\sum_{i=1}^n (\alpha - \gamma/x_i)^2 &= \alpha\end{aligned}$$

from which $1/\hat{x}^* = \frac{1}{n}\sum_{i=1}^n 1/x_i = \bar{x}_H$ so that $\hat{x}^*$ is the harmonic mean. From the second equation $1/\alpha = \bar{x}_H^2/\bar{x}_{2H} - 1$ where $1/\bar{x}_{2H} = \frac{1}{n}\sum_{i=1}^n 1/x_i^2$. so that $\omega_*^2 = \bar{x}_H^2(\bar{x}_H^2/\bar{x}_{2H}^2 - 1)$.

Beta distribution

$$\begin{aligned}\sum_{i=1}^n [(p+q)z_i - p] &= 0\\ \frac{1}{n}\sum_{i=1}^n [(p+q)z_i - p]^2 &= \frac{pq}{p+q+1}\end{aligned}$$

from which follow the explicit formulas $z_* = \bar{z}$ and, denoting by $z_2 = \frac{1}{n}\sum_{i=1}^n z_i^2$ and $\xi = (z_2 - \bar{z})/(\bar{z}^2 - z_2)$, $p = \bar{z}\xi$ and $q = (1-\bar{z})\xi$.

Pareto distribution

$$\hat{x}^* = \bar{x}_H$$

from (4.1). For a fixed a, $c = a/(\hat{x}^* - a)$.

Emerican version of the Pareto distribution

$$\begin{aligned}\sum_{i=1}^n \frac{\alpha x_i - \lambda}{x_i + \lambda} &= 0\\ \frac{1}{n}\sum_{i=1}^n \left(\frac{\alpha x_i - \lambda}{x_i + \lambda}\right)^2 &= \frac{\alpha}{\alpha+2}\end{aligned}$$

Log-gamma distribution

$$\begin{aligned}\sum_{i=1}^n \log x_i &= \alpha/c\\ \frac{1}{n}\sum_{i=1}^n (c\log x_i - \alpha)^2 &= \alpha\end{aligned}$$

Denote by $s_1 = \frac{1}{n}\sum_{i=1}^n \log x_i$ and $s_2 = \frac{1}{n}\sum_{i=1}^n \log^2 x_i$. From the first equation we obtain the sample Johnson mean $\hat{x}^* = e^{s_1}$ and from the second one $1/\hat{\alpha} = s_2/s_1^2 - 1$ so that $\hat{\omega}_*^2 = x_*^2\alpha/c^2 = (s_2 - s_1^2)e^{2s_1}$.

7 EXAMPLE

In a simulation study, samples of length 100 were generated consecutively from distribution listed in rows of the following table, each with values of θ giving $x_*(\theta) = 1$ and $\omega_*^2(\theta) = 1.2$. Both x_* and ω_* were estimated under the assumption of either distribution listed in headlines of columns of the table. In the table are summarized average values of the estimated Johnson means and the square root of the Johnson variance over 5000 samples.

$\hat{x}^*$	gamma	Weibull	lognorm.	beta-pr.	inv. gamma
gamma	1.000	0.94	0.60	0.49	0.12
Weibull	1.06	1.005	0.64	0.53	0.15
lognormal	1.66	1.66	1.010	1.01	0.63
beta-prime	2.00	1.77	1.01	1.008	0.54
inv. gamma	84.4	4.71	1.70	2.13	1.022
$\hat{\omega}_*$					
gamma	1.094	1.06	0.81	0.72	0.31
Weibull	1.17	1.108	0.83	0.75	0.39
lognormal	2.04	1.62	1.082	1.09	0.74
beta-prime	3.52	2.00	1.11	1.113	0.82
inv.gamma	187.	8.52	2.32	3.23	1.117

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