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Kramosil, Ivan
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Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:ics@cs.cas.cz

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Abstract:

Shannon entropy has been proved to be a very useful and powerful tool when quantifying and processing the total amount of randomness contained in a probability distribution. Hence, the effort to suggest and analyze a similar notion for real-valued and lattice-valued possibilistic distributions seems to be quite legitimate and worth being pursued. In this paper we investigate real-valued possibilistic distributions the values $\pi(\omega)$ of which and their complements $1 - \pi(\omega)$ are processed as elements of the complete lattice $\langle [0,1], \leq \rangle$, so that only the operations and relations primary or definable within the framework of complete lattices may be applied. Having proposed an entropy function suitable for this case, the main result obtained below reads that the entropy value of possibilistically independent products of a system of particular real-valued possibilistic distributions is defined by the supremum of entropy values ascribed to particular possibilistic distributions no matter how large the number of distributions in the system of distributions under consideration may be.

Keywords:
possibilistic distribution, possibilistic measure complete lattice, lattice-like processed real-valued possibilistic entropy measures, product of possibilistic distributions

[1]Institute of Computer Science, v.v.i., Academy od Sciences of the Czech Republic, Pod Vodárenskou věží 2, 182 07 Prague 8, Czech Republic, e-mail: kramosil@cs.cas.cz

1 Introduction, Motivation, Preliminaries

Fuzzy sets or, more correctly, real-valued normalized fuzzy subsets of a nonempty space, were conceived by L. A. Zadeh in [10] in a way which can be mathematically formalized by mappings $\pi : \Omega \to [0,1]$ such that $\bigvee_{\omega \in \Omega} \pi(\omega) = 1$, here and below we write $\bigvee, \vee (\bigwedge, \wedge$, resp.) for the supremum and infimum operations induced in $[0,1]$ by the standard linear ordering of real numbers. The total amount of uncertainty (in the sense of fuzziness or vagueness) related to the fuzzy subset π of Ω and contained in a crisp subset A of Ω was defined, again by L. A. Zadeh, in [11] as the real number $\Pi(A) = \bigvee_{\omega \in \Omega} \pi(\omega)$. Due to the elementary properties of real numbers this value is defined for each $\emptyset \neq A \subset \Omega$ and belongs to $[0,1]$, by convention we set $\Pi(\emptyset) = 0$. The total mapping $\Pi : \mathcal{P}(\Omega) \to [0,1]$ is called the *(normalized real-valued) possibility (or possibilistic) measure* induced by π and π itself is called, in this context, *(normalized real-valued) possibilistic distribution* on Ω.

In spite of the operations of addition, series taking and standard integration, playing the main role when processing probability measures, rather the operations of supremum and infimum are the most often occurring ones when processing possibilistic distributions and possibilistic measures by mathematical tools. Because of the well-known fact that supremum and infimum operations can be defined, at least as partial operations, also in some rather general non-numerical structures, e.g., in partially ordered sets (posets), the idea to consider fuzzy sets with non-numerical degrees of fuzziness (degrees of membership) ascribed to elements of the space Ω emerged naturally and rather soon, being mathematically formalized and analyzed, for the first time, by J. A. Goguen in [5]. In order to simplify the mathematical formalization and processing, J. A. Goguen introduced lattice-valued fuzzy subsets of Ω, defined by mappings $\pi^* : \Omega \to T$ such that $\bigvee_{\omega \in \Omega}^{\mathcal{T}} \pi^*(\omega) = \mathbf{1}_{\mathcal{T}}$. Here $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ denotes a *complete lattice,* i.e., a partially ordered set $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ such that the supremum and infimum values $\bigvee^* A$ and $\bigwedge^* A$ are defined in T for each $A \subset T$ (in particular, $\mathbf{1}_{\mathcal{T}} = \bigvee^{\mathcal{T}} T$ and $\oslash_{\mathcal{T}} = \bigvee^{\mathcal{T}} \emptyset (= \bigwedge_{\mathcal{T}} T)$). The aim of this simplification is to eliminate the necessity to prove the definition of $\bigvee^{\mathcal{T}} A$ and/or $\bigwedge^{\mathcal{T}} A$, if it is the case, or to suppose explicitly the existence of these values otherwise. As in the case of real-valued fuzzy sets, $\mathcal{T}$-(valued) fuzzy subset π^* of Ω can be taken as (and called by) $\mathcal{T}$-*(valued) possibilistic distribution* on Ω which defines, setting $\Pi^*(A) = \bigvee^{\mathcal{T}} \{\pi^*(\omega) : \omega \in A\}$ for each $A \subset \Omega$, $\mathcal{T}$–(valued) possibilistic measure Π^* on $\mathcal{P}(\Omega)$.

A more detailed discussion concerning ontological and gnoseological problems related to non-numerically valued fuzzy sets can be found in [5, 2], or elsewhere, [2] can be recommended as an excellent survey of mathematical theory and deep results dealing with lattice-valued fuzzy sets and possibilistic measures induced by them.

2 Towards Lattice-Valued Entropy Function Induced by Possibilistic Distributions

In 1948, C. E. Shannon [8] introduced the notion of entropy as a quantity defining the total amount of uncertainty (in the sense of randomness) contained in a probability distribution $p(\omega_1), p(\omega_2), \dots$ over a finite or countable space $\Omega = \{\omega_1, \omega_2, \dots\}$; let us recall that this entropy, denoted by $H_S(p)$, was defined by

$$H_S(p) = -\sum_{i=1}^{\infty} (\log_2(p(\omega_i)))p(\omega_i). \tag{2.1}$$

The notion of Shannon entropy has been proved to play an important role in information theory and statistical decision functions theory based on Kolmogorov axiomatic theory. In [7] we tried, given a complete lattice $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ and a nonempty space Ω, and taking a $\mathcal{T}$-fuzzy subset of Ω as a $\mathcal{T}$-possiblistic distribution over Ω, to propose a $\mathcal{T}$-valued entropy function ascribing to each such distribution a value from T in a way as close as possible, under the conditions and constraints introduced below, to the Shannon entropy function. The strict restrictive ontological conditions read that only the values from the support set T of the complete lattice $\mathcal{T}$ in question and only the operations and relations either primary in complete lattices or definable on the ground of the primary ones may be taken as formal tools when defining and processing the proposed $\mathcal{T}$-valued entropy function. Hence, no new and ontologically independent relations and operations (as it is, e.g., the case of operation of complement leading from lattices to Boolean algebras) and no mappings of the set T into real numbers or into other formalized structures will be taken into consideration. Let us note that both the most often used mathematical structures for quantification, the already mentioned structure $\langle [0,1], \leq \rangle$, as well as the power-set $\mathcal{P}(X)$ of a fixed nonempty set X partially ordered by the relation of set inclusion, are obviously complete lattices.

Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be a complete lattice, let π be a $\mathcal{T}$-possibilistic distribution on a nonempty space Ω with Π denoting the corresponding possibilistic measure on $\mathcal{P}(\Omega)$, let f be a mapping which takes Ω into T. The value

$$\oint f d\Pi = \bigvee_{t \in T}^{\mathcal{T}} [t \wedge_{\mathcal{T}} \Pi(\{\omega \in \Omega : f(\omega) \geq_{\mathcal{T}} t\})] \tag{2.2}$$

is called the *(Sugeno) integral of f over Ω with respect to* Π *on* $\mathcal{P}(\Omega)$ and with infimum operation $\wedge_{\mathcal{T}}$ taken as the t-norm on T. As proved in [7], in our particular case the relation

$$\oint f d\Pi = \bigvee_{\omega \in \Omega}^{\mathcal{T}} (f(\omega) \wedge_{\mathcal{T}} \pi(\omega)) \tag{2.3}$$

holds.

Aiming to keep the basic methodological paradigma on which the Shannon entropy is based, we defined in [7] a lattice-valued entropy function over the space of lattice-valued possibilistic distributions as the expected value (in the sense of Sugeno integral) of a nonincreasing lattice-valued function of the value $\pi(\omega)$ ascribed to $\omega \in \Omega$ (nonincreasing in the sense of partial ordering relation $\leq_{\mathcal{T}}$). In [7], our attention was focused to the nonincreasing function $g : \Omega \rightarrow T$ defined by

$$f(\omega) = \Pi(\Omega - \{\omega\}) = \bigvee_{\omega_1 \in \Omega, \omega_1 \neq \omega}^{\mathcal{T}} \pi(\omega_1). \tag{2.4}$$

Indeed, if $\pi(\omega_1) \leq_{\mathcal{T}} \pi(\omega_2)$ holds for $\omega_1, \omega_2 \in \Omega$, then the inequality

$$\begin{aligned} f(\omega_1) = \Pi(\Omega - \{\omega_1\}) = \Pi(\Omega - \{\omega_1, \omega_2\}) \vee_{\mathcal{T}} \pi(\omega_2) \geq_{\mathcal{T}} \\ \geq_{\mathcal{T}} \quad \Pi(\Omega - \{\omega_1, \omega_2\}) \vee_{\mathcal{T}} \pi(\omega_1) = \Pi(\Omega - \{\omega_2\}) = f(\omega_2) \end{aligned} \tag{2.5}$$

easily follows. Hence, we defined *$\mathcal{T}$-(valued) entropy of a $\mathcal{T}$-possibilistic distribution* π *over* Ω by

$$H(\pi) = \oint \Pi(\Omega - \{\cdot\}) d\Pi = \bigvee_{\omega \in \Omega}^{\mathcal{T}} (\Pi(\Omega - \{\omega\}) \wedge_{\mathcal{T}} \pi(\omega)). \tag{2.6}$$

Let us note that this definition meets the ontological restrictions introduced above: only the values of π and only the operations of supremum and infimum w.r.to the partial ordering $\leq_{\mathcal{T}}$ are used.

For our further reasoning, the most important results achieved in [7] and dealing with lattice-valued entropy function $H(\pi)$ over lattice-valued possibilistic distribution π over Ω are those focused on coarsenings and products of $\mathcal{T}$-possibilistic distributions in the particular case when the complete lattice $\mathcal{T}$ in question is completely distributive. To recall this notion: $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ is called *completely distributive*, if the relation $s \wedge_{\mathcal{T}} (\bigvee^{\mathcal{T}} A) = \bigvee_{t \in A}^{\mathcal{T}} (s \wedge t)$ holds for each $s \in T$ and each $\emptyset \neq A \subset T$. Let us note that the inequality $s \wedge_{\mathcal{T}} (\bigvee^{\mathcal{T}} A) \geq_{\mathcal{T}} \bigvee_{t \in A}^{\mathcal{T}} (s \wedge_{\mathcal{T}} t)$ obviously holds in each poset $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ for each $s \in T$ and $A \subset T$ for which the suprema and infima occuring in the last inequality are defined. An example of complete lattice which is not completely distributive can be found in [7], both the complete lattices $\langle [0,1], \leq \rangle$ and $\langle \mathcal{P}(X), \subset \rangle$ can be easily proved to be completely distributive.

Fact 1 *(Theorem 3.1 in [7]) Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be a completely distributive complete lattice, let Ω be a nonempty space, let $\Omega^* \subset \mathcal{P}(\Omega)$ be a nontrivial disjoint covering of Ω (i.e., decomposition of Ω), so that*

$\emptyset \neq \Omega_1 \neq \Omega$ for each $\Omega_1 \in \Omega^*, \Omega_1 \cap \Omega_2 = \emptyset$ for different $\Omega_1, \Omega_2 \in \Omega^*$, and $\bigcup_{\Omega_1 \in \Omega^*} \Omega_1 = \Omega$ holds. Let π be a $\mathcal{T}$-possibilistic distribution on Ω, let $\pi^* : \Omega^* \to T$ be defined by $\pi^*(\Omega_i) = \Pi(\Omega_i) = \bigvee_{\omega \in \Omega_i} \pi(\omega)$ for each $\Omega_i \in \Omega^*$. Then π^* defines a $\mathcal{T}$-possibilistic distribution on Ω^* and the inequality $H(\pi) \geq H(\pi^*)$ holds.

Fact 2 *(Theorem 3.2 and its immediate corollary in [7]). Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be a completely distributive complete lattice, for each $i = 1, 2, \ldots, n$, let π_i be a $\mathcal{T}$-possibilistic distribution on a nonempty space Ω_i, let $\pi^n(\omega_1, \omega_2, \ldots, \omega_n) = \bigwedge_{i=1}^{\mathcal{T},n} \pi_i(\omega_i)$ for each $\langle \omega_1, \omega_2, \ldots, \omega_n \rangle \in \Omega_1 \times \Omega_2 \times, \cdots \times \Omega_n$. Then π^n is a $\mathcal{T}$-possibilistic distribution on $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ and the relation $H(\pi^n) = \bigvee_{i=1}^{\mathcal{T},n} H(\pi_i)$ holds.*

As illustrated by a counter-example in [7], Section 4, for infinite system of $\mathcal{T}$-possibilistic distributions π_i on Ω_i the assertion of Fact 2.2 need not hold, as in this case the relation $\pi^*(\omega_1, \omega_2, \ldots) = \bigwedge_{i=1}^{\mathcal{T},\infty} \pi_i(\omega_i)$ need not define a $\mathcal{T}$-possibilistic distribution on the Cartesian product $\mathbf{X}_{i=1}^{\infty} \Omega_i$ of particular spaces Ω_i (like as it is the case with stochastically independent products of infinite systems of probability distributions).

3 Lattice-Like Processed Real-Valued Possibilistic Distributions Endowed by Arithmetical Operation of Complement

Let us return, in this section, to normalized real-valued possibilistic distributions, but keeping in mind, up to one modification to be specified below, the constraints imposed above according to which only the lattice-based relations and constructions dealing with real numbers from $[0, 1]$ may be applied. The modification reads as follows: the lattice relations and operations may be applied not only to the values taken by the possibilistic distribution π in question, i.e., to the real numbers from the set $\{\pi(\omega) : \omega \in \Omega\} \subset [0, 1]$, but also to the values $1 - \pi(\omega)$, hence, to the values in $\{\pi(\omega), 1 - \pi(\omega) : \omega \in \Omega\}$. Under this modification, the value

$$H^*(\pi) = \bigvee_{\omega \in \Omega} [(1 - \pi(\omega)) \wedge \pi(\omega)] \tag{3.1}$$

is defined in a way meeting our conditions. As $H^*(\pi)$ is expected value (in the sense of Sugeno integral) of non-increasing function $1 - \pi(\omega)$ of $\pi(\omega)$, it may be of interest to consider the function H^* as a real-valued, but in a lattice-like way defined and processed, entropy function over real-valued possibilistic distributions on Ω and to analyze, in more detail, its properties.

Before going on with fulfilling this task let us note that in the case of a *probability* distribution $p : \Omega \to [0, 1]$ over a finite or countable space Ω the function H_* defined by

$$H_*(p) = \sum_{\omega \in \Omega} ((1 - p(\omega))p(\omega)) \left(= 1 - \sum_{\omega \in \Omega} (p(\omega))^2 \right), \tag{3.2}$$

syntactically copying $H^*(\pi)$ defined by (3.1) just with supremum replaced by addition and with infimum replaced by product, can be taken as an interesting normalized alternative to the Shannon entropy function $H_{\mathcal{T}}$. Indeed, $H_*(p) \in [0, 1]$ obviously holds, $H_*(p) = 0$ if and only if $p(\omega_0) = 1$ for some $\omega_0 \in \Omega$, and $H_*(p)$ takes its maximum value $1 - (1/n)$ on $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ if and only if $p(\omega_i) = 1/n$ for each $i = 1, 2, \ldots, n$, like as it is the case with Shannon entropy function. Supposing that, for both $j = 1, 2, p_j$ is a probability distribution on a nonepmty finite or countable space Ω_j and setting $p_{12}(\omega_1, \omega_2) = p_1(\omega_1)p_2(\omega_2)$ for every $\langle \omega_1, \omega_2 \rangle \in \Omega_1 \times \Omega_2$ the relation

$$H_*(p_{12}) = 1 - ((1 - H_*(p_1))(1 - H_*(p_2))) \tag{3.3}$$

for the corresponding H_*-entropy values can be easily proved. Indeed, due to (3.2),

$$
\begin{aligned}
1-((1-H_*(p_1))(1-H_*(p_2))) = 1-\left(\sum_{\omega_1\in\Omega_1}(p_1(\omega_1))^2\right)\left(\sum_{\omega_2\in\Omega_2}(p_2(\omega_2))^2\right) = \\
= \quad 1-\sum_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2}(p_1(\omega_1)p_2(\omega_2))^2 = \\
= \quad 1-\sum_{\langle\omega_1,\omega_2\rangle\in\Omega_1\times\Omega_2}(p_{12}(\omega_1,\omega_2))^2 = H_*(p_{12}).
\end{aligned}
\tag{3.4}
$$

The assertion can be proved by induction to each finite system $p_1, p_2, \ldots, p_n$ of probability distribution, each defined on a nonempty finite or countably space $\Omega_i, i = 1, 2, \ldots, n$.

The relation (3.3) can be taken as a normalized ("relativistic-like") addition rule for H_*-entropy of statistically independent product of probability distributions p_1, p_2 obeying the demand that the "sum" of $H_*(p_1)$ and $H_*(p_2)$ should be in $[0, 1]$ (remember the way in which speeds are combined together in relativity theory).

Before focusing our attention, again, to the lattice-like processed real-valued possibilistic distributions taking their values in the complete lattice $\mathcal{T}_0 = \langle [0,1], \leq\rangle$ enriched by the arithmetical complement operation $1-\pi(\omega)$ and to the possibilistically independent products of such distributions, the following rather simple arithmetic assertions will be of use.

Lemma 3.1 *$\mathcal{T}_0 = \langle [0,1], \leq\rangle$ defines a completely distributive complete lattice.*

Proof. For each $A \subset [0,1], \bigvee A$ and $\bigwedge A$ are obviously defined and the inequality $s\wedge(\vee A) \geq \bigvee_{t\in A}(s\wedge t)$ holds for each $s \in [0,1]$. If $s \geq \bigvee A$ is the case, then $s \wedge (\bigvee A) = \bigvee A$, $s \wedge t = t$ for each $t \in A$, so that $\bigvee_{t\in A}(s \wedge t) = \bigvee A$. If $s < \bigvee A$ holds, then there exists $t_0 \in A$ such that $s < t_0$ is valid, hence, the relation $s \wedge t_0 = s = \bigvee_{t\in A}(s\wedge t) = s \wedge (\bigvee A)$ easily follows and the assertion is proved. □

Lemma 3.2 *For each $\emptyset \neq I \subset [0,1]$ the inequality*

$$
\bigwedge I \wedge \left(1-\bigwedge I\right) \leq \bigvee_{x\in I}(x\wedge(1-x)) \tag{3.5}
$$

holds.

Proof. As $1-\bigwedge I = \bigvee_{s\in I}(1-x)$ holds, (3.5) reduces to

$$
\left(\bigwedge_{x\in I} x\right) \wedge \left(\bigvee_{x\in I}(1-x)\right) \leq \bigvee_{x\in I}(x\wedge(1-x)). \tag{3.6}
$$

Applying Lemma 1 to $s = \bigwedge_{x\in I} x$ and $A = \{1-x : x \in I\}$ we obtain that the inequality

$$
\begin{aligned}
\left(\bigwedge_{x\in I} x\right) \wedge \left(\bigvee_{x\in I}(1-x)\right) = \bigvee_{x\in I}\left(\left(\bigwedge_{x\in I} x\right)\wedge(1-x)\right) \leq \\
\leq \bigvee_{x\in I}(x\wedge(1-x))
\end{aligned}
\tag{3.7}
$$

holds. The assertion is proved. □

Lemma 3.3 *For each $x, y \in [0,1]$ such that $|x-y| \leq \varepsilon$ holds, the inequality*

$$
K(x,y) = |((1-x)\wedge x) - ((1-y)\wedge y)| \leq \varepsilon \tag{3.8}
$$

holds as well.

Proof. If $x, y \leq 1/2$ ($x, y \geq 1/2$, resp.) is the case, then $K(x,y) = |x - y|$ ($K(x,y) = |(1-x) - (1-y)| = |y - x| = |x - y|$, resp.) follows, so that $K(x,y) \leq \varepsilon$ holds. Otherwise we may suppose, without any loss of generality, that $x < 1/2$ and $y > 1/2$ is the case, hence, as $|x - y| \leq \varepsilon$ holds, the membership relations $x \in (1/2 - \varepsilon, \varepsilon)$ and $y \in (1/2, 1/2 + \varepsilon)$ follow. In this case, however, $(1-x) \wedge x = x, (1-y) \wedge y = 1 - y \in (1/2 - \varepsilon, 1/2)$ holds, so that $K(x,y) = |x \wedge (1-y)| \leq \varepsilon$ results and the assertion is proved. □

4 Possibilistically Independent Products of Lattice-Like Processed Real-Valued Possibilistic Distributions

Definition 4.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\{\Omega_i : i \in I\}$ be a system of nonempty sets, let $\Omega^I = \mathbf{X}_{i \in I} \Omega_i$ be the Cartesian product of sets Ω_i, hence, Ω^I denotes the set of all mappings ω^I which take I into $\bigcup_{i \in I} \Omega_i$ in such a way that $\omega^I(i) \in \Omega_i$ holds for each $i \in I$. Set, for each $\omega^I \in \Omega^I$,*

$$\pi^I(\omega^I) = \bigwedge_{i \in I} \pi_i(\omega^I(i)). \tag{4.1}$$

As $\mathcal{T}$ is complete lattice, the value $\pi^I(\omega^I)$ is defined for each $\omega^I \in \Omega^I$. The mapping $\pi^I : \Omega^I \to T$, also denoted by $\mathbf{X}_{i \in I}$ is called the possibilistically independent product of $\mathcal{T}$-possibilistic distributions π_i for i ranging over I.

Lemma 4.1 *Let the notations and conditions of Definition 4.1 hold for $\mathcal{T} = \mathcal{T}_0 = \langle [0,1], \leq \rangle$. Then the mapping $\pi^I : \Omega^I \to [0,1]$ defined by (4.1) defines a real-valued possibilistic distribution on Ω^I.*

Proof. As $\bigvee_{\omega \in \Omega} \pi_i(\omega) = 1$ holds for each $i \in I$, there exists, for each $\varepsilon > 0$, and each $i \in I$, an element $\omega_{0,i} \in \Omega_i$ such that $\pi_i(\omega_{0,i}) > 1 - \varepsilon$ holds. For $\omega_0^I \in \Omega^I$ defined by $\omega_0^I(i) = \omega_{0,i}$ for each $i \in I$, (4.1) yields that the inequality $\pi^I(\omega_0^I) \geq 1 - \varepsilon$ is valid, hence, $\bigvee_{\omega^I \in \Omega^I} \pi^I(\omega^I) = 1$ follows and the assertion is proved. □

Theorem 4.1 *Let the conditions of Lemma 4.1 hold, let $\Omega_i = \Omega$ for each $i \in I$, let $H^*(\pi)$ be defined by (3.1) for each $\pi = \pi_i, i \in I$, and $\pi = \pi^I$. Then the relation*

$$H^*(\pi^I) = \bigvee_{i \in I} H^*(\pi_i) \tag{4.2}$$

is valid.

Proof. As shown in the proof of Lemma 4.1, for each $i_0 \in I$ and each $\varepsilon > 0$ there exists $\omega(i_0, \varepsilon) \in \Omega$ such that $\pi_{i_0}(\omega(i_0, \varepsilon)) > 1 - \varepsilon$ holds. Define, for each $\omega \in \Omega$, the mapping $\omega^I(i_0, \varepsilon, \omega) : I \to \Omega$ in this way:

$$\omega^I[i_0, \varepsilon, \omega](i_0) = \omega, \tag{4.3}$$

$$\omega^I[i_0, \varepsilon, \omega](j) = \omega(j, \varepsilon) \text{ for each } j \in I, j \neq i_0. \tag{4.4}$$

Consequently,

$$\pi^I(\omega^I(i_0, \varepsilon, \omega)) = \bigwedge_{j \in I} \pi_j(\omega^I[i_0, \varepsilon, \omega](j)) =$$

$$= \pi_{i_0}(\omega) \wedge \left(\bigwedge_{j \in I, j \neq i_0} \pi_j(\omega(j, \varepsilon)) \right). \tag{4.5}$$

Hence, if $\pi_{i_0}(\omega) \leq 1 - \varepsilon$ holds, then $\pi^I(\omega^I[i_0, \varepsilon, \omega]) = \pi_{i_0}(\omega)$ follows, as $\pi_j(\omega(j, \varepsilon)) > 1 - \varepsilon$ is valid for each $j \in I, j \neq i_0$, due to the choice of $\omega(j, \varepsilon)$. If $\pi_{i_0}(\omega) > 1 - \varepsilon$ is the case, i.e., if $\pi_{i_0}(\omega) \in (1 - \varepsilon, 1]$ holds, then (4.5) yields that also

$$\pi^I(\omega^I[i_0,\varepsilon,\omega]) \geq 1-\varepsilon, \text{ i.e., } \pi^I(\omega^I[i_0,\varepsilon,\omega]) \in [1-\varepsilon,1] \tag{4.6}$$

holds. So, in both the cases the inequality

$$|\pi^I(\omega^I[i_0,\varepsilon,\omega]) - \pi_{i_0}(\omega)| \leq \varepsilon \tag{4.7}$$

is valid. Due to Lemma 3.3, also the inequality

$$\begin{aligned} & K(\pi^I(\omega^I[i_0,\varepsilon,\omega]),\pi_{i_0}(\omega)) = \\ = \; & ([\pi^i(\omega^I[i_0,\varepsilon,\omega]) \wedge (1-\pi^I(\omega^I[i_0,\varepsilon,\omega]))] - \\ & -[\pi_{i_0}(\omega) \wedge (1-\pi_{i_0}(\omega))]) \leq \varepsilon \end{aligned} \tag{4.8}$$

is valid. Hence, for each $i \in I$ and $\omega \in \Omega$, the value $\pi_i(\omega) \wedge (1-\pi_i(\omega))$ can be approximated, up to an arbitrary small difference $\varepsilon > 0$, by the value $\pi^I(\omega^I) \wedge (1-\pi^I(\omega^I))$ for some $\omega^I \in \Omega^I$. So, we may conclude that the inequality

$$\begin{aligned} & H^*(\pi^I) = \bigvee_{\omega^I \in \Omega^I} ((1-\pi^I(\omega^I)) \wedge \pi^I(\omega^I)) \geq \\ \geq \; & \bigvee_{i \in I, \omega \in \Omega} ((1-\pi^i(\omega)) \wedge \pi_i(\omega)) = \bigvee_{i \in I} \left(\bigvee_{\omega \in \Omega} ((1-\pi_i(\omega)) \wedge \pi_i(\omega)) \right) = \\ = \; & \bigvee_{i \in I} H^*(\pi_i) \end{aligned} \tag{4.9}$$

is valid.

Let us prove the inverse inequality $H^*(\pi^I) \leq \bigvee_{i \in I} H^*(\pi_i)$, i.e., the inequality

$$\bigvee_{\omega^I \in \Omega^I} ((1-\pi^I(\omega^I)) \wedge \pi^I(\omega^I)) \leq \bigvee_{i \in I} \left(\bigvee_{\omega \in \Omega} ((1-\pi_i(\omega)) \wedge \pi_i(\omega)) \right). \tag{4.10}$$

What obviously suffices is to prove this inequality for each $\omega^I \in \Omega^I$ in particular, hence, to prove that for each $\omega^I \in \Omega^I$ the relation

$$\begin{aligned} & \pi^i(\omega^I) \wedge (1-\pi^I(\omega^I)) = \left(\bigwedge_{i \in I} \pi_i(\omega^I(i)) \right) \wedge \left(1 - \bigwedge_{i \in I} \pi_i(\omega^I(i)) \right) \leq \\ \leq \; & \bigvee_{i \in I} \left(\bigvee_{\omega \in \Omega} (\pi_i(\omega) \wedge (1-\pi_i(\omega))) \right) \end{aligned} \tag{4.11}$$

holds. Due to Lemma 3.2 we obtain that

$$\begin{aligned} & \left(\bigwedge_{i \in I} \pi^I(\omega^I(i)) \right) \wedge (1 - \bigwedge_{i \in I} \pi_i^I(\omega^I(i))) = \\ = \; & \left(\bigwedge_{i \in I} \pi^I(\omega^I(i)) \right) \wedge \left(\bigvee_{i \in I} (1-\pi_i(\omega^I(i))) \right) \leq \\ \leq \; & \bigvee_{i \in I} [\pi^I(\omega^I(i)) \wedge (1-\pi^I(\omega^I(i)))] \leq \\ \leq \; & \bigvee_{\omega \in \Omega} \left(\bigvee_{i \in I} (\pi_i(\omega) \wedge (1-\pi_i(\omega))) \right) = \\ = \; & \bigvee_{i \in I} \left(\bigvee_{\omega \in \Omega} (\pi_i(\omega) \wedge (1-\pi_i(\omega))) \right) = \bigvee_{i \in I} H^*(\pi_i). \end{aligned} \tag{4.12}$$

The inequality inverse to (4.9) as well as whole the assertion are proved. □

It is perhaps worth being noted explicitly that the simplifying assumption of identical spaces $\Omega_i (= \Omega)$ on which the particular possibilistic distributions $\pi_i, i \in I$, are defined, can be accepted without any loss of generality of the obtained results. Indeed, in general we could take $\Omega = \bigcup_{i \in I} \Omega_i$ and replace each π_i by π_i^0 defined on Ω in such a way that $\pi_i^0(\omega) = \pi_i(\omega)$, if $\Omega \in \Omega_i$, $\pi_i^0(\omega) = 0$, if $\omega \in \Omega - \Omega_i$, so arriving at the same results as in the case with identical Ω_i's.

5 Conclusions

Inspired by a very important property of Shannon entropy function consisting in the additivity of entropy values when considering statistically independent products of particular probability, we tried to define a similar entropy measure for possibilistic distributions, in particular, for the lattice-valued ones. In [7] we show that this task can be more or less successfully solved when replacing the sum of particular entropy values by their supremum (in the sense of the complete lattice in question) and when considering just possibilistically independent products of finitely many lattice-valued possibilistic distributions.

Seeking for some more conditions particular cases under which this result could be proved also for possibilistically independent products of infinite number of particular possibilistic distributions we obtained above that this could be achieved when considering real-valued possibilistic distributions with possibility degrees processed just as elements of the complete lattice $\langle [0,1], \leq \rangle$ enriched by the operation of standard arithmetic complement $1 - x$. It remains as an open and perhaps interesting problem whether the same result could be proved also without the arithmetical complement, when replacing the value $1 - \pi(\omega)$ in the definition of entropy function by the value $\Pi(\Omega - \{\omega\})$. Also some other conditions imposed on the complete lattice $\mathcal{T}$ in question and enabling to strenghen the general results from [7] seem to be worth being sought for and analyzed in more detail.

The references [1, 4, 6] and [9] can serve in order to obtain a more detailed insight into the domains the ideas, notions and results of which are applied or noted in this paper.

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