Johnson System of Parametric Families

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Abstract:

We present a system of parametric unimodal continuous distributions, containing all possible types of the score functions of the prototypes. Since for its construction is used the modified Johnson transformation, we call it a Johnson system. The system contains a great number of practically used probability distributions, some of them in reparametrized forms in order to obtain families with unified meaning of parameters. Finally, we shortly discuss a way for the construction of other, non-Johnson systems of distributions on finite intervals.

Keywords:
description of distributions; basic statistics; score function; point estimates

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1 Introduction

Distribution of random variable X with distribution function F is said to be supported by interval $\mathcal{Q} \subseteq \mathbb{R}$ if density $f(x) = dF(x)/dx$ satisfies relation

$$f(x) \begin{cases} > 0 & \text{if} \quad x \in \mathcal{Q} \\ = 0 & \text{if} \quad x \in \mathbb{R} - \mathcal{Q}. \end{cases}$$

The classification of continuous distributions in the Pearson system [10] is based on the properties of the score function

$$S(x) = -\frac{f'(x)}{f(x)}. \tag{1.1}$$

However, Pearson system does not provide a basis for a systematic account of distributions. The reason is that for distributions with 'partial support' $\mathcal{Q} \neq \mathbb{R}$ the score function does not reflect the properties of the distribution (cf. [1]).

Johnson [5] proposed method for generating distributions of random variable X supported by $\mathcal{Q} = (0, \infty)$ and $\mathcal{Q} = (0, 1)$ by means of transformation $\varphi : \mathbb{R} \to \mathcal{Q}$ defined by

$$\begin{aligned} Y = \varphi^{-1}(X) &= \log X && \text{when} \quad \mathcal{Q} = (0, \infty) \\ Y = \varphi^{-1}(X) &= \log \frac{X}{1 - X} && \text{when} \quad \mathcal{Q} = (0, 1), \end{aligned} \tag{1.2}$$

where Y is a 'prototype' supported by $\mathbb{R}$. He considered, however, only some particular prototypes.

In order to obtain a useful system of parametric distributions we combine both the methods. We use the Pearson's approach to a construction of a system of distributions G on $\mathbb{R}$ with densities g and score functions S, which is complete from the point of view of the behavior of the score functions in infinity. Next, we construct the transformed systems on interval supports $\mathcal{Q}$. On every $\mathcal{Q}$ we obtain random variables $X = \eta^{-1}(Y)$ with distributions $F(x) = G(\eta(x))$ where $\eta : \mathcal{Q} \to \mathbb{R}$ is a version of (1.2) in the form

$$y = \eta(x) = \begin{cases} x & \text{when} \quad (a, b) = \mathbb{R} \\ \log(x - a) & \text{when} \quad -\infty < a < b = \infty \\ \log \dfrac{(x - a)}{(b - x)} & \text{when} \quad -\infty < a < b < \infty \\ \log(b - x) & \text{when} \quad -\infty = a < b < \infty, \end{cases}$$

proposed by Fabián [2]. The density of the transformed distribution is

$$f(x) = g(\eta(x))\eta'(x). \tag{1.3}$$

Besides the density, we characterize transformed distribution by the transformed score function of the prototype,

$$T(x) = S(\eta(x)), \tag{1.4}$$

where S is given by (1.1). (1.4) was detected by fabián [1] to be a significant description of the transformed distributions reflecting their properties. It was shown in [1] that T can be expressed without referring to the prototype distribution by formula

$$T(x) = \frac{1}{f(x)} \frac{d}{dx} \left(-\frac{1}{\eta'(x)} f(x) \right). \tag{1.5}$$

For the sake of simplicity, the explicit formulas for f and T of the transformed distributions are presented only on particular supports, mostly on the Johnson's ones, i.e., $\mathcal{Q} = (0, \infty)$ and $\mathcal{Q} = (0, 1)$.

2 Parent distributions

2.1 Prototypes

We choose the prototypes G supported by $\mathbb{R}$ which cover possible types of the behavior of the score function in infinity:

UE: S unbounded, exponentially increasing
UP: S unbounded, polynomially increasing
BB: S bounded
BR: S bounded redescending
UB: S unbounded when $x \to -\infty$ and bounded when $x \to \infty$
BU: S bounded when $x \to -\infty$ and unbounded when $x \to \infty$.

The simplest score functions and the corresponding densities of the prototypes are given in Table 1.

Table 1a. Prototype distributions on $\mathbb{R}$

type	distribution	$g(x)$	$S(x)$
UE	*	$\frac{1}{K}e^{-\cosh x}$	$\sinh x$
UP	normal	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$	x
UB	extreme value	$e^{-x}e^{-e^{-x}}$	$1-e^{-x}$
BU	Gumbel	$e^{x}e^{-e^{x}}$	e^x-1
BB	logistic	$\frac{e^x}{(1+e^x)^2}$	$\tanh\frac{x}{2}$
BR	Cauchy	$\frac{1}{\pi(1+x^2)}$	$\frac{2x}{1+x^2}$

$K = 2K_0(1)$

All these densities are unimodal. It holds that $g'(0) = 0$ so that

$$S(0) = 0. \tag{2.1}$$

The zero of the score function is a measure of the central tendency of prototype distributions.

2.2 Transformed distributions

The densities of distributions from Table 1a transformed to $\mathcal{Q} = (0, \infty)$ and $\mathcal{Q} = (0, 1)$ are given in Table 1b and 1c, together with the transformed scores of the prototype. It is apparent that the transformation preserves the type of the distribution.

Table 1b. Transformed distributions on $(0, \infty)$

type	distribution	$f(x)$	$T(x)$
UE	GIG	$\frac{1}{Kx}e^{-\frac{1}{2}(x+1/x)}$	$\frac{1}{2}(x-1/x)$
UP	lognormal	$\frac{1}{\sqrt{2\pi}x}e^{-\frac{1}{2}\log^2 x}$	$\log x$
UB	Fréchet	$\frac{1}{x^2}e^{-1/x}$	$1-1/x$
BU	exponential	e^{-x}	$x-1$
B	log-logistic	$\frac{1}{(1+x)^2}$	$\frac{x-1}{x+1}$
BR	log-Cauchy	$\frac{1}{\pi x(1+\log^2 x)}$	$\frac{2\log x}{1+\log^2 x}$

Table 1c. Transformed distributions on $(0,1)$

type	distribution	$f(x)$	$T(x)$
UE		$\frac{1}{Kx(1-x)}e^{-\frac{1}{2}(\frac{x}{1-x}+\frac{1-x}{x})}$	$\frac{x-1/2}{x(1-x)}$
UP	Johnson U_B	$\frac{1}{\sqrt{2\pi}x(1-x)}e^{-\frac{1}{2}\log^2\frac{x}{1-x}}$	$\log\frac{x}{1-x}$
UB		$\frac{1}{x^2}e^{-(1-x)/x}$	$\frac{x-1/2}{x}$
BU		$\frac{1}{(1-x)^2}e^{-x/(1-x)}$	$\frac{x-1/2}{1-x}$
BB	uniform	1	$x-1/2$
BR		$\frac{1}{\pi x(1-x)}\frac{1}{1+\log^2\frac{x}{1-x}}$	$\frac{2\log\frac{x}{1-x}}{1+\log^2\frac{x}{1-x}}$

From (1.4) and (2.1) it follows that

$$T(\eta^{-1}(0)) = 0. \tag{2.2}$$

Point $x^* = \eta^{-1}(0)$ can be considered as the measure of central tendency of the transformed distributions. It holds that $x^* = 1$ and $x^* = 1/2$ for all parent distributions on $(0,\infty)$ and $(0,1)$, respectively.

3 Parametric families

General parametric families of the Johnson system are constructed in three steps.

3.1 Introduction of the shape parameters of the prototypes

The score functions listed in Table 1 were provided by shape parameters $\alpha > 0$ and $\nu > 0$ (if possible) in such a way that $S(0;\alpha,\nu) = 0$. The results of the procedure are given in Table 2.

Table 2. System of Johnson prototype families with shape parameters α,ν.

type	$g(z;\alpha,\nu)$	$S(z;\alpha,\nu)$
UE	$\frac{\nu^{\rho/2}z^{\rho-1}}{2K_\rho(\alpha\sqrt{\nu})}e^{-\frac{\alpha}{2}(e^z+\nu e^{-z})}$	$\frac{\alpha}{2}(e^z-\nu e^{-z})-\rho$
UP	$\frac{1}{2^\lambda\Gamma(\lambda)}e^{-\frac{1}{2}\|z\|^{1+\alpha}}$	$\frac{\alpha+1}{2}\ \mathrm{sgn}\ z\ \|z\|^\alpha$
UB	$\frac{\alpha^\alpha}{\Gamma(\alpha)}e^{-\alpha}e^{-\alpha e^{-z}}$	$\alpha(1-e^{-z})$
BU	$\frac{\alpha^\alpha}{\Gamma(\alpha)}e^{\alpha z}e^{-\alpha e^z}$	$\alpha(z-1)$
BB	$\frac{1}{\nu^\alpha B(\nu\alpha,\alpha)}\frac{e^{\nu\alpha z}}{(e^z+1/\nu)^{[1+\nu]\alpha}}$	$\alpha\frac{z-1}{z+1/\nu}$
BR	$\frac{1}{B(\frac{1}{2},\alpha-\frac{1}{2})}\frac{1}{(1+z^2)^\alpha}$	$\frac{2\alpha z}{1+z^2}$

B and Γ are the beta and gamma functions, K_ρ is the Bessel function of the third kind, $\rho = \alpha(1-\nu)/2$ and $\lambda = (\alpha+2)/(\alpha+1)$.

Table 2 appears to be a basic scheme from which the general parametric families of the prototypes can be derived (Section 3.2), as well as the corresponding transformed families on arbitrary supports (Section 3.3).

For distributions with a symmetric parent, α means the excess. Parameter $\nu \in (0,\infty)$ characterizes the non-symmetry. This is apparent from Fig.1, showing densities of type BB,

$$g_{1,\nu}(y) = e^{\nu y}/(e^y+1/\nu)^{1+\nu},$$

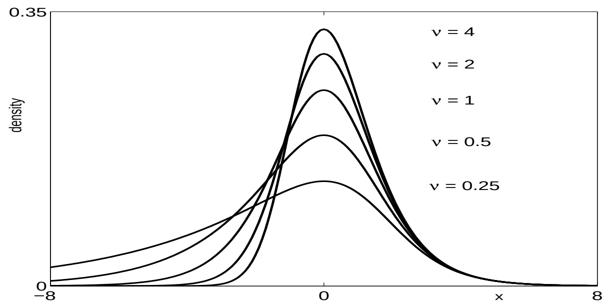

Figure 1. Densities of prototypes of type BB with various values of the parameter of non-symmetry ν

for some values of ν. Parameter α of non-symmetric (and mutually symmetric) distributions UB and BU characterizes both the excess and the non-symmetry simultaneously.

3.2 Introduction of the location and scale parameters of the prototypes

Let $\mu \in \mathbb{R}$ be the location and $\sigma > 0$ the scale parameters. If we put $z = w$ in Table 2, where

$$w = \frac{y-\mu}{\sigma} \tag{3.1}$$

is the pivotal variable ([4]), we obtain parametric distributions $G_{\mu,\sigma,\alpha,\nu}$ with densities and score functions

$$g_{\mu,\sigma,\alpha,\nu}(y) = \sigma^{-1} g_{\alpha,\nu}(w) \tag{3.2}$$

$$S_{\mu,\sigma,\alpha,\nu}(y) = -\frac{1}{g_{\alpha,\nu}(w)}\frac{d}{dy} g_{\alpha,\nu}(w) = \frac{1}{\sigma} S_{\alpha,\nu}(w). \tag{3.3}$$

μ is the modal value of $G_{\mu,\sigma,\alpha,\nu}$. Obviously, $S_{\mu,\sigma,\alpha,\nu}(\mu) = 0$. μ is taken as a measure of central tendency of $G_{\mu,\sigma,\alpha,\nu}$. Let σ, α and ν be given constants, let random variables $X_1, ..., X_n$ be distributed according to $G_{\mu,\sigma,\alpha,\nu}$ and $x_1, ..., x_n$ be their values. The bounded or unbounded score functions indicate whether the estimate $\hat{\mu} : \sum_{i=1}^n S_{\mu,\sigma,\alpha,\nu}(x_i) = 0$ of the measure of the central tendency is robust or sensitive to outlier observations.

3.3 Transformed families

The transformed location of the prototype,

$$t = \eta^{-1}(\mu), \tag{3.4}$$

will be called a *Johnson parameter* of the transformed distribution $F_{\eta^{-1}(\mu),\sigma,\alpha,\nu}(x) = G_{\mu,\sigma,\alpha,\nu}(\eta(x))$. The explicit forms of the Johnson parameter on different supports are

$$t = \begin{cases} \mu & \text{if} \quad \mathcal{Q} = \mathcal{R} \\ e^{\mu} + a & \text{if} \quad \mathcal{Q} = (a, \infty) \\ \frac{be^{\mu}+a}{1+e^{\mu}} & \text{if} \quad \mathcal{Q} = (a, b) \\ b - e^{\mu} & \text{if} \quad \mathcal{Q} = (-\infty, b). \end{cases}$$

t can be considered as a measure of central tendency of the transformed parametric distributions (see [2]) and the bounded or unbounded transformed score functions of the prototype indicate the robustness or sensitivity of its estimates to outlier observations.

By (3.4), (1.3), (3.2) and (1.4), (3.3), the density and the transformed score function of the prototype of distribution $F_{t,\sigma,\alpha,\nu}(x)$ are

$$f_{t,\beta,\alpha,\nu}(x) = \beta g_{\alpha,\nu}(\eta(w))\eta'(x) \tag{3.5}$$

$$T_{t,\beta,\alpha,\nu}(x) = \beta S_{\alpha,\nu}(\eta(w)) \tag{3.6}$$

where $\beta = 1/\sigma$ and

$$\eta(w) = \frac{\eta(x) - \eta(t)}{\sigma}, \tag{3.7}$$

which can be taken as the pivotal variable on $\mathcal{Q}$. By setting $z = \eta(w)$ in Table 2 and by using relations (3.5) and (3.6) we obtain Johnson systems of transformed distributions on arbitrary support.

4 Johnson system on $(0, \infty)$

Let us study in more details the Johnson system on $\mathcal{Q} = (0, \infty)$. Here $\eta(x) = \log x, \eta'(x) = 1/x$ and the pivotal variable (3.7) is

$$z = \eta(w) = \log\left(\frac{x}{t}\right)^{\beta}. \tag{4.1}$$

Note that the Johnson parameter of distribution supported by $(0, \infty)$ is often termed by the scale parameter. However, we take the value $1/\beta$ as the scale.

From Table 2 we derive explicit forms of the densities and transformed score functions of the prototypes of the six Johnson parametric families and detect their members used in practice.

Type UP. This family is the lognormal distribution

$$f_{t,\beta}(x) = \frac{1}{\sqrt{2\pi}x} e^{-\frac{1}{2}\log^2\left(\frac{x}{t}\right)}$$

$$T_{t,\beta}(x) = \beta \log\left(\frac{x}{t}\right)^{\beta}.$$

For the sake of simplicity we do not consider a more general prototype $g(y) \sim e^{-\frac{1}{2}\left|\frac{y-\mu}{\sigma}\right|^{\alpha}}$ and a generalized UP family on $(0, \infty)$

$$f_{t,\beta,\alpha}(x) = \frac{1}{2^{\lambda}\Gamma(\lambda)x} e^{-\frac{1}{2}(\log|\frac{x}{t}|^{\beta})^{1+\alpha}}$$

where $\lambda = (\alpha + 2)/(\alpha + 1)$.

Type UB. The family with non-symmetric prototypes skewed to the right with transformed scores of the prototype bounded at infinity is given by

$$f_{t,\beta,\alpha}(x) = \frac{\beta\alpha^{\alpha}}{\Gamma(\alpha)x}\left(\frac{x}{t}\right)^{-\beta\alpha} e^{-\alpha\left(\frac{x}{t}\right)^{-\beta}}$$

$$T_{t,\beta,\alpha}(x) = \alpha\beta(1 - (x/t)^{-\beta}).$$

The member $f_{t,\beta,1}(x)$ is the Fréchet distribution.

Type BU. The family with non-symmetric prototypes skewed to the left with transformed scores of the prototype bounded at zero is given by

$$f_{t,\beta,\alpha}(x) = \frac{\beta\alpha^{\alpha}}{\Gamma(\alpha)x}\left(\frac{x}{t}\right)^{\beta\alpha} e^{-\alpha\left(\frac{x}{t}\right)^{\beta}} \tag{4.2}$$

$$T_{t,\beta,\alpha}(x) = \alpha\beta((x/t)^{\beta} - 1).$$

(4.2) is called by [8] a generalized gamma family. Members of the family are the following distributions

Weibull	$f_{t,\beta,1}(x)$
gamma	$f_{\alpha/\gamma,1,\alpha}(x)$
chi-squared	$f_{\nu,1,\nu/2}(x)$
Rayleigh	$f_{a\sqrt{2},2,1}(x)$
Maxwell	$f_{a\sqrt{3},2,3/2}(x)$.

Type BB. The family of distributions with heavy-tailed densities and bounded transformed scores of the prototype is given by

$$\begin{aligned} f_{t,\beta,\alpha,\nu}(x) &= \frac{\beta}{\nu^{\alpha} B(\nu\alpha,\alpha)x} \frac{(x/t)^{\beta\nu\alpha}}{[(x/t)^{\beta}+1/\nu]^{(1+\nu)\alpha}} \\ T_{t,\beta,\alpha,\nu}(x) &= \alpha\beta\frac{(x/t)^{\beta}-1}{(x/t)^{\beta}+1/\nu}. \end{aligned} \tag{4.3}$$

In [8], a family with densities

$$f^{TB}_{\gamma,\beta,\varepsilon,\delta}(x) = \frac{\beta}{B(\varepsilon,\delta)x}\frac{(x/\gamma)^{\beta\varepsilon}}{[(x/\gamma)^{\beta}+1]^{\varepsilon+\delta}}$$

is called the transformed beta family. By (1.5), the corresponding transformed scores of the prototypes are

$$T^{TB}_{\gamma,\beta,\varepsilon,\delta}(x) = \frac{\beta\delta(x/\gamma)^{\beta}-\varepsilon}{(x/\gamma)^{\beta}+1}. \tag{4.4}$$

Since $T^{TB}_{\gamma,\beta,\varepsilon,\delta}(\gamma) \neq 0$ if $\varepsilon \neq \delta$, parameter γ is not a Johnson parameter. By comparing (4.4) with (4.3) we obtain

$$f^{TB}_{\gamma,\beta,\varepsilon,\delta}(x) = f_{\gamma(\varepsilon/\delta)^{1/\beta},\beta,\delta,\varepsilon/\delta}(x).$$

Belonging to this type are distributions

log-logistic	$f_{t,\beta,1,1}(x)$
beta-prime	$f_{p/q,1,q,p/q}(x)$
Fisher-Snedecor	$f_{1,1,\nu_2/2,\nu_1/\nu_2}(x)$
Burr III	$f_{k^{1/c},c,1,k}(x)$
Burr XII	$f_{k^{-1/c},c,k,1/k}(x)$
Lomax	$f_{1/\alpha,1,\alpha,1/\alpha}(x)$.

Type BR. The distributions of this type are characterized by heavy-tailed densities and redescending transformed scores of the prototype,

$$\begin{aligned} f_{t,\beta,\alpha}(x) &= \frac{1}{B(\frac{1}{2},\alpha-\frac{1}{2})}\frac{1}{[1+\log^2(x/t)^{\beta}]^{\alpha}} \\ T_{t,\beta,\alpha}(x) &= \frac{2\alpha\log(x/t)^{\beta}}{1+\log^2(x/t)^{\beta}}. \end{aligned} \tag{4.5}$$

The members are the log-Cauchy distribution, $f_{t,\beta,1}(x)$, and the Student z distribution, $f_{c^{1/2},1,(c+1)/2}(x)$ (cf. [9]). The prototype of a more general family of this type is mentioned in [7], pp.327.

Type UE. Belonging to this type is a generalized inverse Gaussian family [7] with densities

$$f^{GIG}_{p,q,\lambda}(x) = \frac{(p/q)^{\lambda/2}}{2K_{\lambda}(\sqrt{pq})}x^{\lambda-1}e^{-\frac{1}{2}(px+q/x)},$$

$p, q, \lambda \in (0, \infty)$. By (1.5),

$$T^{GIG}_{p,q,\lambda}(x) = \frac{1}{2}(px - q/x) - \lambda. \tag{4.6}$$

As $T^{GIG}_{p,q,\lambda}(1) = 0$ if $\lambda = (p-q)/2$, (4.6) can be generalized in this concrete case for the Johnson and β parameters. By setting $p = \alpha$ and $q = \nu\alpha$ we obtain a family with densities descending steeply to zero and with exponentially increasing transformed scores of the prototype

$$\begin{aligned} f_{t,\beta,\alpha,\nu}(x) &= \frac{\nu^{\rho/2}(x/t)^\rho}{2K_\rho(\alpha\sqrt{\nu})x} e^{-\frac{\alpha}{2}\left[(x/t)^\beta + \nu(x/t)^{-\beta}\right]} \\ T_{t,\beta,\alpha,\nu}(x) &= \frac{\alpha}{2}\left[(x/t)^\beta - \nu(x/t)^{-\beta}\right] - \frac{\alpha}{2}(1-\nu), \end{aligned} \tag{4.7}$$

where $\rho = (\nu+1)\alpha/2$. The prototype of (4.7) is apparently a distribution with density

$$g_{\mu,\sigma,\alpha,\nu}(y) = \frac{\nu^{\rho/2}}{2K_\rho(\alpha\sqrt{\nu})} e^{\rho(y-\mu)} e^{-\frac{\alpha}{2}\left(e^{\frac{y-\mu}{\sigma}} + \nu e^{-\frac{y-\mu}{\sigma}}\right)}.$$

The two frequently used families of the type UE, the Wald and inverse Gaussian, was chosen perhaps owing to their simple normalizing constants. Both of them cannot be further generalized for the t and β parameters. A composite distribution belonging to type UE is the Birnbaum-Saunders distribution [7] with transformed score function of the parent distribution in the form $T(x) = \frac{1}{2}(x - \frac{1}{x}) + \frac{1}{2} - \frac{x}{1+x}$, which is of rather complex form, satisfying, however, to the condition (2.2) and, therefore, ready to be generalized for the pivotal variable (4.1).

5 Systems on finite intervals

5.1 Johnson system on $(0, 1)$

By using relations (3.5) and (3.6) where $\eta'(x) = 1/[x(1-x)]$ and by the use of Table 2 where we put

$$z = \frac{\eta(x) - \eta(t)}{\sigma} = \log\left(\frac{x(1-t)}{t(1-x)}\right)^\beta,$$

we obtain the Johnson system on $(0, 1)$. Distributions of this system are not used excepting the Johnson's U_B distribution, given by

$$\begin{aligned} f_{t,\beta}(x) = \frac{1}{1-x} f^{UP}_{t,\beta,1}(x) &= \frac{\beta}{\sqrt{2\pi}x(1-x)} e^{-\frac{1}{2}\log^2\left(\frac{x(1-t)}{t(1-x)}\right)^\beta} \\ T_{t,\beta}(x) = T^{UP}_{t,\beta,1}(x) &= \beta \log\left(\frac{x(1-t)}{t(1-x)}\right)^\beta, \end{aligned}$$

and the beta distribution

$$\begin{aligned} f_{p,q}(x) = \frac{1}{1-x} f^{BB}_{p/(p+q),1,q,p/q}(x) &= \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1} \\ T_{p,q}(x) = T^{BB}_{p/(p+q),1,q,p/q}(x) &= (p+q)x - p. \end{aligned}$$

Taking into account three supports discussed till now, we can speak about 'triplets' of distributions consisting from the prototype and two transformed distributions. The triplet studied in [5] is the normal, lognormal and Johnson U_B distributions, in [11] the logistic, log-logistic and uniform distributions and the triplet

$$g(x) = \frac{1}{2}e^{-|x|} \qquad f(x) = \begin{cases} 1/2 & x \le 1 \\ 1/2x^2 & x > 1 \end{cases} \qquad h(x) = \begin{cases} x/[2(1-x)] & x \le 0.5 \\ (1-x)/2x & x > 0.5 \end{cases}$$

with Laplace prototype is studied in [6]. We do not include it for the sake of simplicity into the Johnson system described in Table 2. Many other such triplets can be considered, as for instance the generalized logistic of the fourth kind ([7]) with density $f_{p,q}(y) = \frac{1}{B(p,q)} \frac{e^{py}}{(e^y+1)^{p+q}}$, beta-prime with density $f_{p,q}(x) = \frac{1}{B(p,q)} \frac{x^{p-1}}{(x+1)^{p+q}}$ and the beta distributions.

5.2 Other systems

Whereas $\eta(x) = \log x$ is the only reasonable mapping $\eta : (0,\infty) \to \mathbb{R}$, it is easy to find other monotone mappings $\eta : (a,b) \to \mathbb{R}$, leading to 'non-Johnson' systems of distributions on a finite interval. We briefly mention three such systems, resulting from transformations

$$\begin{array}{llrcl} \text{(i)} & \eta : (0,1) \to \mathbb{R} & \eta(x) & = & -\log(-\log x) \\ \text{(ii)} & \eta : (-1,1) \to \mathbb{R} & \eta(x) & = & \tanh^{-1} x \\ \text{(iii)} & \eta : (-\pi/2,\pi/2) \to \mathbb{R} & \eta(x) & = & \tan x \end{array}$$

(i) Let us set $q = -\log x$ so that $\eta'(x) = 1/xq$. As the prototypes, the distributions given in Table 1 can be used. The parent distributions can be provided by shape parameters and by 'pivotal variable on $(0,1)$', which is in this case $z = \log(t/q)^\beta$, where $t = -\log\kappa$ and $\kappa = e^{-e^{-\mu}}$ is the transformed location of the prototype. Since the function (i) is non-symmetric, this system is probably of little use.

(ii) Since $\tanh^{-1}(x) = \frac{1}{2}\log\frac{x-1}{x+1}$, this system is similar to the Johnson system on $\mathcal{Q} = (-1,1)$.

(iii) In this system $\eta'(x) = \cos^{-2} x$. However, it is not apparent what prototypes to use. In order to get simple forms of parent densities, we choose simple forms of functions T, which is on support $(-\pi/2,\pi/2)$ given, using (1.5), by

$$T(x) = \sin 2x - \cos^2 x f'(x)/f(x). \tag{5.1}$$

Some proposals are listed in the second column of Table 3. In the third column are the densities computed by integration of (5.1) and in the next two columns the densities and score functions of the prototypes.

Table 3. Parent distributions of '$\tan x$' system and their prototypes.

type	$T(x)$	$f(x)$	$g(y)$	$S(y)$
UE	$\tan x$	$\frac{1}{\sqrt{2\pi}\cos^2 x} e^{-\frac{1}{2}\tan^2 x}$	$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$	y
BB	$\sin x$	$\frac{1}{2K_1(1)\cos^2 x} e^{-1/\cos x}$	$\frac{1}{2K_1(1)} e^{-\sqrt{1+y^2}}$	$\frac{y}{\sqrt{1+y^2}}$
BR	$\sin 2x$	$1/\pi$	$\frac{1}{\pi}\frac{1}{1+y^2}$	$\frac{2y}{1+y^2}$
$BR-$	$\sin 2x - \cos^2 x$	$\frac{1}{c}e^x$	$\frac{1}{c}\frac{1}{1+y^2} e^{\tan^{-1} y}$	$\frac{2y-1}{1+y^2}$
$BR+$	$\sin 2x + \cos^2 x$	$\frac{1}{c}e^{-x}$	$\frac{1}{c}\frac{1}{1+y^2} e^{-\tan^{-1} y}$	$\frac{2y+1}{1+y^2}$

$c = e^{\pi/2} - e^{-\pi/2}$.

In Table 3, type BR is the parent of the Burr XI distribution and the prototype BR_+ is the parent of the Pearson IV distribution (cf. [7]). The score function of prototypes BR_+ and BR_- do not satisfy condition (2.1) so that the pertaining densities cannot be provided by the location and scale parameters. Let us put

$$S_{G_+}(y) = \frac{2y}{[1+(y+1/2)^2]}, \qquad S_{G_-}(y) = \frac{2y}{[1+(y-1/2)^2]}.$$

The corresponding densities are by (1.1)

$$g_+(y) = \frac{1}{c[1+(y+1/2)^2]} e^{\tan^{-1}(y+1/2)}$$

and

$$g_-(y) = \frac{1}{c[1 + (y - 1/2)^2]} e^{-\tan^{-1}(y-1/2)},$$

where $c = e^{\pi/2} - e^{-\pi/2}$. g_+ and g_- are the heavy-tailed asymmetric parents of the type BR (Fig.2), which can be included into the prototype distributions of the Johnson system given in Table 1.

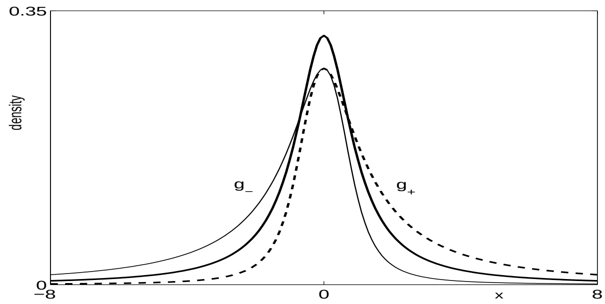

Figure 2. Densities $g_-, g_{0,1,1}$ (Cauchy distribution) and g_+

The parent densities could be further provided by some shape parameters and, at last, by pivotal variable (3.1) and the transformed densities by the 'pivotal variable on $(-\pi/2, \pi/2)$' which is, by (3.7),

$$z = \beta(\tan x - \tan t) = \frac{\beta \sin(x - t)}{\cos x \cos t}.$$

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