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The minC Combination of Belief Functions: derivation and formulas.[1]

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Abstract:

Principal ideas of the minC combination are recalled. A mathematical structure of generalized frames of discernment is analysed and formalized. A generalized schema for a computation of the minC combination is presented. Conflicting belief masses redistribution among non-conflicting focal elements is overviewed. Final general formulas for computation of the minC combination are presented.

Some examples of computation of the minC combination follow. A brief comparison of the minC combination with other combination rules is presented. Related works and approaches and some open problems for a future research are referred in the end.

Keywords:
Belief function, Combination of belief functions, Dempster-Shafer theory, Conflict, Potential conflict, Contradiction, minC combination.

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1 Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing. Belief functions enable representation of incomplete and uncertain knowledge, belief updating and combination of evidence. Belief functions were originally introduced as a principal notion of Dempster-Shafer Theory (DST) or the Mathematical Theory of Evidence [20].

For a combination of beliefs Dempster's rule of combination is used in DST. Under strict probabilistic assumptions, its results are correct and probabilistically interpretable for any couple of belief functions. Nevertheless these assumptions are rarely fulfilled in real applications. It is not uncommon to find examples where the assumptions are not fulfilled and where results of Dempster's rule are counter-intuitive, e.g. see [2, 3], thus a rule with more intuitive results is required in such situations.

Hence, a series of modifications of Dempster's rule were suggested and alternative approaches were created. The classical ones are the Dubois-Prade's rule [15] and the Yager's rule of belief combination rule [24]. Others include a wide class of weighted operators [18], the Transferable Belief Model (TBM) using the so called non-normalized Dempster's rule [23], disjunctive (or dual Demspter's) rule of combination [14], combination 'per elements' with its special case — minC combination, see [7], and other combination rules. It is also necessary to mention the method for application of Dempster's rule in the case of partially reliable input beliefs [16].

The minC combination was originally presented in [4] and in its full version [7]. A motivation and ideas of the minC combination are presented there. The actual combination is investigated and presented only for three-element frames of discernment, nevertheless even this presentation is rather a description of ideas how to compute the minC combination than a presentation of applicable formula(s) for computing. An introduction of such formulas is a topic of the present contribution.

The minC combination has two basic steps: 1) a generalized level on so-called generalized frames of discernment, which allows positive conflicting belief masses, 2) reallocation of these conflicting belief masses to non-conflicting focal elements, i.e., a transformation of generalized results to classic frame of discernment.

Necessary preliminaries are presented in Section 2. Section 3 briefly recalls the principal results of [7].

A mathematical structure of generalized frames of discernment is analysed and formalized in Section 4. A generalized schema for a computation of the minC combination is presented in Section 5. Section 6 brings a detail overview of conflicting belief masses redistribution among non-conflicting focal elements. Final general formulas for computation of the minC combination are presented in Section 7.

For examples of computation of the minC combination see Section 8. Section 9 presents a brief comparison of the minC combination with other combination rules. Related works and approaches are referred in Section 10. The section displays also an open problems for a future research. A concluding Section 11 closes the contribution.

2 Preliminaries

2.1 Basic notions

Let us assume an exhaustive finite *frame of discernment* $\Omega = \{\omega_1, ..., \omega_n\}$, whose elements are mutually exclusive.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$, such that $\sum_{A\subseteq\Omega} m(A) = 1$, the values of bba are called *basic belief masses (bbm)*.[3] A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$, belief function Bel uniquely corresponds to bba m and vice-versa. $\mathcal{P}(\Omega)$ is often denoted also by 2^Ω. A *focal element* is a subset X of the frame of discernment Ω, such that $m(X) > 0$. If a focal element is a one-element subset of Ω we are referring to a *singleton*.

[3] $m(\emptyset) = 0$ is often assumed in accordance with Shafer's definition [20]. A classical counter example is Smets' Transferable Belief Model (TBM) which admits positive $m(\emptyset)$ as it assumes $m(\emptyset) \geq 0$.

Dempster's (conjunctive) rule of combination $\oplus$ is given as
$(m_1 \oplus m_2)(A) = \sum_{X\cap Y=A} K m_1(X)m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X\cap Y=\emptyset} m_1(X)m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [20]; putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\odot$, see e. g. [23].

Yager's rule of combination $\circledY$, see [24], is given as
$(m_1 \circledY m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y=A} m_1(X)m_2(Y)$ for $\emptyset \neq A \subset \Theta$,
$(m_1 \circledY m_2)(\Theta) = m_1(\Theta)m_2(\Theta) + \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset} m_1(X)m_2(Y)$, and
$(m_1 \circledY m_2)(\emptyset) = 0$;

Dubois-Prade's rule of combination $\circledD$ is given as
$(m_1 \circledD m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y=A} m_1(X)m_2(Y) + \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset, X\cup Y=A} m_1(X)m_2(Y)$ for $\emptyset \neq A \subseteq \Theta$, and $(m_1 \circledD m_2)(\emptyset) = 0$, see [15].

We say that two basic belief masses are *conflicting* if they are assigned to disjoint focal elements, $m_1(X)$ is conflicting with $m_2(Y)$ for $X, Y \subset \Omega$ whenever $X \cap Y = \emptyset$. Two belief functions represented by bba's m_1, m_2 are in *full conflict / contradiction* if it holds that $\sum_{X\cap Y=\emptyset} m_1(X)m_2(Y) = 1$, i.e. whenever $\sum_{X\cap Y\neq\emptyset} m_1(X)m_2(Y) = 0$.

An algebra $\mathcal{L} = (L, \wedge, \vee)$ is called a *lattice* if $L \neq \emptyset$ and $\wedge,\ \vee$ are two binary operations on L with the following properties: $x \wedge x = x$, $x \vee x = x$ (idempontency), $x \wedge y = y \wedge x$, $x \vee y = y \vee x$ (commutativity), $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$ (associativity), and $x \wedge (y \vee x) = x$, $x \vee (y \wedge x) = x$ (absorption).
If the operations $\wedge,\ \vee$ satisfy also distributivity, i.e. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ we speak about a *distributive lattice*.
If there are moreover minimal $\perp$ and maximal $\top$ elements in L, i.e. if $x \wedge \perp = \perp$ and $x \vee \top = \top$, and if for any $x \in L$ exist $\overline{x} \in L$ such that $x \vee \overline{x} = \top$ and $x \wedge \overline{x} = \perp$, i.e. if a distributive lattice is *complementary*, we speak about *Boolean lattice*.
We can equivalently write any element of $X \in L$ in *conjunctive normal form* (CNF):
$X = \bigwedge_{i=1,...,m}(\bigvee_{j=1,...,k_i} X_{ij})$ for some m, $k_1, ..., k_m$, $X_{ij} \in L$, i.e. intersection of joins. Dually, X in *disjunctive normal form* is join of intersections $X = \bigvee_{i=1,...,m}(\bigwedge_{j=1,...,k_i} X_{ij})$.

3 An idea of the minC combination

This section briefly present an idea of the minC (minimal conflict or minimal contradiction)[4] combination from [7], thus it is possible to look at [7] for detail.

3.1 Introductory example

Let us assume two believers (belief agents, agents who have their own belief) and their beliefs. We do not know where their beliefs come from or how they were constructed, and we are not interested in such an information for our purposes. For the sake of simplicity we consider a three-element space of discernment $\Omega = \{A, B, C\}$. Let the beliefs of our agents be represented by the following basic belief assignments: $m_1(\{A\}) = a_1 = 0.9$, $m_1(\{C\}) = c_1 = 0.1$ and $m_2(\{B\}) = b_2 = 0.9$, $m_2(\{C\}) = c_2 = 0.1$. Using the Dempster's rule of combination we get $m(\{C\}) = 1$, $m(X) = 0$ for $\{C\} \neq X \subset \Omega$.

We obtain this result because the set $\{C\}$ is the only consensus of both believers, i.e., only the product $c_1 \cdot c_2$ is assigned to a subset of Ω and the other ones produce a conflict (or contradiction) which is normalized, i.e. assigned again to the same subset $\{C\}$ of Ω in this case

This result is correct if both the input belief functions are fully reliable, correctly constructed from probabilistically independent sources.

On the other hand if the above assumptions are not fulfilled, both the believers may be surprised or disappointed with the resulting combined belief. The belief masses assigned to $\{C\}$ by both of them were small. Why then is the result of the combination $m(\{C\}) = 1$, and $m(X) = 0$ for $X \neq \{C\}$? It is possible to explain such a belief combination result as "When two parties fight, the third one prospers".

[4]The term "contradiction" is used in [4, 7], whereas we use "conflict" here because it is more frequent in literature.

Both believers put 0 to one of the elements, but it is not necessary to interpret that as an idea/belief of an absolute exclusion of the element. Maybe the corresponding element is more acceptable (or comparatively acceptable) than $\{C\}$. (0 is not considered by believers as a certain exclusion here.)

There are two different interpretations of such an example. The first interpretation of our example is based on barely-distinguishable elements A and B. Let us not distinguish them now. In this situation we obtain the following belief assignments on coarsened 2-element frame of discernment $\Omega' = \{AB, C\}$: $m'_1(\{AB\}) = ab'_1 = 0.9$, $m'_1(\{C\}) = c'_1 = 0.1$ and $m'_2(\{AB\}) = ab'_2 = 0.9$, $m'_2(\{C\}) = c'_2 = 0.1$. Thus, using Dempster's rule, we get $\frac{81}{82}$ assigned to $\{AB\}$ and only $\frac{1}{82}$ assigned to $\{C\}$, which is a completely different result.

The second interpretation is based on a mistake or some kind of misbelief in some of the initial beliefs (or in both of them). In this case the second result can actually be better than the original one, even if it is not a completely correct belief (but it is natural because the misbelief was transferred).

The presented example is very simple illustrative one, for more complex examples see [3].

3.2 Looking for an associative combination of belief functions

In order to overcome the above problems it was looked for a new associative rule of combination. All the problems come from processing / normalization of conflicting belief masses $m_1(X)m_2(Y)$ where $X \cap Y = \emptyset$. It is also the reason why the non-normalized Dempster's rule is used in Smets' Transferable Belief Model (TBM), but this approach only postpones normalization from the credal level to the decisional one, for detail see [7]. The proportionalized combination [3] makes a proportionalization of $m_1(X)m_2(Y)$ among X, Y, and $X \cup Y$, unfortunately, it is not associative.

The minC approach generalizes both the above approaches: similarly to TBM, positive conflicting bbm's are allowed in its internal level, but different types of conflicts are distinguished as in the proportionalized combination.

The result is an associative combination of *generalized basic belief assignments* on a *generalized frame of discernment* which includes also so-called conflicting sets. After a combination conflicting belief masses are proportionalized among subsets of Ω. Such a proportionalization breaks associativity, hence we have to make all the combination on the generalized level at first, and perform a proportionalization after it, in the end of a combination process. Moreover, it is recommended to keep also the internal generalized results to be prepared for a possible combination with another new input bba.

3.3 Conflict and potential conflicts

A *conflict* (contradiction) between two or more disjunctive subsets of a frame of discernment is denoted with symbol $\times$. A basic idea of the minC approach is that conflict $X \times Y$ between sets X and Y is different from conflict $X \times Z$ between X and Z and both of them are different from $Y \times Z$ and from conflict $X \times Y \times Z$ among all X, Y and Z, when three of more belief functions are combined, for different disjoint sets $X, Y, Z \subset \Omega$. Not to have an infinite set of different conflicts, classes or *types of conflicts* are defined in [4, 7]. It was investigated which conflicts are in the same type and which ones are in different types there. The investigation of conflicts was performed and presented on three-element frame of discernment $\Omega = \{A, B, C\}$ in [4, 7]. We have to note, that there is only one type of conflict $\times$ on the belief functions defined on a two-element frame of discernment, $\times$ corresponds to $m(\emptyset)$ there; hence the generalized level of minC combination fully coincides with the (non-normalized) conjunctive rule there.

When looking for an associative combination, the problems appeared also when multiples of bba's $m_1(X)m_2(Y)$ were assigned to $X \cap Y$ for X, Y such that $X \cap Y \neq \emptyset$ & $X \not\subset Y$ & $Y \not\subset X$. There is no problem when two FB's are combined, but problems arise when three or more FB's should be associatively step-wise combined. To overcome these problems, a *potential conflict* (potential contradiction) $X \times Y$ was defined for $X, Y \in \Omega$, where $X \cap Y \neq \emptyset$ & $X \not\subset Y$ & $Y \not\subset X$. If disjunctivity of conflicting sets should be underlined, we refer to *pure conflicts*.

As an example of a potential conflict we can present $\{A, B\} \times \{B, C\}$, which is not a conflict in the case of combination of two beliefs ($\{A, B\} \cap \{B, C\} = \{B\} \neq \emptyset$), but it can cause a conflict in a later combination with another belief, e.g. real conflict $\{A,B\}\times\{B,C\}\times\{A,C\}$ because there is $\{A,B\}\cap\{B,C\}\cap\{A,C\}=\emptyset$ which is different from $B\times AC$.

For easier handling of conflicts a consideration of *conflicts per elements* is defined and used in the minC approach: a conflict $\{A\}\times\{B,C\}$ is considered as a two-element set $\{A\times B,\ A\times C\}$ of *elementary contradictions* of singletons. In general $X\times Y$ is considered as a set of elementary conflicts $\{x_1\times y_1,...,x_1\times y_s,\ ...,\ x_r\times y_1,...,x_r\times y_s\}$ for $X=\{x_1,...,x_r\}\subseteq\Omega$, $Y=\{y_1,...,y_s\}\subseteq\Omega$. As consideration conflicts per elements is used both for pure and potential conflicts, *trivial conflicts* $\omega_i\times\omega_i$ appear in set expressions of potential conflicts $X\times Y$ for ω_i from $X\cap Y$. Trivial conflicts are not conflicts in fact as belief mass of singleton is not in conflict with itself, thus we can unify $\omega\times\omega$ with ω for any $\omega\in\Omega$, and consequently also $\omega_i\times\omega_i\times...\times\omega_i$ with ω_i and $\omega_i\times\omega_j\times\omega_j\times...\times\omega_j$ with $\omega_i\times\omega_j$ in general.

Using the consideration of conflicts per elements we can observe that pure conflicts are sets of (non-trivial) elementary conflicts, potential conflicts are sets of both trivial and non-trivial elementary conflicts, i.e. both elements of Ω and trivial conflicts between/among them.

Because of the assumption of commutativity and associativity of a final combination rule, also a relation of conflict should be commutative and associative. Thus $\omega_i\times\omega_j$ is assumed to be equivalent to $\omega_j\times\omega_i$, i.e. they are conflicts of the same type: $\omega_i\times\omega_j\sim\omega_j\times\omega_i$, and $(\omega_i\times\omega_j)\times\omega_k\sim\omega_i\times(\omega_j\times\omega_k)\sim\omega_i\times\omega_j\times\omega_k$. Consequently we obtain e.g. $\omega_1\times\omega_2\times\omega_2\times\omega_1\sim\omega_1\times\omega_2\not\sim\omega_1\times\omega_2\times\omega_3$, etc. for elementary conflicts; and $\{\omega_1\}\times\{\omega_2\}\times\{\omega_2\}\times\{\omega_1\}\sim\{\omega_1\}\times\{\omega_2\}\sim\{\omega_2\}\times\{\omega_1\}\sim\{\omega_1\}\times\{\omega_1,\omega_2\}\times\{\omega_2\}\not\sim\{\omega_1\}\times\{\omega_2,\omega_3\}\not\sim\{\omega_1\}\times\{\omega_2\}\times\{\omega_3\}$, etc., for conflicts in general.

It is time to present the last but the principal assumption for conflicts, which appears in the name minC combination: to decrease a number of conflicting belief masses and to decrease a number of types of (non-elementary) conflicts only *minimal conflicts* with respect to conflictness of included elementary conflicts are considered, where $\omega_i<_c\omega_i\times\omega_j<_c\omega_i\times\omega_j\times\omega_k$, etc. I.e. non-conflicting elements are preferred to elementary conflicts which include them and 'shorter' elementary conflict are preferred to 'longer' ones in which the 'shorter' ones are included. Thus we obtain e.g. $\{\omega_1,\omega_2\}\times\{\omega_1,\omega_2\}=\{\omega_1\times\omega_1,\omega_1\times\omega_2,\omega_2\times\omega_1,\omega_2\times\omega_2\}\sim\{\omega_1,\omega_1\times\omega_2,\omega_2\}\sim\{\omega_1,\omega_2\}$ as non-conflicting ω_1 and ω_2 (i.e., trivial conflicts) are less conflicting than $\omega_1\times\omega_2$. Analogously we obtain $X\sim X\times X$ for any $X\subseteq\Omega$ as it was assumed. And the same holds also for any pure or potential conflict X.

3.4 Generalized level of minC combination

Equivalence of conflicts and its classes — types of conflicts, i.e., generalized frame of discernment generated from three-element classic frame $\Omega=\{A,B,C\}$, were found in [4, 7], and rules were established to determine to which element of the generalized frame multiples of generalized bbm's should be assigned when generalized bba is combined with classical one or when two generalized bba's are combined. There are the following 8 types of conflicts $A\times B,\ A\times C,\ B\times C,\ A\times BC,\ B\times AC,\ C\times AB,\ \times,\ \Box$, and the following 3 types of potential conflicts $\Box A,\ \Box B,\ \Box C$, where $A\times B$ is an abbreviation for $\{A\}\times\{B\}=\{A\times B\}$, $A\times BC$ is an abbreviation for $\{A\}\times\{B,C\}=\{A\times B,A\times C\}$, $\times$ is an abbreviation for $A\times B\times C$, $\Box$ is an abbreviation for $AB\times AC\times BC\ \sim\ \{A\times B,A\times C,B\times C\}$, $\Box A$ is an abbreviation for $AB\times AC\ \sim\ \{A,B\times C\}$, etc.

Rules for assigning of multiples of gbbm's are presented in several lemmata and summarized in tables there. We recall the table for combination of a generalized basic belief assignment with a classic one, see Table 3.1, for a table for combination of two generalized bba's see [4, 7].

A final value $m(A)$ of the generalized bba m is obtained as a sum of all multiples corresponding to cells in the table which contains A. For example we can compute:

$m(X)=\sum_{\substack{U\cap V=X,\\ U\subseteq V\vee V\subseteq U}} m_1(U)\cdot m_2(V)+(m_1(X)\cdot m_2(\Box X)+m_1(\Box X)\cdot m_2(X))$, where $X,U,V\subseteq\Omega,\ |X|=1$;

$m(X)=\sum_{\substack{U\cap V=X,\\ U\subseteq V\vee V\subseteq U}} m_1(U)\cdot m_2(V)$, where $X,U,V\subseteq\Omega,\ |X|>1$;

$m(\Box X)=\sum_{\substack{U\cap V=X,\\ U\not\subseteq V\,\&\,V\not\subseteq U}} m_1(U)\cdot m_2(V)+\sum_{X\subset U}(m_1(U)\cdot m_2(\Box X)+m_1(\Box X)\cdot m_2(U))+m_1(\Box X)\cdot m_2(\Box X)$, where $X,U,V\subseteq\Omega,\ |X|=1$; etc.

Specially,

$m(\{A\})=\sum_{\substack{U\cap V=\{A\},\\ U\subseteq V\vee V\subseteq U}} m_1(U)\cdot m_2(V)+(m_1(\{A\})\cdot m_2(\Box A)+m_1(\Box A)\cdot m_2(\{A\}))=m_1(\{A\})\cdot m_2(\{A\})+$

Table 3.1: The table of combination of a generalized bba with a classical one on $\Omega = \{A, B, C\}$.

	$\{A\}$	$\{B\}$	$\{C\}$	$\{B,C\}$	$\{A,C\}$	$\{A,B\}$	$\{A,B,C\}$
$\{A\}$	$\{A\}$	$A\times B$	$A\times C$	$A\times BC$	$\{A\}$	$\{A\}$	$\{A\}$
$\{B\}$	$A\times B$	$\{B\}$	$B\times C$	$\{B\}$	$B\times AC$	$\{B\}$	$\{B\}$
$\{C\}$	$A\times C$	$B\times C$	$\{C\}$	$\{C\}$	$\{C\}$	$C\times AB$	$\{C\}$
$\{B,C\}$	$A\times BC$	$\{B\}$	$\{C\}$	$\{B,C\}$	$\Box C$	$\Box B$	$\{B,C\}$
$\{A,C\}$	$\{A\}$	$B\times AC$	$\{C\}$	$\Box C$	$\{A,C\}$	$\Box A$	$\{A,C\}$
$\{A,B\}$	$\{A\}$	$\{B\}$	$C\times AB$	$\Box B$	$\Box A$	$\{A,B\}$	$\{A,B\}$
$\{A,B,C\}$	$\{A\}$	$\{B\}$	$\{C\}$	$\{B,C\}$	$\{A,C\}$	$\{A,B\}$	$\{A,B,C\}$
$A\times B$	$A\times B$	$A\times B$	$\times$	$A\times B$	$A\times B$	$A\times B$	$A\times B$
$A\times C$	$A\times C$	$\times$	$A\times C$	$A\times C$	$A\times C$	$A\times C$	$A\times C$
$B\times C$	$\times$	$B\times C$	$B\times C$	$B\times C$	$B\times C$	$B\times C$	$B\times C$
$A\times BC$	$A\times BC$	$A\times B$	$A\times C$	$A\times BC$	$A\times BC$	$A\times BC$	$A\times BC$
$B\times AC$	$A\times B$	$B\times AC$	$B\times C$	$B\times AC$	$B\times AC$	$B\times AC$	$B\times AC$
$C\times AB$	$A\times C$	$B\times C$	$C\times AB$	$C\times AB$	$C\times AB$	$C\times AB$	$C\times AB$
$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$\Box$	$A\times BC$	$B\times AC$	$C\times AB$	$\Box$	$\Box$	$\Box$	$\Box$
$\Box A$	$\{A\}$	$B\times AC$	$C\times AB$	$\Box$	$\Box A$	$\Box A$	$\Box A$
$\Box B$	$A\times BC$	$\{B\}$	$C\times AB$	$\Box B$	$\Box$	$\Box B$	$\Box B$
$\Box C$	$A\times BC$	$B\times AC$	$\{C\}$	$\Box C$	$\Box C$	$\Box$	$\Box C$

$m_1(\{A\})\cdot m_2(\{A,B\}) + m_1(\{A\})\cdot m_2(\{A,C\}) + m_1(\{A\})\cdot m_2(\{A,B,C\}) + m_1(\{A,B\})\cdot m_2(\{A\}) +$
$m_1(\{A,C\})\cdot m_2(\{A\}) + m_1(\{A,B,C\})\cdot m_2(\{A\}) + m_1(\{A\})\cdot m_2(\Box A) +$
$m_1(\Box A)\cdot m_2(\{A\})$,
$m(\Box A) = \sum_{\substack{U\cap V=\{A\},\\ U\not\subseteq V \& V\not\subseteq U}} m_1(U)\cdot m_2(V) + \sum_{\{A\}\subset U}(m_1(U)\cdot m_2(\Box A) + m_1(\Box A)\cdot m_2(U)) +$
$m_1(\Box A)\cdot m_2(\Box A) = m_1(\{A,B\})\cdot m_2(\{A,C\}) + m_1(\{A,C\})\cdot m_2(\{A,B\}) + m_1(\Box A)\cdot m_2(\{A,B\}) +$
$m_1(\Box A)\cdot m_2(\{A,C\}) + m_1(\Box A)\cdot m_2(\{A,B,C\}) + m_1(\{A,B\})\cdot m_2(\Box A) + m_1(\{A,C\})\cdot m_2(\Box A) +$
$m_1(\{A,B,C\})\cdot m_2(\Box A) + m_1(\Box A)\cdot m_2(\Box A)$, etc.

Unfortunately there is neither general formula for computing $m(A)$ for arbitrary A from the generalized frame nor any formula nor table for combination of generalized basic belief assignments generated from more-element frames of discernment in [4] nor in [7]. Formulation of such formulas is a principal part of this paper, see Sections 5 and 7.

3.5 Final results of the minC combination

As it was already mentioned, reallocation of the conflicting masses to non-conflicting focal elements is the final step of the minC combination. This step is very easy for potential conflicts which are simply relocated to their non-conflicting parts (to the sets of their trivial elementary conflicts). A proportionalization of pure conflicts follows it. Three proportionalizations are very briefly presented in [4] and [7]. We do not recalled them here because a presentation of different proportionalizations (and distributions more generally) is presented in Section 6 in more detail. The basis of a present new presentation of the minC combination is an analysis and description of a mathematical structure of generalized frames of discernment, see the next section.

The generalized level of minC combination gives non-negative weights to all elements of a generalized frame of discernment, i.e., also to types of conflicts and types of potential conflicts.

The generalized level of minC combination is associative and commutative operation and it commutes also with coarsening of frame of discernment.

Unfortunately proportionalizations of conflicting belief masses break associativity of the minC combination. Hence all the input bba's must be combined on the generalized level at first, and a proportionalization may not be performed before finishing of the generalized level combination. Thus it is useful to keep both the final proportionalized results and the working generalized level ones, because of to be prepared for possible additional source of belief, which we possibly need to combine together with the present input beliefs, i.e., with the present result of the combination respectively.

4 Structure of generalized frame of discernment

Let us look for a mathematical structure of minC generalized frame of discernment Ω_g generated by $\Omega = \{\omega_1, ..., \omega_n\}$. In the previous section we have recalled the special case of a generalized frame of discernment generated by classic 3-element frame of discernment $\Omega = \{\omega_1, ..., \omega_3\} = \{a, b, c\}$. The table shows us to which element of Ω_g should be assigned multiples $m_1(X)m_2(Y)$ on generalized level for any $X, Y \in \Omega_g$. Thus the table defines a mapping $t : \Omega_g \times \Omega_g \longrightarrow \Omega_g$ for $|\Omega| = 3$.

What are general properties of mapping t ? t should be:
– idempotent – as we want to assign $m_1(X)m_2(X)$ back to X
– commutative – as we want to obtain commutative combination, thus also $m_1(X)m_2(Y)$ and $m_1(Y)m_2(X)$ should be assigned to the same $t(X, Y) = t(Y, X)$
– associative – the property follows again from desired property of a final combination rule.

What are the elements of Ω_g ?
1) $X \subseteq \Omega$, i.e., sets of elements $\{\omega_{X1}, ..., \omega_{Xk_X}\}$,
2) pure conflicts: $X \times Y$ for $X \cap Y = \emptyset$, i.e. e.g, $\{\omega_i, \omega_j\}$ or more generally $\{\omega_{X11} \times ... \times \omega_{X1k_{X1}}, \omega_{X21} \times ... \times \omega_{X2k_{X2}}, ..., \omega_{Xm1} \times ... \times \omega_{Xmk_{Xm}}\}$, i.e., sets of non-trivial elementary conflicts,
3) potential conflicts $X \cup Y$, where $X \subset \Omega$ and Y is a pure conflict, i.e.: $\{\omega_{X1}, ..., \omega_{Xk_X}, \omega_{X11} \times ... \times \omega_{X1k_{X1}}, \omega_{X21} \times ... \times \omega_{X2k_{X2}}, ..., \omega_{Xm1} \times ... \times \omega_{Xmk_{Xm}}\}$, i.e. set of both trivial and non-trivial elementary conflicts.
Thus the elements of the generalized frame of discernment are constructed from elements of Ω using sets (grouping several elements together) and of idempotent, commutative and associative operation t.

We can show that the minC generalized frame of discernment, i.e., a structure of pure and potential conflicts, forms a distributive lattice $\mathcal{L}(\Omega) = (L(\Omega), \wedge, \vee)$, where operation meet $\wedge$ coincides with the above operation t, i.e. $X \wedge Y = t(X, Y)$, e.g. $\{\omega_i\} \wedge \{\omega_j\} = t(\{\omega_i\}, \{\omega_j\}) = \{\omega_i\} \times \{\omega_j\} = \{\omega_i \times \omega_j\}$ for $0 \leq i \neq j \leq n$,
and operation join $\vee$ performs creation of sets of elements of Ω, i.e. for example, $\{\omega_i\} \vee \{\omega_j\} = \{\omega_i, \omega_j\}$) for $0 \leq i \neq j \leq n$.

In full generality we have:
$X \vee Y = \{w \mid w \in X \text{ or } w \in Y \text{ and } (\neg\exists w')(w' \in X \cup Y,\ w' \leq_c w)\}$,
$X \wedge Y = \{w \mid w \in X \cap Y$ or $[w = \omega_{w1} \times \omega_{w2} \times ... \times \omega_{wk_w}$, where $(\exists x \in X)(x \leq_c w), (\exists y \in Y)(y \leq_c w)$ and $(\neg\exists w' \leq_c w)((\exists x \in X)(x \leq_c w'), (\exists y \in Y)(y \leq_c w'))]\}$, i.e., meet $X \wedge Y$ contains $w \in X \cap Y$ and elementary conflicts of elements from X with elements from Y, where only minimal conflict w.r.t. to $\leq_c$ are considered.
Where it is further defined: $x \times x = x$, $y \times x = x \times y$, and $w_1 = x_{11} \times x_{12} \times ... \times x_{1k_1} \leq_c x_{21} \times x_{22} \times ... \times x_{2k_2} = w_2$ iff $(\forall x_{1k})(\exists x_{2m})(x_{1k} = x_{2m})$, i.e., w_1 is less conflicting than w_2 or w_1 is subconflict of elementary conflict w_2 (w_2 contains all elements from w_1 + possibly some other(s)). Thus any trivial conflict $\omega_i \times \omega_i$ is $\leq_c$-less than any elementary conflict which contain ω_i, i.e. non-conflicting elements (trivial conflicts) are preferred in $\wedge$.

We can notice that $\vee$ coincides with $\cup$ when it is applied to non-conflicting arguments (i.e. to arguments which are sets of trivial elementary conflicts): when all $w \in X$ are non-conflicting, i.e., $w = \omega_i$ for some i, there cannot be any less conflicting element in $X \cup Y$, hence $w \in X \rightarrow w \in X \vee Y$, and similarly for $w \in Y$. In general $X \vee Y$ is $X \cup Y$ from where all non-minimal conflicts w.r.t. $\leq_c$ are removed.

We can also notice that $\wedge$ coincides with $\cap$ when if $X \subseteq Y$ or $Y \subseteq X$:
let $X \subseteq Y$, $w \in X$ be also in $X \cap Y = X$, as we suppose only minimal conflicts in X, all $w \in X$ are also in $X \wedge Y$ (neither any elementary conflict $w' \in X$ nor any $w'' \in Y$ can be less conflicting than w), thus no element of X is removed; and for any $w_{xy} = \omega_{x_1} \times ... \times \omega_{x_m} \times \omega_{y_1} \times ... \times \omega_{y_k}$ we have $w_x = \omega_{x_1} \times ... \times \omega_{x_m} \leq_c w_{xy}$, where w_x is already in $X = X \cap Y$.

Note that we really need to distinguish between operations $\wedge$ and $\cap$, because $\{\omega_i\} \cap \{\omega_j\} = \emptyset$

whereas $\{\omega_i\} \wedge \{\omega_j\} = \{\omega_i \times \omega_j\} \neq \emptyset$. When identifying $\wedge$ and $\cap$ we put all the conflicts together into one class and we obtain the non-normalized Dempster's rule.

We have already shown reflexivity, commutativity and associativity of $\wedge$ (as properties of $t(\_,\_)$), the same holds also for $\vee$;
reflexivity: $X \vee Y = \{w \,|\, w \in X \text{ and } \neg(\exists w')(w' \in X,\, w' <_c w)\} = X$, because we suppose only $\leq_c$ minimal elementary conflicts in X;
commutativity: $X \vee Y = \{w \,|\, (w \in X \text{ or } w \in Y) \text{ and } \neg(\exists w')\,(w' \in X \cup Y,\, w' <_c w)\} = \{w \,|\, (w \in Y \text{ or } w \in X) \text{ and } \neg(\exists w')\,(w' \in Y \cup X)\, w' <_c w)\} = Y \vee X$;
associativity: $(X \vee Y) \vee V = X \vee (Y \vee V)$, analogically.

The rest is to prove absorption and distributivity of operations $\wedge$ and $\vee$.
$X \wedge (X \vee Y)$: $w \in X \wedge (X \vee Y)$ iff $w \in X \cap (X \vee Y)$ or $w = \omega_{x_1} \times ... \times \omega_{x_m} \times \omega_{y_1} \times ... \times \omega_{y_k}$, thus iff $w \in X \cap (X \cup Y)$ possibly without some non-minimal elementary conflict(s) (from $\vee$) and possibly with some other elementary conflict(s) (from $\wedge$); $X \cap (X \cup Y) = X$, there in no $<_c$ elementary conflict in $X \vee Y$, thus nothing is removed, and all elementary conflicts $w = \omega_{x_1} \times ... \times \omega_{x_m} \times \omega_{y_1} \times ... \times \omega_{y_k}$ are already in X, hence $X \wedge (X \vee Y) = X$.
$X \vee (X \wedge Y)$: $w \in X \vee (X \wedge Y)$ iff $w \in X \cup (X \wedge Y)$ possibly without some non-minimal elementary conflict(s), thus iff $w \in X \cup (X \cap Y)$ possibly without some non-minimal elementary conflict(s) and possibly with some other(s) $w = \omega_{x_1} \times ... \times \omega_{x_m} \times \omega_{y_1} \times ... \times \omega_{y_k}$; $X \cup (X \cap Y) = X$, elementary conflicts possibly added from $X \wedge Y$ cannot be minimal in X, thus they are immediately removed, there is no $\leq_c$ elementary conflict in $X \wedge Y$ than in X, hence nothing from X is removed, $X \vee (X \wedge Y) = X$, and absorption is proved.

Distributivity: $X \wedge (Y \vee Z)$: $w \in X \wedge (Y \vee Z)$ iff $w \in X \cap (Y \cup Z)$ where non-minimal elementary conflicts from $Y \cup Z$ are removed and minimal elementary conflicts $\omega_{x_1} \times ... \times \omega_{x_i} \times \omega_{y_1} \times ... \times \omega_{y_j}$ and $\omega_{x_1} \times ... \times \omega_{x_k} \times \omega_{z_1} \times ... \times \omega_{z_l}$ are added; $(X \wedge Y) \vee (X \wedge Z)$: $w \in (X \wedge Y) \vee (X \wedge Z)$ iff $w \in (X \cap Y) \cup (X \cup Z) = X \cap (Y \cup Z)$ where minimal elementary conflicts $\omega_{x_1} \times ... \times \omega_{x_i} \times \omega_{y_1} \times ... \times \omega_{y_j}$ are added to $X \cap Y$ and $\omega_{x_1} \times ... \times \omega_{x_k} \times \omega_{z_1} \times ... \times \omega_{z_l}$ are added to $X \cap Z$ and finally non-minimal elementary conflicts from $(X \wedge Y) \cup (X \wedge Z)$ are removed, i.e., elementary conflicts $w_{XY} \in X \wedge Y$, such that there exists $w_{XZ} \in X \wedge Z$, $w_{XZ} <_c w_{XY}$, and elementary conflicts $w_{XZ} \in X \wedge Z$, such that there exists $w_{XY} \in X \wedge Y$, $w_{XY} <_c w_{XZ}$ are removed; there is $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$, the same elementary conflicts are added in both the cases, thus it remains to show that same elementary conflict are removed in both the cases. If elementary conflict w which is not minimal in $Y \cup Z$ appears in $(X \wedge Y) \cup (X \wedge Z)$ it can be minimal there only if it differs from the corresponding minimal elementary conflict with some ω_is from X hence it is among elementary conflicts added to $X \cap (Y \vee Z)$ i.e. it is returned to $X \cap (Y \cup Z)$. On the other side if elementary conflict w is non-minimal in $(X \wedge Y) \cup (X \wedge Z)$ it is either non-minimal also in $Y \cup Z$ or it is some of conflicts added to $(X \cap Y) \cup (X \cap Z)$, hence it is also added back to $X \cap (Y \cup Z)$. Hence the same elementary conflicts are added and removed in both the cases and we have proved $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$.
A proof of the second equality $X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$ is analogous.

Thus we have verified that $\wedge$ and $\vee$ satisfy all the properties of lattice operations meet and join, hence $\mathcal{L}(\Omega) = (L(\Omega), \wedge, \vee)$ is really a distributive lattice. Thus the following statement holds.

Statement 1 *MinC generalized frame of discernment generated by classic Shaferian frame of discernment Ω forms a distributive lattice $\mathcal{L}(\Omega) = (\Omega, \wedge, \vee)$, where operations $\wedge$ and $\vee$ are defined as it is above.*

We can notice that the absorption property corresponds with elimination of non-minimal conflicts in the minC combination, let e.g. $X = \{\omega_1\}$, $Y = \{\omega_2\}$. From equation $X \vee (X \wedge Y) = X$ we have $\{\omega_1\} \vee (\{\omega_1\} \wedge \{\omega_2\}) = \{\omega_1, \omega_1 \times \omega_2\} = \{\omega_1\}$, and from equation $X \wedge (X \vee Y) = X$ we obtain $\{\omega_1\} \wedge (\{\omega_1\} \vee \{\omega_2\}) = \{\omega_1\} \times \{\omega_1, \omega_2\} = \{\omega_1,\, \omega_1 \times \omega_2\} = \{\omega_1\}$. Similarly, we can notice that the distributivity equation $(X \vee Y) \wedge (X \vee Z) = X \vee (Y \wedge Z)$ corresponds with arising of potential conflict in the minC combination, see e.g. $(\{\omega_1\} \vee \{\omega_2\}) \wedge (\{\omega_1\} \vee \{\omega_3\}) = \{\omega_1, \omega_2\} \times \{\omega_1, \omega_3\} = \{\omega_1,\, \omega_1 \times \omega_3,\, \omega_2 \times \omega_1,\, \omega_2 \times \omega_3\} = \{\omega_1,\, \omega_2 \times \omega_3\} = \{\omega_1\} \vee (\{\omega_2\} \wedge \{\omega_3\})$. Of course also the other distributivity equation $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$ is in accordance with minC combination: $\{\omega_1\} \wedge (\{\omega_2\} \vee \{\omega_2\}) = \{\omega_1\} \times \{\omega_2, \omega_3\} = \{\omega_1 \times \omega_2,\, \omega_1 \times \omega_3\} = (\{\omega_1\} \wedge \{\omega_2\}) \vee (\{\omega_1\} \wedge \{\omega_3\})$.

Hence all the properties of lattice operations either correspond to some properties of the minC combinations or they are simply in accordance with the minC combination.

The above results correspond with an observation from [21] that a minC generalized frame of discernment generated by classic 3-element frame Ω corresponds with a DSm hyper-power set D^Θ, i.e. with Dedekind lattice $(\Theta, \cap, \cup)$ generated by 3-element frame Θ. But we must pay attention that it can be $\theta_i \cap \theta_j \neq 0$ for $i \neq j$ in DSm theory in general, as overlapping of elements is allowed in there, whereas it is always $\omega_i \cap \omega_j = 0$ for $i \neq j$ in our classic approach. Hence we really need an extra meet operation $\wedge$ different from intersection $\cap$ here, as it is already mentioned above.

We have to mention that the elementary conflict ordering $\leq_c$ is a partial reverse of lattice ordering $\leq$ defined as $x \leq y$ iff $x \wedge y = x$ or dually as $x \leq y$ iff $x \vee y = y$, see e.g. $x \wedge y \wedge z \leq x \wedge y$ as $(x \wedge y \wedge z) \wedge (x \wedge y) = x \wedge y \wedge z$ whereas $x \wedge y \wedge z = x \times y \times z \geq_c x \times y = x \wedge y$, there is no $\leq_c$ ordering of sets of elementary conflicts thus $\{x, y\}$ and $\{x, y, z\}$ are $\leq_c$-non-comparable whereas $x \vee y \leq x \vee y \vee z$, and similarly $x \wedge y \wedge w \leq (x \wedge y) \vee (v \wedge w)$ for $\leq_c$-non-comparable sets of elementary conflicts $\{x \times y \times w\}$ and $\{x \times y, v \times w\}$.

5 General schema of minC combination

We can present a general schema of minC combination now. m^0 is computed on $\mathcal{L}(\Omega)$, such that $m_1(X)m_2(Y)$ should be allocated to $t(X, Y) = X \wedge Y$. $\mathcal{L}(\Omega)$ is closed with respect to $\wedge$ thus there is no problem with empty set, hence we can simply use a generalization of non-normalized conjunction rule to $\mathcal{L}(\Omega)$:

$$m^0(A) = \sum_{X,Y \in \mathcal{L}(\Omega),\, X \wedge Y = A} m_1(X)m_2(Y)$$

for all $A \in \mathcal{L}(\Omega)$.

It is obvious that m^0 is commutative, associative, and non-idempotent operation. It holds neutrality of the vacuous belief function VBF as $X \wedge (\omega_1 \vee ... \vee \omega_n) = X$ for any $X \in \mathcal{L}(\Omega)$.

Hence we have a simple and effective expression of minC on the generalized level. The rest is expression of potential conflicts relocation and of proportionalization of pure conflicts.

Potential conflict relocation. If we combine only 2 belief functions, i.e. m^0 is computed directly from two classic FB's, the situation is quite simple: $m_i(W) = 0$ for $W \in \mathcal{L}(\Omega) \setminus \mathcal{P}(\Omega)$ and potential conflicts arise just for $X, Y \subset \Omega$, such that $X \cap Y \neq 0$, $X \not\subseteq Y$, $Y \not\subseteq X$, hence we can simply relocate $m^0(X \times Y)$ to $X \cap Y$. Thus we can write $m^1(A) = m^0(A) + \sum_{X \cap Y = A,\, X \not\subseteq Y\, Y \not\subseteq X} m_1(X)m_2(Y) = \sum_{X \cap Y = A} m_1(X)m_2(Y)$ for any $\emptyset \neq A \in \mathcal{P}(\Omega)$.

To keep as much as associativity as possible in a combination process, we have to compute all combinations on the generalized level before potential conflict relocation. When processing more than two input belief functions, let us say k, we cannot use m_i values as here m^0 is combination of m_k with a result of generalized combination of the other $k-1$ inputs (or more generally a combination of results of generalized combinations of $0 < m < 1$ and $k - m$ inputs). Thus we have to compute m^1 directly from m^0.

If a relocated potential conflict P arise as $P_1 \times ... \times P_s = P_1 \wedge ... \wedge P_s$ for $P_i \subseteq \Omega$ we can relocate $m(P)$ to $P_1 \cap ... \cap P_s$ analogically to the previous case. We can express any P_i as $\{\omega_{i1}, ..., \omega_{is_i}\}$, and it holds that $P_1 \cap ... \cap P_s \neq \emptyset$, otherwise P would be a pure conflict. As we can rewrite any expression of any element of $\mathcal{L}(\Omega)$ to conjunctive normal form, we can use the above idea for any potential conflict X in the following way. We express P in conjunctive normal form $CNF(P) = P_1 \wedge P_2 \wedge ... \wedge P_k$ and relocate $m(P)$ to $P_1 \cap P_2 \cap ... \cap P_k$. For simplification of formulas we define $\bigcap X$ as $\bigcap X = X_1 \cap X_2 \cap ... \cap X_k$ for any $X \in \mathcal{L}(\Omega)$, where $X_1 \wedge X_2 \wedge ... \wedge X_k$ is $CNF(X)$.

Hence we can express m^1 as it follows:
$m^1(A) = m^0(A) + \sum_{X \neq A,\ X \in \mathcal{L}(\Omega),\ \bigcap X = A} m^0(X) = \sum_{X \in \mathcal{L}(\Omega),\ \bigcap X = A} m^0(X)$, for all $\emptyset \neq A \in \mathcal{P}(\Omega)$.

We have $\sum_{X \in \mathcal{P}(\Omega)} m^1(X) \leq 1$ as gbbm's of pure conflict have not yet been assigned to elements of $\mathcal{P}(\Omega)$. We can set $m^1(A) = 0$ for any $A \in \mathcal{L}(\Omega) \setminus \mathcal{P}(\Omega)$ such that $\bigcap A \neq \emptyset$, because its gbbm has already been relocated to $\bigcap A \in \mathcal{P}(\Omega)$. Nothing is done with pure conflict in this step, thus we can write $m^1(A) = m^0(A)$ for A such that $\bigcap A = \emptyset$. Hence we obtain $\sum_{X \in \mathcal{L}(\Omega)} m^1(X) = 1$.

Final classic bba m we obtain after proportionalization of gbbm's of pure conflicts:

$$
\begin{aligned}
m(A) &= m^1(A) + \sum_{\substack{X\in\mathcal{L}(\Omega)\\ \bigcap X=\emptyset,\ A\subseteq\bigcup X}} prop(A,X)\, m^0(X) \\
&= \sum_{\substack{X\in\mathcal{L}(\Omega)\\ \bigcap X=A}} m^0(X) + \sum_{\substack{X\in\mathcal{L}(\Omega)\\ \bigcap X=\emptyset,\ A\subseteq\bigcup X}} prop(A,X)\, m^0(X)
\end{aligned}
$$

for all $\emptyset \neq A \in \mathcal{P}(\Omega)$, where $prop(A,X)m^0(X)$ is a proportion of gbbm of pure conflict X which should be assigned to A, and $\bigcup X$ is defined as $\bigcup X = X_1 \cup X_2 \cup ... \cup X_k$, where $CNF(X) = X_1 \wedge X_2 \wedge ... \wedge X_k$. $m(\emptyset) = 0$ $(= m^0(\emptyset) = m^1(\emptyset))$.

It holds that $\sum_{X\in\mathcal{P}(\Omega)} m^1(X) = 1$: it was $\sum_{X\in\mathcal{L}(\Omega)} m^0(X) = 1$, all $A \in \mathcal{L}(\Omega)\backslash\mathcal{P}(\Omega)$ such that $\bigcap A \neq \emptyset$ has been relocated to elements of $\mathcal{P}(\Omega)$ already in gbba m^1, and all such that $\bigcap A = \emptyset$ have been proportionalized among elements of $\mathcal{P}(\Omega)$ within construction of the final bba m. Thus m is a correctly defined classic bba defined on $\mathcal{P}(\Omega)$.

When considering various proportionalizations of conflicting belief masses, the minC combination is a family of different rules in fact. Every particular one is given by its corresponding proportionalization.

There are several different proportionalizations suggested in [4, 5, 7, 8], we will discuss them in the following section.

6 Distribution of pure conflicts

As we already have a simple expression for both the generalized level of minC combination and the potential conflicts relocation, we can concentrate ourselves to proportionalization and to reallocation of conflicting belief mass in general. An original simple idea of proportionalization was used in the proportionalized rule of combination [3]. $m_1(X)m_2(Y)$ is distributed among $X, Y, X \cup Y$ in proportion of $m_{\textcircled{\cap}}(X), m_{\textcircled{\cap}}(Y)$ and $m_{\textcircled{\cap}}(X \cup Y)$. This idea was included also among several proportionalizations published in the original papers about minC combination [4, 7] and in the paper about combination per elements [5]. Let us make an overview of all these proportionalizations together with conflict reallocation in classic combination rules. As the conflict reallocation in the classic rules is computed from input belief masses and similarly proportionalizations in [4, 7] was presented only for a case of combination of two belief functions, we suppose that m^0 is a result of combination of 2 FB's now. Thus there is a one-to-one correspondence between type of pure conflict and a couple of subsets of the frame of discernment.

Let us distribute bbm $m^0(X \times Y)$ of a pure conflict $X \times Y$ of two sets $X, Y \subset \Omega$, i.e. $X \cap Y = \emptyset$. There are at least 10 following variants of conflict distribution:

(d0) Yager's approach: whole $m^0(X \times Y)$ is simply relocated to Ω,

(d1) Demspter-Shafer approach: $m^0(X \times Y)$ is distributed (normalized, proportionalized) among all $W \subseteq \Omega$,

(d2) $m^0(X \times Y)$ is proportionalized among all singletons $\{\omega_i\}$, where $\omega_i \in \Omega$,

(d3) Dubois-Prade approach: whole $m^0(X \times Y)$ is relocated to $X \cup Y$, see also (1) in [5],

(d4) $m^0(X \times Y)$ is proportionalized among singletons $\{\omega_i\}$, where $\omega_i \in X \cup Y$, see (0) in [5],

(d5) $m^0(X \times Y)$ is proportionalized between X and Y, see (2) in [5],

(d6) $m^0(X \times Y)$ is proportionalized among X, Y and $X \cup Y$, see (a) in [4, 7],

(d7) $m^0(X \times Y)$ is proportionalized among all $W \in \mathcal{P}(X) \cup \mathcal{P}(Y)$, see (3) in [5],

(d8) $m^0(X \times Y)$ is proportionalized among all $W \in \mathcal{P}(X \cup Y)$, see (c) in [4, 7] and (4) in [5], (d9) $m^0(X \times Y)$ is proportionalized among all sets $\{\omega_i\}$, $\{\omega_j\}$, $\{\omega_i, \omega_j\}$, where $\omega_i \in X, \omega_j \in Y$, see (b) in [4, 7].

All these distributions is easily generalizable to the case n-ary combination, where basic belief mass $m_1(X_1)m_2(X_2)...m_n(X_n)$ of conflict $X_1 \times ... \times X_n$ should be distributed. As it was mentioned above we have a different situation in reallocation of pure conflict in the minC combination as it is binary operation and we have to distribute $m^0(X)$ of pure conflict not knowing the original input bbm's. E.g. pure conflicts $X_1 = \{\omega_1, \omega_2, \omega_3, \omega_4\} \times \{\omega_3, \omega_4\} \times \{\omega_5, \omega_6\}$, $X_2 = \{\omega_3, \omega_4\} \times \{\omega_5, \omega_6\}$, and $X_3 = \{\omega_3, \omega_4\} \times \{\omega_5, \omega_6, \omega_7, \omega_8\}$ are equivalent, they have the same CNF $(\omega_3 \vee \omega_4) \wedge (\omega_5 \vee \omega_6)$, and we cannot recognize from which input bbm's it has been arised. We can reformulate the above distributions (d0) – (d9) as it follows:

(d0) whole $m^0(X)$ is simply relocated to Ω, as before,

(d1) Demspter-Shafer approach: $m^0(X)$ is distributed (normalized, proportionalized) among all $W \subseteq \Omega$, as before,

(d2) $m^0(X)$ is proportionalized among all singletons $\{\omega_i\}$, where $\omega_i \in \Omega$, as before,

(d3) whole $m^0(X)$ is relocated to $\bigcup X$, see (1) in [5],

(d4) $m^0(X)$ is proportionalized among singletons $\{\omega_i\}$, where $\omega_i \in \bigcup X$, see (0) in [5],

(d5) $m^0(X)$ is proportionalized among X_i, where $CNF(X) = X_1 \wedge ... \wedge X_k$, see (2) in [5],

(d6) $m^0(X)$ is proportionalized among $W \in \mathcal{P}(C(X))$, where $C(X) = \{X_1, ..., X_k\}$ such that $CNF(X) = X_1 \wedge ... \wedge X_k$,

(d7) $m^0(X)$ is proportionalized among all $W \in \bigcup_1^k \mathcal{P}(X_i)$, where $X_i \in C(X)$, see (3) in [5],

(d8) $m^0(X)$ is proportionalized among all $W \in \mathcal{P}(\bigcup_1^k X_i)$, where $X_i \in C(X)$, see (4) in [5],

(d9) (d9) has a generalization for n-ary combination, but there is no reasonable generalization of (d9) when $m^0(X)$ should be proportionalized without knowledge of all the original individual input belief masses.

ad (d1): we have to note that (d0) differs from Yager's approach here, as a conflicting bmm combined with any bbm's is conflicting again a the resulting multiple of bbm's is relocated to Ω here, whereas conflicting multiple of two elements is immediately relocate to Ω and after to any bbm which comes with another input bba.
ad (d3): we have to note that similarly (d3) differs from Dubois-Prade approach here. Let us asume e.g. conflicts X_1, X_2, X_3 from the above example. $\bigcup X_1 = \bigcup X_2 = \bigcup X_3$ as $CNF(X_1) = CNF(X_2) = CNF(X_3)$, hence $m^0(X_i)$ is always relocated to $\{\omega_3, \omega_4, \omega_5, \omega_6\} = \omega_3 \vee \omega_4 \vee \omega_5 \vee \omega_6$, whereas $m^0(X_1)$ should be located to $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$ and $m^0(X_3)$ should be relocated to $\{\omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}$ with the n-ary Dubois-Prade rule.

We can reformulate the distributions (d0) – (d4) as it follows:
$distr_0(\Omega, X) = 1,\ distr_0(A, X) = 0$ for $A \neq \Omega$,
$prop_1(A, X) = \frac{m^1(A)}{\sum_{\cap W \neq \emptyset} m^0(W)}$, for $\emptyset \neq A \in \mathcal{P}(\Omega)$, $\emptyset \neq W \in \mathcal{P}(\Omega)$,
$prop_2(\{\omega_i\}, X) = \frac{m^1(\{\omega_i\})}{\sum_1^n m^0(\{\omega_i\})},\ distr_2(A, X) = 0$ for $|A| > 1$,
$distr_3(\bigcup X, X) = 1,\ distr_2(A, X) = 0$ for $A \neq \bigcup X$,
$prop_4(\{\omega\}, X) = \frac{m^1(\{\omega\})}{\sum_{\omega' \in \cup X} m^0(\{\omega'\})},\ distr_4(A, X) = 0$ otherwise.

As it was mentioned in [5] proportionalization $prop_4$ given by distribution (d4) is too sensitive to bbm's of singleton, if nothing is assigned to singletons in m^1, i.e., if nothing is assigned do singletons and to potential conflicts containing just one non-conflicting element in m^0 $m(X)$ cannot be proportionalized, thus this proportionalization is not very useful, the same holds also for $prop_2$ corresponding to distribution (d2). Distributions (d0) and (d3) relocate whole bbm of the conflict to one subset of Ω thus they are not proportionalizations in fact, on the other hand they are simple from the computational point of view. Proportionalization $prop_1$ corresponding to (d1) in normalization in fact, thus it enables only another expression for Dempster's rule. Hence the distributions (d0) – (d5) are not very interesting from the point of view of minC which idea is based on proportionalization of different types of conflicts. Nevertheless we can all of them use in the minC combination, it is enough to generalize our general schema in the following way

$$m(A) \ = \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = A}} m^0(X) \ + \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = \emptyset,\, A \subseteq \bigcup X}} distr(A, X)\, m^0(X)$$

for all $\emptyset \neq A \in \mathcal{P}(\Omega)$, where $distr(A, X)$ is either an expression for proportionalization or another reallocation of conflicting bbm $m^0(X)$. $m(\emptyset) = 0$.

6.1 Proportionalization of pure conflicts

Let us turn our attention to more prospective distributions (d5) – (d8) now. In all of them $m^0(X)$ is proportionalized among some sets W_i from $\mathcal{P}(\bigcup X)$ according their bbm's $m^1(W_i)$, if all these bbm's are equal to zero, i.e., if proportionalization ratio is equal to "$\frac{0}{0}$", we have to define how allocate same portions of $m^0(X)$ to sets W_i. There are 2 ways how to do it: 1) divide $m^0(X)$ by the number w of sets W_i and relocate $\frac{m^0(X)}{w}$ to every W_i or 2) assign whole $m^0(X)$ to union of the sets W_i, i.e. all the subsets of the union obtain 0 bbm, i.e., the same (zero) portion of $m^0(X)$, and whole $m^0(X)$ is a part of plausibility of whole $\bigcup X$ and all its subsets. According to the way of solving proportions "$\frac{0}{0}$" we distinguish 2 variants of all proportionalizations(d5) – (d8). Similarly we can solve "$\frac{0}{0}$" also in (d2) and (d4), where we divide $m^0(X)$ by n or by $|\bigcup X|$ respectively, or we relocate whole $m^0(X)$ to Ω or to $\bigcup X$. On the other hand we do not solve "$\frac{0}{0}$" for $prop_1$ as it is the only distribution which keeps associativity of non-normalized conjunctive rule, and "$\frac{0}{0}$" arises in the case of $prop_1$ only for input beliefs which are in full conflict. Hence we do not make any definition breaking associativity of combination using $prop_1$.

$prop_{5i}(A, X) = \frac{m^1(A)}{\sum_{W \in C(X)} m^1(W)}$, for $A \in C(X)$, where $\sum_{W \in C(X)} m^1(W) > 0$,
$prop_{5i}(A, X) = 0$, for $A \notin C(X)$, where $\sum_{W \in C(X)} m^1(W) > 0$,
$prop_{51}(A, X) = \frac{1}{|C(X)|}$, for $A \in C(X)$, $\sum_{W \in C(X)} m^1(W) = 0$,
$prop_{51}(A, X) = 0$, for $A \notin C(X)$, $\sum_{W \in C(X)} m^1(W) = 0$,
$prop_{52}(C(X), X) = 1$, where $\sum_{W \in C(X)} m^1(W) = 0$,
$prop_{52}(A, X) = 0$, for $A \neq C(X)$, $\sum_{W \in C(X)} m^1(W) = 0$,

$prop_{6i}(A, X) = \frac{m^1(A)}{\sum_{W \in \mathcal{P}(C(X))} m^1(W)}$, for $A \in \mathcal{P}(C(X))$, where $\sum_{W \in \mathcal{P}(C(X))} m^1(W) > 0$,
$prop_{6i}(A, X) = 0$, for $A \notin \mathcal{P}(C(X))$, where $\sum_{W \in \mathcal{P}(C(X))} m^1(W) > 0$,
$prop_{61}(A, X) = \frac{1}{|\mathcal{P}(C(X))|}$, for $A \in \mathcal{P}(C(X))$, $\sum_{W \in \mathcal{P}(C(X))} m^1(W) = 0$,
$prop_{61}(A, X) = 0$, for $A \notin \mathcal{P}(C(X))$, $\sum_{W \in \mathcal{P}(C(X))} m^1(W) = 0$,
$prop_{62}(\mathcal{P}(C(X)), X) = 1$, where $\sum_{W \in \mathcal{P}(C(X))} m^1(W) = 0$,
$prop_{62}(A, X) = 0$, for $A \neq \mathcal{P}(C(X))$, $\sum_{W \in \mathcal{P}(C(X))} m^1(W) = 0$,

$prop_{7i}(A, X) = \frac{m^1(A)}{\sum_{W \in \bigcup_1^k \mathcal{P}(X_i)} m^1(W)}$, for $A \in \bigcup_1^k \mathcal{P}(X_i)$, where $X_i \in C(X)$ and $\sum_{W \in \bigcup_1^k \mathcal{P}(X_i)} m^1(W) > 0$,
$prop_{7i}(A, X) = 0$, for $A \notin \bigcup_1^k \mathcal{P}(X_i)$, where $X_i \in C(X)$ and $\sum_{W \in \bigcup_1^k \mathcal{P}(X_i)} m^1(W) > 0$,
$prop_{71}(A, X) = \frac{1}{|\bigcup_1^k \mathcal{P}(X_i)|}$, for $A \in \bigcup_1^k \mathcal{P}(X_i)$, where $X_i \in C(X)$ and $\sum_{W \in \bigcup_1^k \mathcal{P}(X_i)} m^1(W) = 0$,
$prop_{71}(A, X) = 0$, for $A \notin \bigcup_1^k \mathcal{P}(X_i)$, where $X_i \in C(X)$ and $\sum_{W \in \bigcup_1^k \mathcal{P}(X_i)} m^1(W) = 0$,
$prop_{72}(\bigcup_1^k \mathcal{P}(X_i), X) = 1$, where $X_i \in C(X)$ and $\sum_{W \in \bigcup_1^k \mathcal{P}(X_i)} m^1(W) = 0$,
$prop_{72}(A, X) = 0$, for $A \neq \bigcup_1^k \mathcal{P}(X_i)$, where $X_i \in C(X)$ and $\sum_{W \in \bigcup_1^k \mathcal{P}(X_i)} m^1(W) = 0$,

$prop_{8i}(A,X) = \frac{m^1(A)}{\sum_{W\in\mathcal{P}(\bigcup X)} m^1(W)}$, for $A \in \mathcal{P}(\bigcup X)$, where $\sum_{W\in\mathcal{P}(\bigcup X)} m^1(W) > 0$,
$prop_{8i}(A,X) = 0$, for $A \notin \mathcal{P}(\bigcup X)$, where $\sum_{W\in\mathcal{P}(\bigcup X)} m^1(W) > 0$,
$prop_{81}(A,X) = \frac{1}{|\mathcal{P}(\bigcup X)|}$, for $A \in \bigcup_1^k \mathcal{P}(X_i)$, where $\sum_{W\in\bigcup_1^k \mathcal{P}(X_i)} m^1(W) = 0$,
$prop_{81}(A,X) = 0$, for $A \notin \bigcup X$, where $\sum_{W\in\bigcup_1^k \mathcal{P}(X_i)} m^1(W) = 0$,
$prop_{82}(\mathcal{P}(\bigcup X),X) = 1$, where $\sum_{W\in\mathcal{P}(\bigcup X)} m^1(W) = 0$,
$prop_{82}(A,X) = 0$, for $A \neq \bigcup X$, where $\sum_{W\in\mathcal{P}(\bigcup X)} m^1(W) = 0$,

Using the above proportionalizations we have 8 different variants of minC combination for $i = 5,6,7,8$, $j = 1,2$. We can express them with a general formula as it follows:

$$\begin{aligned} m(A) \;&=\; m^1(A) \;+ \sum_{\substack{X\in\mathcal{L}(\Omega)\\ \bigcap X=\emptyset,\ A\subseteq\bigcup X}} prop_{ij}(A,X)\,m^0(X) \\ &= \sum_{\substack{X\in\mathcal{L}(\Omega)\\ \bigcap X=A}} m^0(X) \;+ \sum_{\substack{X\in\mathcal{L}(\Omega)\\ \bigcap X=\emptyset,\ A\subseteq\bigcup X}} prop_{ij}(A,X)\,m^0(X) \end{aligned}$$

for all $\emptyset \neq A \in \mathcal{P}(\Omega)$,
$m(\emptyset) = 0$.

Proportionalizations $prop_{6j}$ correspond with the original idea of proportionalization of a conflict $U \times V$ among U, V, and $U \cup V$ when just two beliefs are combined. Proportionalizations $prop_{5j}$ proportionalize the same conflict only among $U$and V. Proportionalizations $prop_{7j}$ distribute conflict X among subsets of $W_i \in C(X)$ and $prop_{8j}$ among all subsets of $\bigcup X$.

Distribution 1) of $m^0(X)$ in the case of "$\frac{0}{0}$" ratio to k same pieces more corresponds to the original idea of proportionalization as all the corresponding set obtain the same part of $m^0(X)$.

Let us note that distribution 2) of $m^0(X)$ in the case of "$\frac{0}{0}$" (i.e. relocation of whole $m^0(X)$) is the same for all proportionalizations $prop_{42}, \ldots.., prop_{82}$. We have: $\mathcal{P}(X) = \{Y | Y \subseteq X\} = \bigcup_{Y\subseteq X} Y = X$ and $X = X_1 \wedge \ldots. \wedge X_k$. Further $C(X) = \{X_1, \ldots, X_k\} = X_1 \cup \ldots \cup X_k = \bigcup X$, $\mathcal{P}(C(X)) = C(X) = \bigcup X$, $\bigcup \mathcal{P}(X_i) = \bigcup X_i = C(X) = \bigcup X$, $\mathcal{P}(\bigcup X) = \bigcup X$, and $\bigcup_{\omega_i \in \cup X} \{\omega_i\} = \bigcup X$. This relocation of m^0 to the only set corresponds to the fact that all $m^0(W)$ are zero for the sets W among which $m^0(X)$ should be redistributed, thus we distribute also same zero proportions of $m^0(X)$ to these sets and the 'remaining' whole $m^0(X)$ is assigned to $\bigcup X$ as we have no other reason to distribute it to less subsets.

7 Final formulas for minC combination

The proportionalizations $prop_{6j}$ and $prop_{8j}$ are more favourable to the others. They also correspond to proportionalizations (a) and (c) in the original papers [4, 7] about the minC combination. Thus, we present the minC combination rule parameterized with $prop_{61}$ and $prop_{82}$ for example of the final formulas for the minC combination. Let us substitute $prop_{61}$ and $prop_{82}$ for $prop_{ij}$ to the general schema of the minC combination:

$$m_{ij}(A) \;= \sum_{\substack{X\in\mathcal{L}(\Omega)\\ \bigcap X=A}} m^0(X) \;+ \sum_{\substack{X\in\mathcal{L}(\Omega)\\ \bigcap X=\emptyset,\ A\subseteq\bigcup X}} prop_{ij}(A,X)\,m^0(X)$$

for all $\emptyset \neq A \in \mathcal{P}(\Omega)$,
$m_{ij}(\emptyset) = 0$.

For proportionalization $prop_{61}$ we obtain:

$$m_{61}(A) \;= \sum_{\substack{X\in\mathcal{L}(\Omega)\\ \bigcap X=A}} m^0(X) \;+ \sum_{\substack{X\in\mathcal{L}(\Omega)\\ \bigcap X=\emptyset,\ A\subseteq\bigcup X}} prop_{61}(A,X)\,m^0(X)$$

for all $\emptyset \neq A \in \mathcal{P}(\Omega)$,
$m_{61}(\emptyset) = 0$.

Using the formulas for proportinalization $prop_{61}$ we obtain the following:

$$m_{61}(A) \;=\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = A}} m^0(X) \;+\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = \emptyset,\, A \subseteq \bigcup X}} \frac{m^1(A)}{\sum_{W \in \mathcal{P}(C(X))} m^1(W)}\, m^0(X)$$

for $A \in \mathcal{P}(C(X))$, where $\sum_{W \in \mathcal{P}(C(X))} m^1(W) > 0$,

$$m_{61}(A) \;=\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = A}} m^0(X) \;+\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = \emptyset,\, A \subseteq \bigcup X}} \frac{1}{|\mathcal{P}(C(X))|}\, m^0(X)$$

for $A \in \mathcal{P}(C(X))$, $\sum_{W \in \mathcal{P}(C(X))} m^1(W) = 0$,

$$m_{61}(A) \;=\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = A}} m^0(X)$$

for $A \notin \mathcal{P}(C(X))$,
$m_{61}(\emptyset) = 0$.

For proportionalization $prop_{82}$ we obtain:

$$m_{82}(A) \;=\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = A}} m^0(X) \;+\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = \emptyset,\, A \subseteq \bigcup X}} prop_{82}(A, X)\, m^0(X)$$

for all $\emptyset \neq A \in \mathcal{P}(\Omega)$,
$m_{82}(\emptyset) = 0$.

Using the formulas for proportinalization $prop_{82}$ we obtain the following:

$$m_{82}(A) \;=\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = A}} m^0(X) \;+\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = \emptyset,\, A \subseteq \bigcup X}} \frac{m^1(A)}{\sum_{W \in \mathcal{P}(\bigcup X)} m^1(W)}\, m^0(X)$$

for $A \in \mathcal{P}(\bigcup X)$, where $\sum_{W \in \mathcal{P}(\bigcup X)} m^1(W) > 0$,

$$m_{82}(\mathcal{P}(\bigcup X)) \;=\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = A}} m^0(X) \;+\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = \emptyset,\, A \subseteq \bigcup X}} m^0(X)$$

where $\sum_{W \in \mathcal{P}(\bigcup X)} m^1(W) = 0$,

$$m_{82}(A) \;=\; \sum_{\substack{X \in \mathcal{L}(\Omega) \\ \bigcap X = A}} m^0(X)$$

otherwise,
i.e., for $A \notin \mathcal{P}(\bigcup X)$, and for $A \neq \bigcup X$, where $\sum_{W \in \mathcal{P}(\bigcup X)} m^1(W) = 0$,
$m_{82}(\emptyset) = 0$.

Let us remember: $m^0(A) = \sum_{X,Y \in \mathcal{L}(\Omega),\, X \wedge Y = A} m_1(X) m_2(Y)$ for all $A \in \mathcal{L}(\Omega)$, where m_i are classic or generalized bba's. $m^1(A) = \sum_{X \in \mathcal{L}(\Omega),\, \bigcap X = A} m^0(X)$, for all $\emptyset \neq A \in \mathcal{P}(\Omega)$, when computed from gbba m^0, or equivalently, $m^1(A) = \sum_{X,Y \in \mathcal{L}(\Omega),\, X \cap Y = A} m_1(X) m_2(Y)$, for all $\emptyset \neq A \in \mathcal{P}(\Omega)$, when computed from two classic or generalized bba's m_i.

8 Examples

Let us show computation of minC combination on several examples. We will start with the well known Zadeh's example. Further we present minC results for Smets' bomb example and one more example which distinguishes different proportionalizations.

8.1 Zadeh's example

Let us have two following input bba's m_1 and m_2 on a frame $\Omega = \{\omega_1, \omega_2, \omega_3\}$:
$m_1(\{\omega_1\}) = 0.99,\ m_1(\{\omega_2\}) = 0,\ m_1(\{\omega_3\}) = 0.01,$
$m_2(\{\omega_1\}) = 0,\ m_2(\{\omega_2\}) = 0.99,\ m_2(\{\omega_3\}) = 0.01.$
On the generalized level on $\mathcal{L}(\Omega)$ we obtain
$m^0(\omega_1) = 0,\ m^0(\omega_2) = 0,\ m^0(\omega_3) = 0.0001,$
$m^0(\omega_1 \wedge \omega_2) = 0.9801, m^0(\omega_1 \wedge \omega_3) = 0.0099,\ m^0(\omega_2 \wedge \omega_3) = 0.0099.$
Because $m^0(\{\omega_3\}) > 0$ and $\{\omega_3\}$ is the only non-conflicting focal, both $m^0(\omega_1 \wedge \omega_3)$ and $m^0(\omega_2 \wedge \omega_3)$ are whole relocated to ω_3 for any proportionalization $prop_{ij}$, $i = 5, ..., 8, j = 1, 2$: as $\{\omega_3\} \in C(\omega_1 \wedge \omega_3) = \{\{\omega_1\}, \{\omega_3\}\} = \bigcup_{i=1}^{k} \mathcal{P}(X_i) = \mathcal{P}(\{\omega_1\}) \cup \mathcal{P}(\{\omega_3\}) = \{\{\omega_1\}\} \cup \{\{\omega_3\}\} = \{\{\omega_1\}, \{\omega_3\}\} \subset \mathcal{P}(C(\omega_1 \wedge \omega_3)) = \{\{\omega_1\}, \{\omega_3\}, \{\omega_1, \omega_3\}\} = \bigcup \omega_1 \wedge \omega_3$, and analogically $\{\omega_3\} \in C(\omega_2 \wedge \omega_3) = \bigcup_{i=1}^{k} \mathcal{P}(X_i) \subset \mathcal{P}(C(\omega_2 \wedge \omega_3)) = \bigcup \omega_2 \wedge \omega_3$.
As $m^0(X) = 0$ for all $X \in \bigcup \omega_1 \wedge \omega_2$, a proportionalization ratio is not defined (it is "$\frac{0}{0}$") for $\omega_1 \wedge \omega_2$, thus $m^0(\omega_1 \wedge \omega_2)$ is either distributed in the same portions between $\{\omega_1\}, \{\omega_2\}$ with $prop_{i1}$ with $prop_{i1}$ for $i = 5, 7$, or among $\{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}$ with $prop_{i1}$ for $i = 6, 8$, or it is whole relocated to $\{\omega_1, \omega_2\}$ with $prop_{i2}$ for $i = 5, ..., 8$.
Hence we obtain resulting:
$m_{i1}(\{\omega_1\}) = 0.49005,\ m_{i1}(\{\omega_2\}) = 0.49005,\ m_{i1}(\{\omega_1, \omega_2\}) = 0,\ m_{i1}(\{\omega_3\}) = 0.0199,$ for $i = 5, 7$,
$m_{i1}(\{\omega_1\}) = 0.3267,\ m_{i1}(\{\omega_2\}) = 0.3267,\ m_{i1}(\{\omega_1, \omega_2\}) = 0.3267,\ m_{i1}(\{\omega_3\}) = 0.0199,$ for $i = 6, 8$,
$m_{i2}(\{\omega_1\}) = 0,\ m_{i2}(\{\omega_2\}) = 0,\ m_{i2}(\{\omega_1, \omega_2\}) = 0.9801,\ m_{i2}(\{\omega_3\}) = 0.0199,$ for $i = 5, ..., 8$
according to used proportionalization $prop_{ij}$.

8.2 Smets' bomb example

Let us have a frame $\Omega = \{\omega_1,\ \omega_2,\ \omega_3\} = \{SmallBomb, BigBomb, FalseBomb\}$ and two observers. The first observer reports a small bomb, i.e. $m_1(\{\omega_1\}) = 1$, whereas the second reports a big bomb, i.e. $m_2(\{\omega_2\}) = 1$.
On the generalized level we obtain $m^0(\omega_1 \wedge \omega_2) = 1$ and after redistribution of conflicting belief masses we obtain resulting:
$m_{i1}(\{\omega_1\}) = 0.5,\ m_{i1}(\{\omega_2\}) = 0.5,\ m_{i1}(\{\omega_1, \omega_2\}) = 0$, using proportionalization $prop_{i1}$ for $i = 5, 7$, or
$m_{i1}(\{\omega_1\}) = 0.3333,\ m_{i1}(\{\omega_2\}) = 0.3333,\ m_{i1}(\{\omega_1, \omega_2\}) = 0.3333$, using proportionalization $prop_{i1}$ for $i = 6, 8$, or
$m_{i2}(\{\omega_1\}) = 0\ m_{i2}(\{\omega_2\}) = 0,\ m_{i2}(\{\omega_1, \omega_2\}) = 1$, using proportionalization $prop_{i2}$ for $i = 5, 6, 7, 8$.
$m_{ij}(X) = 0$ for all other $X \in \mathcal{P}(\Omega)$ for all $i = 5, ..., 8,\ j = 1, 2$.
All the results correspond with our expectation that the observed object is a real bomb (not a false bomb).

8.3 A new example

Let us have two input bba's m_1 and m_2 on a frame $\Omega = \{\omega_1, \omega_2, \omega_3\}$ again:
$m_1(\{\omega_1\}) = 0.5,\ m_1(\{\omega_1, \omega_2\}) = 0.3,\ m_1(\{\omega_1, \omega_2, \omega_3\}) = 0.2,$
$m_2(\{\omega_3\}) = 0.6,\ m_2(\{\omega_1, \omega_2, \omega_3\}) = 0.4.$

On the generalized level on $\mathcal{L}(\Omega)$ we obtain four non-conflicting focal elements:
$m^0(\omega_1) = 0.20,\ m^0(\omega_3) = 0.12,\ m^0(\omega_1 \vee \omega_2) = 0.12,\ m^0(\omega_1 \vee \omega_2 \vee \omega_3\}) = 0.08,$

and two conflicting focal elements $m^0(\omega_1 \wedge \omega_3) = 0.30$, $m^0((\omega_1 \vee \omega_2) \wedge \omega_3\}) = 0.18$, there are no potential conflicts, thus $m^1(X) = m^0(X)$.

$\sum_{X \in C(\omega_1 \wedge \omega_3)} m^1(X) = m^0(\omega_1) + m^0(\omega_3) > 0$ and $m^1(\omega_1 \vee \omega_3) = m^0(\omega_1 \vee \omega_3) = 0$, hence $m^0(\omega_1 \wedge \omega_3)$ is proportionalized between $\{\omega_1\}$ and $\{\omega_3\}$ for any proportionalization $prop_{ij}$ for $i = 5, ..., 8$, $j = 1, 2$, as zero portion of $m^0(\omega_1 \wedge \omega_3)$ is relocated to $\{\omega_1, \omega_3\}$ with $prop_{ij}$ in this case[5] .

$\sum_{X \in C((\omega_1 \vee \omega_2) \wedge \omega_3)} m^1(X) = m^0(\omega_1 \vee \omega_2) + m^0(\omega_3) > 0$ and $m^1(\omega_2) = m^1(\omega_1 \vee \omega_3) = m^1(\omega_2 \vee \omega_3) = 0$, thus $m^0((\omega_1 \vee \omega_2) \wedge \omega_3)$ is proportionalized:
between $\{\omega_1, \omega_2\}$ and $\{\omega_3\}$ in ratio $1 : 1$ with proportionalization $prop_{5j}$ for $j = 1, 2$,
among $\{\omega_1, \omega_2\}$, $\{\omega_3\}$ and $\{\omega_1, \omega_2, \omega_3\}$ in ratio $3 : 3 : 2$ with proportionalization $prop_{6j}$ for $j = 1, 2$,
among $\{\omega_1\}$, $\{\omega_1, \omega_2\}$, and $\{\omega_3\}$ in ratio $5 : 3 : 3$ with proportionalization $prop_{7j}$ for $j = 1, 2$,
or among $\{\omega_1\}$, $\{\omega_3\}$ $\{\omega_1, \omega_2\}$, and $\{\omega_1, \omega_2, \omega_3\}$ in ratio $5 : 3 : 3 : 2$ with proportionalization $prop_{8j}$[6] for $j = 1, 2$, respectively.

Thus we have $m_{5j}(\{\omega_1\}) = 0.20 + \frac{5}{8}\,0.30$, $m_{5j}(\{\omega_3\}) = 0.12 + \frac{3}{8}\,0.30 + \frac{1}{2}\,0.18$, $m_{5j}(\{\omega_1, \omega_2\}) = 0.12 + \frac{1}{2}\,0.18$, $m_{5j}(\{\omega_1, \omega_2, \omega_3\}) = 0.08$, for $prop_{5j}$, and analogically for the other proportionalizations.

Hence we obtain resulting:
$m_{5j}(\{\omega_1\}) = 0.3875$, $m_{5j}(\{\omega_3\}) = 0.3225$, $m_{5j}(\{\omega_1, \omega_2\}) = 0.2100$, $m_{5j}(\{\omega_1, \omega_2, \omega_3\}) = 0.0800$,
$m_{6j}(\{\omega_1\}) = 0.3875$, $m_{6j}(\{\omega_3\}) = 0.3000$, $m_{6j}(\{\omega_1, \omega_2\}) = 0.1875$, $m_{6j}(\{\omega_1, \omega_2, \omega_3\}) = 0.1250$,
$m_{7j}(\{\omega_1\}) \approx 0.4693$, $m_{7j}(\{\omega_3\}) \approx 0.2816$, $m_{8j}(\{\omega_1, \omega_2\}) \approx 0.1692$, $m_{7j}(\{\omega_1, \omega_2, \omega_3\}) = 0.0800$,
$m_{8j}(\{\omega_1\}) \approx 0.4567$, $m_{8j}(\{\omega_3\}) \approx 0.2740$, $m_{8j}(\{\omega_1, \omega_2\}) \approx 0.1615$, $m_{8j}(\{\omega_1, \omega_2, \omega_3\}) \approx 0.1077$,
$m_{ij}(\{\omega_2\}) = m_{ij}(\{\omega_1, \omega_3\}) = m_{ij}(\{\omega_2, \omega_3\}) = 0$ for any $i = 5, ..., 8$, $j = 1, 2$.

We can notice that there is no difference between using proportionalizations $prop_{i1}$ and $prop_{i2}$ for $i = 5, ...6$, in this example, it is due to the fact that $\sum_{X \in C(X)} > 0$ for all conflicting focal elements $X \in \mathcal{L}(\Omega)$.

For distinguishing of all 8 different proportionalizations $prop_{ij}$ for $i = 5, ..., 8$, $j = 1, 2$, we need a more complicated example, we need to add at least another two elements to Ω.

9 A comparison with another combination rules

Let us start with a simple case of 2-element frame of discernment $\Omega_2 = \{\omega_1, \omega_n\}$. There is only one type of conflict $\omega_1 \times \omega_2$ on Ω_2, thus the generalized level of minC coincides with the non-normalized conjunctive (non-normalized Dempster's) rule $\circledcirc$ there. MinC with any proportionalization $prop_{ij}$ for $i = 1, 6 - 8$, $j = 1, 2$ for any set of input belief functions which are not in full conflict (contradiction), i.e. $m^0(\{\omega_1 \times \omega_2\}) < 1$, coincides with Dempster's rule and with proportionalized combination from [3]. Because the idea of proportionalizations of conflicts was only briefly sketched in [7], the problem of fully conflicting belief functions is not discussed there at all. Dempster's rule is not and it cannot be defined for belief functions which are in full contradiction, whereas minC combination is defined there, because it has to solve similarly contradictions which are not full, see e.g. Zadeh's example.

On general n-element frame of discernment $\Omega_n = \{\omega_1, ..., \omega_n\}$ the minC combination with proportionalization $proj_{1j}$ for $j = 1, 2$ coincides with Dempster's rule $\oplus$ for any set of inputs which are not in full conflict.

The minC combination with proportionalization $proj_{0j}$ for $j = 1, 2$ for any two input belief functions coincides with Yager's rule Ⓨ. It can differ from Ⓨ when more input belief functions are combined, as all combination is performed on the generalized level at first, and conflict mass is assigned to Ω_n in the end when all combination on the generalized level is already finished. It coincides with k-ary Yager's rule when k input belief functions is combined.

The minC combination with proportionalization $proj_{3j}$ for $j = 1, 2$ for any two input belief functions coincides with Dubois-Prade's rule Ⓓ. Analogically to the case of $prop_{0j}$, it can differ from Ⓓ when more input belief functions are combined, as all combination is performed on the generalized

[5]To strictly follow the formulas from the previous section, it would be more correct to say that $m^0(\omega_1 \wedge \omega_3)$ is proportionalized among $\{\omega_1\}$, $\{\omega_3\}$ and $\{\omega_1, \omega_3\}$ in ratio $5 : 3 : 0$ for proportionalizations $prop_{ij}$ for $i = 6, 8$, $j = 1, 2$.

[6]Let us note again that more correct it would be to say that $m^0((\omega_1 \vee \omega_2) \wedge \omega_3)$ is proportionalized among $\{\omega_1\}$, $\{\omega_2\}$, $\{\omega_3\}$ $\{\omega_1, \omega_2\}$, $\{\omega_1, \omega_3\}$, $\{\omega_2, \omega_3\}$, and $\{\omega_1, \omega_2, \omega_3\}$ in ratio $5 : 0 : 3 : 3 : 0 : 0 : 2$ with proportionalization $prop_{8j}$ for $j = 1, 2$.

level and conflict mass is assigned to unions in the end when all combination on the generalized level is already finished. It coincides neither with k-ary Dubois-Prade's rule when k input belief functions is combined, because $\bigcup(\bigcup_{i=1}^{k} X_i) \subseteq \bigcup_{i=1}^{k} X_i$ (not equal) in general, see also Section 6.

The minC combination with proportionalization $proj_{6j}$ for $j = 1, 2$ for any two input belief functions coincides with the proportionalized combination from [3]. Analogically to the cases of $prop_{0j}$ and $prop_{3j}$, it can differ from the proportionalized combination when more input belief functions are combined, as all combination is performed on the generalized level and conflict mass of pure conflicts is proportionalized in the end when all combination on the generalized level is already finished.

It coincides neither with k-ary proportionalization rule when k input belief functions is combined, because $\mathcal{P}(C(X_1 \cup ... \cup X_k)) \subseteq \mathcal{P}(X_1 \cup ... \cup X_k)$ in general.

10 Related works

In this section we briefly mention another related approaches, namely approaches where a lattice structure is used for combination of belief functions. We also briefly recall an atempt to simplify a computation of the minC combination using coarsening and refinement.

We have to start with the already mentioned proportionalized combination [3] which was the predecessor of the minC combination which uses a proportionalization of conflicts; it was defined on the classic structure of the power set of a frame of discernment $\mathcal{P}(\Omega)$. Beside it, we have to mention also a combination 'per elements' from [5]; unfortunately mathematical structure of this combination has not been investigated till now.

Among lattice-based combinations we have to mention Besnard et al. approach [1] at first. Their lattice *propositional space* $\langle\Omega, \mathcal{R}\rangle$ corresponds with a minC generalized frame of discernment. But its motivation is a possibility to represent belief that two or more elements can appear simultaneously. Moreover, to express that some elements are mutually exclusive, there is an exclusivity relation $\mathcal{E}$ on Ω^2. If ω_1 and ω_2 are mutually exclusive (i.e., $(\omega_1, \omega_2) \in \mathcal{E}$) it holds that $\omega_1 \wedge \omega_2 \approx \perp$, where $\perp$ is the least element of $\mathcal{R}$ and $\approx$ is an equivalence relation, thus all conflicts are equivalent in this approach. Belief functions, plausibility and commonality functions are defined on *evidential structure* $\langle\Omega, \mathcal{R}, \mathcal{E}\rangle$. As Besnard's approach was developed from TBM approach it is allowed that belief mass of $\perp$, i.e, conflicting belief mass, is positive.

Thus even if the propositional space corresponds to the generalized frame of discernment Besnard et al. approach is quite different. We have to remember that all elements are supposed to be mutually exclusive in the minC approach, and that the lattice structure serves on working internal level for distinguishing of different types of conflicts. No belief functions, plausibility nor commonality functions are defined on a minC generalized frame of discernment. Belief masses of conflicts are supposed to be 0, thus a proportionalization of conflicting masses is used in the minC approach to redistribute them among non-conflicting elements, i.e., among elements of $\mathcal{P}(\Omega)$.

Further we have to mention DSm (Dezert-Smarandache of Demspter-Shafer modified) theory. DSm approach is based on an assumption that elements of the frame of discernment can be overlapping, it is analogous to simultaneous appearance of elements from Ω. The lattice structure D^Ω is called Dedekind lattice or hyper-power set of Ω in DSm theory. A structure, where all its elements can be mutually overlapping, is called the free DSm model and it corresponds to Besnard's propositional space. There are allowed so called *exclusivity constraints* in DSm approach which defines which elements are mutually exclusive; a set of these constraints corresponds to exclusivity relation $\mathcal{E}$ in Besnard's evidential structure. A DSm model with constraints is called *hybrid DSm model* and it corresponds with Besnard's evidential structure.

Independently developed DSm approach is very close to Besnard's approach, but not the same as it is developed from the classic Dempster-Shafer theory, where $m(\emptyset) = 0$, where $\emptyset$ is the least element of the hyper-power set D^Ω, this differs Besnard and DSm approaches. DSm theory is more general, more elaborated, it allows e.g. a situation with concurrent appearance of any two elements but three or more elements cannot appear simultaneously, i.e., the expressivity of DSm constraints is higher than that of Besnard's exclusivity relation on Ω^2. Moreover the *non-existential constraints* are used in DSm theory for total exclusion of some elements from Ω during combination, this enables the so-called dynamic fusion. To by-pass non-associativity, n-ary combination rules are used in DSm theory. A detail comparison of these very similar but different approaches would be an interesting task for a future research.

Even if assumptions of the minC approach and of DSm theory are completely different the minC generalized frame of discernment coincides with DSm hyper-power set (DSm free model). And DSm classic rule for combination of belief functions defined on the free DSm model coincides with computation of m^0 on the minC generalized level. A comparison of the minC combination with the DSm hybrid (DSmH) rule on Shafer's DSm model has been published in [21], see Chapter 10 [8].

Generalization of minC to DSm hyper-power sets, i.e. a combination of two lattice-based approaches is just under development, see [11].

All of the combinations using lattice-based structure have high computing complexity in general, because the size of a lattice generated from Ω rapidly increases with $n = |\Omega|$. Therefore an idea of coarsening of input belief functions to simple 2-element frames, computation of generalized level of the minC combination there, and refinement of results back to the original n-element frame of discernment was investigated in [6], see also [9]. This idea is based on commutation of the generalized level of minC combination with refinement/coarsening and on simplicity of the combination on 2-element frames. It was proved in [6] that a belief function Bel on n-element frame Ω of discernment uniquely correspond to a set of $2^{n-1} - 1$ belief functions on 2-element frames constructed by coarsening of Ω (by splitting of Ω to two complementary parts). We can apply the same idea of splitting of a frame of discernment also to minC generalized frames of discernment $\mathcal{L}(\Omega)$, unfortunately such an expression of a generalized basic belief assignment on n-element frame using generalized assignments on 2-element splitted frames is not 1 to 1 in the case of minC generalized frames, and no procedure how to reconstruct resulting generalized bba from those splitted ones has not been found for general belief functions.

A question of determination of a class of belief functions to which the above idea of refinement/coarsening is applicable is a still open problem for a future research. Another open problem is determination of a special class of belief functions where a complexity of the minC do not increase so rapidly as it does in a general case.

11 Conclusion

A mathematical structure of the minC combination has been analysed in this contribution. It is a lattice based structure. Based on it, formulas for computing of the minC combination has been introduced. The new formulas express the minC combination for belief functions defined on arbitrary finite frame of discernment. When considering various proportionalizations of conflicting belief masses, the minC combination is a family of different rules.

An analysis of particular proportionalizations and more general redistributions of conflicting belief masses has been presented.

Beside the analysis and new presentation of the minC combination, several important related approaches is mentioned and their principal properties are compared.

The presented results can serve as a basis for more particular comparisons of alternative approaches: of the minC combination, DSm hybrid rule, Proportional Conflict Redistribution (PCR) rule, and of other rules from alternative approaches.

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