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Generalizations and Extensions of Lattice-Valued Possibilistic Measures, Part II[1]

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Abstract:

In this technical report, the systematic investigation of lattice-valued possibilistic measures, opened in the first part of this report (cf. Technical Report No. 952, ICS AS CR, December 2005) is pursued and focused towards possibilistic variants of the notions of outer (upper) and lower (inner) possibilistic measures induced by a given partial possibilistic measure. The notion of Lebesgue measurability for possibilistic measures is defined and it is generalized to the notion of almost-measurability in the sense that the values of inner and outer measure ascribed to a set are not identical, as demanded in the case of Lebesgue measurability, but these two values do not differ "too much" from each other in the sense definable within the lattice structure. In the rest of this report, the probabilistic model of decision making under uncertainty is modified to the case of lattice-valued possibilistic measures, so arriving at the notion of possibilistic decision function. The Bayesian and the minimax principles when quantifying the qualities of particular possibilistic decision functions are also analyzed and applied to some cases, e.g., to the case of possibilistic decision functions for state identification.

Keywords:
Lattice-valued possibilistic measures, inner (lower) and outer (upper) possibilistic measures, approximation and completion of partial lattice-valued possibilistic measures, decision making under uncertainty, possibilistic decision function, possibilistic loss function.

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17 Measurability in the Lebesgue Sense Induced by Lattice-Valued Possibilistic Measures

This chapter may be taken as a continuation of Chapter 15, as it goes on with the investigation of inner and outer measures induced by lattice-valued monotone and possibilistic measures. Namely, our attention will be focused to the idea common in the standard real-valued measure theory (cf. [20], e.g.), when the measure defined on a σ-field of subsets of a basic space is extended to those subsets, for which the values of their inner and outer measures, induced by the measure in question, coincide (are identical). In the most common case of Borel measure, ascribing the length $|b-a|$ to each semi-open interval $\langle a, b)$ of the real line and extended uniquely to the system of Borel subsets of this line, the further extension leads to the system of subsets measurable in the Lebesgue sense. As a matter of fact, this system really extends the system of Borel subsets, but still far not each subset of the real line is measurable in the Lebesgue sense. Let us try to apply this idea to partial lattice-valued monotone and possibilistic measures.

Definition 17.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be a nonempty set, let $\mathcal{R}$ be a nonempty system of subsets of Ω, let Π be a $\mathcal{T}$-monotone measure on $\mathcal{R}$. A subset $A \subset \Omega$ is called* measurable in the Lebesgue sense, *if the equality $\Pi_\star(A) = \Pi^\star(A)$ holds.*

Denoting by $\mathcal{M}(\Pi, \mathcal{R})$ the system of all $\mathcal{L}$-measurable subsets of Ω, the inclusion $\mathcal{R} \subset \mathcal{M}(\Pi, \mathcal{R})$ is obvious for each $\mathcal{T}$-monotone measure Π. In general, however, the inequality $\mathcal{R} \neq \mathcal{M}(\Pi, \mathcal{R})$ is the case, i.e., $\mathcal{M}(\Pi, \mathcal{R})$ non-trivially extends $\mathcal{R}$. Indeed, let $A \subset B \subset C \subset \Omega$ be such that $A \neq B \neq C, A, C$ are in $\mathcal{R}, \Pi(A) = \Pi(C)$, and B is not in $\mathcal{R}$. Then $\Pi(A) \leq \Pi_\star(B) \leq \Pi^\star(B) = \Pi(C)$ holds, so that $\Pi_\star(B) = \Pi^\star(B)$, and $B \in \mathcal{M}(\Pi, \mathcal{R}) - \mathcal{R}$ follows.

A system $\mathcal{S}$ of subsets of Ω is called *closed with respect to unions (finite unions, countable unions*, resp.), if for each subsystem $\emptyset \neq \mathcal{R} \subset \mathcal{S}$ (each finite $\emptyset \neq \mathcal{R} \subset \mathcal{S}$, each countable $\emptyset \neq \mathcal{R} \subset \mathcal{S}$, resp.) the set $\bigcup \mathcal{R} = \bigcup_{R \in \mathcal{R}} R$ is in $\mathcal{S}$.

Theorem 17.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\mathcal{R}$ be a system of subsets of a nonempty set Ω which contains $\emptyset$ and Ω and which is closed with respect to unions, let Π be a complete partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$. Then also $\mathcal{M}(\Pi, \mathcal{R})$ is closed with respect to unions.*

Proof. Let $\mathcal{A} \subset \mathcal{M}(\Pi, \mathcal{R})$ be a non-empty system of $\mathcal{L}$-measurable subsets of Ω, so that $\Pi_\star(A) = \Pi^\star(A)$ holds for each $A \in \mathcal{A}$. Consequently, for each $A \in \mathcal{A}$ and for each $A_1, A_2 \in \mathcal{R}$ such that $A_1 \subset A \subset A_2$ holds, we obtain, due to the assumptions, that the unions $\bigcup_{A \in \mathcal{A}} A_1$ and $\bigcup_{A \in \mathcal{A}} A_2$ are in $\mathcal{R}$ and the inclusion $\bigcup_{A \in \mathcal{A}} A_1 \subset \bigcup \mathcal{A} \subset \bigcup_{A \in \mathcal{A}} A_2$ is valid. Hence, an easy calculation yields that

$$
\begin{aligned}
& \Pi\left(\bigcup_{A \in \mathcal{A}} A_1\right) = \bigvee_{A \in \mathcal{A}} \Pi(A_1) \leq \bigvee \left\{\Pi(B) : B \subset \bigcup \mathcal{A},\ B \in \mathcal{R}\right\} = \\
= \ & \Pi_\star\left(\bigcup \mathcal{A}\right) \leq \Pi^\star\left(\bigcup \mathcal{A}\right) = \bigwedge \left\{\Pi(B) : B \supset \bigcup \mathcal{A},\ B \in \mathcal{R}\right\} \leq \\
\leq \ & \Pi\left(\bigcup_{A \in \mathcal{A}} A_2\right) = \bigvee_{A \in \mathcal{A}} \Pi(A_2). \qquad (17.1)
\end{aligned}
$$

This inequality being valid for each $A_1, A_2 \in \mathcal{R}$ such that $A_1 \subset A \subset A_2$ holds, it remains to be valid when replacing $\Pi(A_1)$ by $\bigvee\{\Pi(B) : B \subset A,\ B \in \mathcal{R}\}$, and $\Pi(A_2)$ by $\bigwedge\{\Pi(B) : B \supset A,\ B \in \mathcal{R}\}$, so that we obtain the inequality

$$
\bigvee_{A \in \mathcal{A}} \left(\bigvee\{\Pi(B) : B \subset A,\ B \in \mathcal{R}\}\right) \leq \Pi_\star\left(\bigcup \mathcal{A}\right) \leq \Pi^\star\left(\bigcup \mathcal{A}\right) \leq \bigvee_{A \in \mathcal{A}} \left(\bigwedge\{\Pi(B) : B \supset A,\ B \in \mathcal{R}\}\right) \qquad (17.2)
$$

which can be re-written, using the definitions of inner and outer measures, as

$$\bigvee_{A\in\mathcal{A}} \Pi_\star(A) \le \Pi_\star\left(\bigcup\mathcal{A}\right) \le \Pi^\star\left(\bigcup\mathcal{A}\right) \le \bigvee_{A\in\mathcal{A}} \Pi^\star(A). \tag{17.3}$$

As $\Pi_\star(A) = \Pi^\star(A)$ holds for every $A \in \mathcal{A}$, the equality $\Pi_\star(\bigcup\mathcal{A}) = \Pi^\star(\bigcup\mathcal{A})$ follows, so that $\bigcup\mathcal{A} \in \mathcal{M}(\Pi,\mathcal{R})$ holds and the assertion is proved. □

In general, the system of subsets of Ω which are measurable in the Lebesgue sense with respect to Π and $\mathcal{R}$ is not closed on complements unless some more conditions are imposed on the partial $\mathcal{T}$-possibilistic measure Π on $\mathcal{R}$. In order to describe these conditions explicitly, some auxiliary notions will be of use.

Let $\mathcal{T} = \langle T, \le\rangle$ be a complete lattice. A function $f : T \to T$ is called *non-increasing and continuous* on T, if $t_1 \le t_2$ implies $f(t_1) \ge f(t_2)$ for each $t_1, t_2 \in T$, and if for each $\emptyset \ne S \subset T$ the relations

$$\bigvee_{t\in S} f(t) = f\left(\bigwedge_{t\in S} t\right) \left(= f\left(\bigwedge S\right), \text{abbreviately}\right), \tag{17.4}$$

$$\bigwedge_{t\in S} f(t) = f\left(\bigvee_{t\in S} t\right) \left(= f\left(\bigvee S\right), \text{abbreviately}\right), \tag{17.5}$$

hold. A partial $\mathcal{T}$-monotone measure Π defined on $\mathcal{R} \subset \mathcal{P}(\Omega)$ is called *extensional with respect to complement*, if there exists a non-increasing and continuous function f on T such that $\Pi(\Omega - A) = f(\Pi(A))$ for each $A \in \mathcal{R}$ such that $\Omega - A$ is also in $\mathcal{R}$.

Theorem 17.2 *Let $\mathcal{T} = \langle T, \le\rangle$ be a complete lattice, let Ω be a nonempty set, let $\mathcal{R}$ be a system of subsets of Ω which contains $\emptyset$ and Ω and is closed with respect to complement, i.e., $\Omega - A \in \mathcal{R}$ holds for each $A \in \mathcal{R}$. Let Π be a partial $\mathcal{T}$-monotone measure on $\mathcal{R}$ which is extensional with respect to complement. Then the system of all subsets of Ω which are measurable in the Lebesgue sense with respect to Π and $\mathcal{R}$ is also closed with respect to complement, so that $\Omega - A \in \mathcal{M}(\Pi,\mathcal{R})$ holds for each $A \in \mathcal{M}(\Pi,\mathcal{R})$.*

Proof. Let $f : T \to T$ be a non-increasing continuous function on T such that $\Pi(\Omega - A) = f(\Pi(A))$ holds for each $A \in \mathcal{R}$. As $\mathcal{R}$ is closed with respect to complements, each set in $\mathcal{R}$ can be written as the complement of another set from $\mathcal{R}$, i.e., $A = \Omega - (\Omega - A)$, so that the following reasoning is easily to verify. For each $A \subset \Omega$,

$$\begin{aligned}
\Pi_\star(\Omega - A) &= \bigvee\{\Pi(B) : B \subset \Omega - A,\ B \in \mathcal{R}\} = \\
&= \bigvee\{\Pi(\Omega - B) : \Omega - B \subset \Omega - A,\ B \in \mathcal{R}\} = \\
&= \bigvee\{f(\Pi(B)) : B \supset A,\ B \in \mathcal{R}\} = \\
&= f\left(\bigwedge\{\Pi(B) : B \supset A,\ B \in \mathcal{R}\}\right) = f(\Pi^\star(A)).
\end{aligned} \tag{17.6}$$

Dually,

$$\begin{aligned}
\Pi^\star(\Omega - A) &= \bigwedge\{\Pi(B) : B \supset \Omega - A,\ B \in \mathcal{R}\} = \\
&= \bigwedge\{\Pi(\Omega - B) : \Omega - B \supset \Omega - A,\ B \in \mathcal{R}\} = \\
&= \bigwedge\{f(\Pi(B)) : B \subset A,\ B \in \mathcal{R}\} = \\
&= f\left(\bigvee\{\Pi(B) : B \subset A,\ B \in \mathcal{R}\}\right) = f(\Pi_\star(A)).
\end{aligned} \tag{17.7}$$

Hence, if $A \in \mathcal{M}(\Pi,\mathcal{R})$, then $\Pi_\star(A) = \Pi^\star(A)$, so that also $\Pi_\star(\Omega - A) = \Pi^\star(\Omega - A)$, consequently, $\Omega - A \in \mathcal{M}(\Pi,\mathcal{R})$. The assertion is proved. □

Theorem 17.3 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be a nonempty set, let $\mathcal{R}$ be a system of subsets of Ω which contains $\emptyset$ and Ω and is closed with respect to unions and complements, so that $\bigcup \mathcal{A} \in R$ and $\Omega - A \in \mathcal{R}$ holds for each $A \in \mathcal{R}$. Let Π be a complete partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$ which is extensional with respect to complements. Then the system of all subsets of Ω which are measurable in the Lebesgue sense with respect to Π and $\mathcal{R}$ is closed on intersections, so that $\bigcap \mathcal{A} \in \mathcal{M}(\Pi, \mathcal{R})$ holds for each $A \subset \mathcal{M}(\Pi, \mathcal{R})$.*

Proof. The conditions of Theorem 17.2 being fulfilled, the equalities

$$\Pi_\star(\Omega - A) = f(\Pi^\star(A)),\ \Pi^\star(\Omega - A) = f(\Pi_\star(A)) \tag{17.8}$$

are valid for every $A \subset \Omega$, where $f : T \to T$ is the non-increasing function which defines the extensionality of Π, i.e., such that $\Pi(\Omega - A) = f(\Pi(A))$ holds for each $A \in \mathcal{R}$. Also the conditions of Theorem 17.1 are satisfied, so that, for each $\mathcal{A} \subset \mathcal{M}(\Pi, \mathcal{R})$ and each $A \in \mathcal{M}(\Pi, \mathcal{R})$ also $\bigcup \mathcal{A} \in \mathcal{M}(\Pi, \mathcal{R})$ and $\Omega - A \in \mathcal{M}(\Pi, \mathcal{R})$. Consequently, for the system $\mathcal{A}^- = \{\Omega - A : A \in \mathcal{A}\}$ the inclusion $\mathcal{A}^- \subset \mathcal{M}(\Pi, \mathcal{R})$ follows, hence, also $\bigcup \mathcal{A}^-$ is in $\mathcal{M}(\Pi, \mathcal{R})$, what implies the identity $\Pi_\star(\bigcup \mathcal{A}^-) = \Pi^\star(\bigcup \mathcal{A}^-)$. For the intersection $\bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} A$ we obtain that

$$\begin{aligned} \Pi_\star\left(\bigcap \mathcal{A}\right) &= f(\Pi^\star\left(\Omega - \bigcap \mathcal{A}\right)) = f(\Pi^\star\left(\bigcup_{A \in \mathcal{A}} (\Omega - A)\right) = \\ &= f(\Pi^\star\left(\bigcup \mathcal{A}^-\right)) = f(\Pi_\star\left(\bigcup \mathcal{A}^-\right)) = f(\Pi_\star\left(\Omega - \bigcap \mathcal{A}\right)) = \Pi^\star\left(\bigcap \mathcal{A}\right), \end{aligned} \tag{17.9}$$

hence, $\bigcap \mathcal{A} \in \mathcal{M}(\Pi, \mathcal{R})$ follows and the assertion is proved. □

Consequently, if $\mathcal{R}$ is an ample field, i.e., a system of subsets of Ω which is closed with respect to unions, intersections and complements, and if the conditions of Theorem 19.3 are fulfilled, then also the system $\mathcal{M}(\Pi, \mathcal{R})$ of all subsets of Ω which are measurable in the Lebesgue sense with respect to Π and $\mathcal{R}$ also defines an ample field.

18 Almost-Measurability Induced by Lattice-Valued Possibilistic Measures

Aiming to copy the notion of almost-measurability as introduced above for real-valued partial monotone and possibilistic measures, we arrive at the following definition which will serve as the outgoing point for our further considerations.

Definition 18.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be a nonempty set, let $\mathcal{R}$ be a system of subsets of Ω containing $\emptyset$ and Ω, let Π be a partial $\mathcal{T}$-monotone measure on $\mathcal{R}$, let $\Pi_\star$ ($\Pi^\star$, resp.), be the inner (the outer, resp.) monotone measure induced by Π on $\mathcal{P}(\Omega)$, let $t \in T$. A set $A \subset \Omega$ is called* t-almost measurable *with respect to $\mathcal{R}$ and Π, if the inequality $\varrho(\Pi_\star(A),\ \Pi^\star(A)) \leq t$ holds.*

Let us recall that, by (13.20), for each $t_1, t_2 \in T$,

$$\varrho(t_1, t_2) = (t_1 \wedge t_2^C) \vee (t_1^C \wedge t_2) \tag{18.1}$$

and

$$t^C = \bigvee \{s \in T : s \wedge t = \mathbf{0}_\mathcal{T}\} \tag{18.2}$$

for each $t \in T$. Hence,

$$\varrho(\Pi_\star(A),\ \Pi^\star(A)) = (\Pi_\star(A) \wedge (\Pi^\star(A))^C) \vee ((\Pi_\star(A))^C \wedge \Pi^\star(A)), \tag{18.3}$$

and we will seeking, below, how to simplify this expression. If $t = \mathbf{1}_\mathcal{T}$, every $A \subset \Omega$ is trivially almost t-measurable, if $t = \mathbf{0}_\mathcal{T}$, t-almost measurability reduces to the $\mathcal{L}$-measurability (measurability in the Lebesgue sense) introduced and investigated in Chapter 17. For each $A \subset \Omega$ the value

$\varrho(\Pi_\star(A),\ \Pi^\star(A))$ will be called the *(degree of) discrepancy* of the set A and denoted by $d_{\mathcal{R},\Pi}(A)$ (or simply by $d(A)$, if $\mathcal{R}$ and Π are fixed or given by the context). Hence, a subset of Ω is t-almost measurable, if its discrepancy is smaller than or equal to the value t.

Let $\mathcal{T} = \langle \mathcal{P}(X), \subset \rangle$ be the complete lattice defined by the power-set $\mathcal{P}(X)$ of all subsets of a space X, partially ordered by the relation of set-theoretic inclusion. In this case, $\varrho(A, B) = A \div B$ for each $A, B \subset X$, so that $\varrho(A, B) = B - A$, if $A \subset B \subset X$ holds. Inspired by this relation and by the fact that $\Pi_\star(A) \leq \Pi^\star(A)$ holds for each partial $\mathcal{T}$-monotone measure Π on $\mathcal{R} \subset \mathcal{P}(\Omega)$ and for each $A \subset \Omega$, we would like to simplify the relation (18.3) to

$$\varrho(\Pi_\star(A),\ \Pi^\star(A)) = \Pi^\star(A) \wedge (\Pi_\star(A))^C. \tag{18.4}$$

However, this cannot be reached in general without having introduced some more assumptions, as the relation $t_1 \wedge (t_2)^C = \mathbf{0}_\mathcal{T}$ for $t_2 \leq t_1$ and, in particular, the relation $t_1 \wedge (t_1)^C = \mathbf{0}_\mathcal{T}$ does not hold in general in any complete lattice (let us recall the counter-example with $T = \{\mathbf{0}_\mathcal{T}, t_1, t_2, t_3, \mathbf{1}_\mathcal{T}\}$ and with $\mathbf{0}_\mathcal{T} < t_i < \mathbf{1}_\mathcal{T}$ being the only valid partial-ordering relations on T, cf. Chapter 13 for more detail). Leaving aside its relations to notions introduced in the foregoing chapters, the following direct definition will be of use below.

Definition 18.2 *A complete lattice $\mathcal{T}\langle T, \leq \rangle$ is called* semi-Boolean*, if $t \wedge t^C = \mathbf{0}_\mathcal{T}$ holds for each $t \in T$. If, moreover, $t \vee t^C = \mathbf{1}_\mathcal{T}$ holds for each $t \in T$, the complete lattice $\mathcal{T}$ is called* Boolean-like.

Lemma 18.1 *If a complete lattice $\mathcal{T} = \langle T, \leq \rangle$ is completely distributive in the sense that*

$$s \wedge \left(\bigvee S\right) = \bigvee_{t \in S} (s \wedge t) \tag{18.5}$$

holds for each $s \in T$ and each $\emptyset \neq S \subset T$, then $\mathcal{T}$ is semi-Boolean (let us note that the relation $s \wedge (\bigvee S) \geq \bigvee_{t \in S}(s \wedge t)$ holds in general, as $s \wedge (\bigvee S) \geq s \wedge t$ is the case for every $t \in S$).

Proof. If $\mathcal{T}$ is completely distributive, then for every $t \in T$,

$$t \wedge t^C = t \wedge \bigvee \{s \in T : s \wedge t = \mathbf{0}_\mathcal{T}\} = \bigvee \{s \wedge t : s \wedge t = \mathbf{0}_\mathcal{T}\} = \mathbf{0}_\mathcal{T}. \tag{18.6}$$

□

Lemma 18.2 *Let $\mathcal{T}$ be a semi-Boolean complete lattice, let Ω be a nonempty set and $\{\emptyset, \Omega\} \subset \mathcal{R} \subset \mathcal{P}(\Omega)$ a system of sets, let $\Pi : \mathcal{R} \to T$ be a partial $\mathcal{T}$-monotone measure on $\mathcal{R}$. Then, for every $A \subset \Omega$ and every $A_1, A_2 \in \mathcal{R}$ such that the inclusions $A_1 \subset A \subset A_2$ hold, the relation*

$$d_{\mathcal{R},\Pi}(A) \leq \Pi(A_2) \wedge (\Pi(A_1))^C \tag{18.7}$$

is valid.

Proof. As $\Pi_\star(A) \leq \Pi^\star(A)$ holds and $\mathcal{T}$ is semi-Boolean, we obtain that

$$\Pi_\star(A) \wedge (\Pi^\star(A))^C \leq \Pi^\star(A) \wedge (\Pi^\star(A))^C = \mathbf{0}_\mathcal{T}, \tag{18.8}$$

so that (18.1) reduces to (18.4). If $A_1, A_2 \in \mathcal{R},\ A_1 \subset A \subset A_2$ holds, then

$$\Pi(A_1) \leq \Pi_\star(A) \leq \Pi^\star(A) \leq \Pi(A_2) \tag{18.9}$$

follows. However, if $t_1, t_2 \in T$ are such that $t_1 \leq t_2$ is the case, then for each $s \in T$ such that $s \wedge t_2 = \mathbf{0}_\mathcal{T}$ also $s \wedge t_1 = \mathbf{0}_\mathcal{T}$ holds, hence, the inequality

$$t_1^C = \bigvee \{s \in T : s \wedge t_1 = \mathbf{0}_\mathcal{T}\} \geq \bigvee \{s \in T : s \wedge t_2 = \mathbf{0}_\mathcal{T}\} = t_2^C \tag{18.10}$$

follows. In particular, $(\Pi(A_1))^C \geq (\Pi_\star(A))^C$ holds, so that

$$d_{\mathcal{R},\Pi}(A) = \varrho(\Pi_\star(A),\ \Pi^\star(A)) = \Pi^\star(A) \wedge (\Pi_\star(A))^C \leq \Pi(A_2) \wedge (\Pi(A_1))^C \tag{18.11}$$

results. The assertion is proved. □

Theorem 18.1 *Let the notations and conditions of Lemma 4.2 hold, let the complete lattice $\mathcal{T} = \langle T, \leq \rangle$ be Boolean-like. Then, for any $A \subset \Omega$,*

$$d_{\mathcal{R},\Pi}(A) = \bigwedge_{A_1 \subset A,\ A_1 \in \mathcal{R}} \left(\bigwedge_{A_2 \supset A,\ A_2 \in \mathcal{R}} (\Pi(A_2) \wedge (\Pi(A_1))^C) \right). \tag{18.12}$$

Proof. Combining the definition of discrepancy $d_{\mathcal{R},\Pi}(A)$ with (18.7), we obtain that

$$\begin{aligned}
& \Pi^\star(A) \wedge (\Pi_\star(A))^C = d_{\mathcal{R},\Pi}(A) \leq \bigwedge_{A_1 \subset A,\ A_1 \in \mathcal{R}} \left(\bigwedge_{A_2 \supset A,\ A_2 \in \mathcal{R}} (\Pi(A_2) \wedge (\Pi(A_1))^C) \right) = \\
= & \bigwedge_{A_1 \subset A,\ A_1 \in \mathcal{R}} \left[(\Pi(A_1))^C \wedge \bigwedge_{A_2 \supset A,\ A_2 \in \mathcal{R}} \Pi(A_2) \right] = \bigwedge_{A_1 \subset A,\ A_1 \in \mathcal{R}} \left[(\Pi(A_1))^C \wedge \Pi^\star(A) \right] = \\
= & \Pi^\star(A) \wedge \bigwedge_{A_1 \subset A,\ A_1 \in \mathcal{R}} (\Pi(A_1))^C \leq \Pi^\star(A) \wedge \left(\bigvee_{A_1 \subset A,\ A_1 \in \mathcal{R}} \Pi(A_1) \right)^C = \\
= & \Pi^\star(A) \wedge (\Pi_\star(A))^C.
\end{aligned} \tag{18.13}$$

The assertion is proved. □

Given a complete lattice $\mathcal{T} = \langle T, \leq \rangle$, a system $\mathcal{R}$ of subsets of a nonempty space Ω such that $\{\emptyset, \Omega\} \subset \mathcal{R}$, a partial $\mathcal{T}$-monotone measure Π on $\mathcal{R}$ and a fixed element $t \in T$, let us denote by $\mathcal{L}(\mathcal{R}, \Pi, t) \subset \mathcal{P}(\Omega)$ the system of all t-almost measurable subsets of Ω. Hence, in symbols,

$$\mathcal{L}(\mathcal{R}, \Pi, t) = \{A \subset \Omega : d_{\mathcal{R},\Pi}(A) \leq t\}, \tag{18.14}$$

hence,

$$\mathcal{L}(\mathcal{R}, \Pi, t) = \{A \subset \Omega : \Pi^\star(A) \wedge (\Pi_\star(A))^C \leq t\} \tag{18.15}$$

supposing that $\mathcal{T}$ is completely distributive. Obviously, for each $t_1, t_2 \in T$ such that $t_1 \leq t_2$ holds the inclusions

$$\mathcal{R} \subset \mathcal{M}(\mathcal{R}, \Pi) = \mathcal{L}(\mathcal{R}, \Pi, \mathbf{0}_{\mathcal{T}}) \subset \mathcal{L}(\mathcal{R}, \Pi, t_1) \subset \mathcal{L}(\mathcal{R}, \Pi, t_2) \subset \mathcal{L}(\mathcal{R}, \Pi, \mathbf{1}_{\mathcal{T}}) = P(\Omega) \tag{18.16}$$

are valid.

Theorem 18.2 *Under the notations and conditions just introduced, the set-theoretic relations valid for the systems of t-almost measurable subsets of Ω partially copy the lattice operations defined in $\mathcal{T}$, namely, for each $S \subset T$,*

$$\bigcap_{t \in S} \mathcal{L}(\mathcal{R}, \Pi, t) = \mathcal{L}\left(\mathcal{R}, \Pi, \bigwedge S\right), \tag{18.17}$$

$$\bigcup_{t \in S} \mathcal{L}(\mathcal{R}, \Pi, t) \subset \mathcal{L}\left(\mathcal{R}, \Pi, \bigvee S\right), \tag{18.18}$$

let us recall that $\bigvee S = \bigvee_{t \in S} t$ and $\bigwedge S = \bigwedge_{t \in S} t$.

Proof. As for each $t \in S$ the relation $\bigwedge S \leq t \leq \bigvee S$ trivially holds, (18.16) implies that the set inclusions

$$\mathcal{L}\left(\mathcal{R}, \Pi, \bigwedge S\right) \subset \mathcal{L}(\mathcal{R}, \Pi, t) \subset \mathcal{L}\left(\mathcal{R}, \Pi, \bigvee S\right) \tag{18.19}$$

and, consequently,

$$\mathcal{L}\left(\mathcal{R},\Pi,\bigwedge S\right)\subset\bigcap_{t\in S}\mathcal{L}(\mathcal{R},\Pi,t),\ \mathcal{L}\left(\mathcal{R},\Pi,\bigvee S\right)\supset\bigcup_{t\in S}\mathcal{L}(\mathcal{R},\Pi,t) \tag{18.20}$$

are also valid. If $A\in\bigcap_{t\in S}\mathcal{L}(\mathcal{R},\Pi,t)$, then $d_{\mathcal{R},\Pi}(A)\leq t$ holds for each $t\in S$, so that $d_{\mathcal{R},\Pi}(A)\leq\bigwedge S$ and $A\in\mathcal{L}(\mathcal{R},\Pi,\bigwedge S)$ follow due to the definition of infimum in $\mathcal{T}$. Hence, the inversion of the first inclusion in (18.20) and, consequently, (18.17) follow, so that the assertion is proved. □

It is perhaps worth noting explicitly, that the inclusion inverse to (18.18) does not hold in general. Indeed, let the complete lattice $\mathcal{T}=\langle T,\leq\rangle$ be continuous in $\mathbf{1}_{\mathcal{T}}$ in the sense that $\bigvee\{t\in T: t<\mathbf{1}_{\mathcal{T}}\}=\mathbf{1}_{\mathcal{T}}$ (let us recall the semi-open interval $[0,1)=\{x\in[0,1]: x<1\}$ with respect to the standard ordering $\leq$). Take $\mathcal{R}=\{\emptyset,\Omega\}$ with $\Pi(\emptyset)=\mathbf{0}_{\mathcal{T}}=\bigwedge T$ and $\Pi(\Omega)=\mathbf{1}_{\mathcal{T}}=\bigvee T$, let card $(\Omega)\geq 2$ hold, so that there exist nonempty proper subsets of Ω. Consequently, for every $A\subset\Omega,\ \emptyset\neq A\neq\Omega$,

$$\Pi_\star(A)=\bigvee\{\Pi(B): B\subset A,\ B\in\mathcal{R}\}=\Pi(\emptyset)=\mathbf{0}_{\mathcal{T}}, \tag{18.21}$$

$$\Pi^\star(A)=\bigwedge\{\Pi(B): B\supset A,\ B\in\mathcal{R}\}=\Pi(\Omega)=\mathbf{1}_{\mathcal{T}}. \tag{18.22}$$

Moreover,

$$(\mathbf{0}_{\mathcal{T}})^C=\bigvee\left\{s\in T: s\wedge\left(\bigwedge T\right)=\bigwedge T\right\}=\bigvee T=\mathbf{1}_{\mathcal{T}}, \tag{18.23}$$

$$(\mathbf{1}_{\mathcal{T}})^C=\bigvee\left\{s\in T: s\wedge\left(\bigvee T\right)=\bigwedge T\right\}=\bigwedge T=\mathbf{0}_{\mathcal{T}}. \tag{18.24}$$

As $\Pi_\star(\emptyset)=\Pi^\star(\emptyset)=\mathbf{0}_{\mathcal{T}}$ and $\Pi_\star(\Omega)=\Pi^\star(\Omega)=\mathbf{1}_{\mathcal{T}}$, we obtain that

$$d_{\mathcal{R},\Pi}(\emptyset)=\Pi^\star(\emptyset)\wedge(\Pi_\star(\emptyset))^C=\mathbf{0}_{\mathcal{T}}\wedge\mathbf{1}_{\mathcal{T}}=\mathbf{0}_{\mathcal{T}}, \tag{18.25}$$

$$d_{\mathcal{R},\Pi}(\Omega)=\Pi^\star(\Omega)\wedge(\Pi_\star(\Omega))^C=\mathbf{1}_{\mathcal{T}}\wedge\mathbf{0}_{\mathcal{T}}=\mathbf{0}_{\mathcal{T}}, \tag{18.26}$$

and

$$d_{\mathcal{R},\Pi}(\mathcal{A})=\Pi^\star(\mathcal{A})\wedge(\Pi_\star(\mathcal{A}))^C=\mathbf{1}_{\mathcal{T}}\wedge\mathbf{0}_{\mathcal{T}}=\mathbf{1}_{\mathcal{T}} \tag{18.27}$$

for every $\emptyset\neq A\neq\Omega,\ A\subset\Omega$. Hence, for every $t\in T,\ t<\mathbf{1}_{\mathcal{T}}$,

$$\mathcal{L}(\mathcal{R},\Pi,t)=\{A\subset\Omega: d_{\mathcal{R},\Pi}(\mathcal{A})\leq t\}=\{\emptyset,\Omega\}, \tag{18.28}$$

but

$$\mathcal{L}(\mathcal{R},\Pi,\mathbf{1}_{\mathcal{T}})=\mathcal{P}(\Omega), \tag{18.29}$$

so that, for $S=T-\{\mathbf{1}_{\mathcal{T}}\}\subset T$, we obtain that

$$\begin{aligned}\bigcup_{t\in S}\mathcal{L}(\mathcal{R},\Pi,t) &= \bigcup_{t<\mathbf{1}_{\mathcal{T}}}\mathcal{L}(\mathcal{R},\Pi,t)=\{\emptyset,\Omega\}\neq\mathcal{L}\left(\mathcal{R},\Pi,\bigvee S\right)=\\ &= \mathcal{L}\left(\mathcal{R},\Pi,\bigvee_{t<\mathbf{1}_{\mathcal{T}}}t\right)=\mathcal{L}(\mathcal{R},\Pi,\mathbf{1}_{\mathcal{T}})=\mathcal{P}(\Omega).\end{aligned} \tag{18.30}$$

It is almost obvious that the non-symmetric role of union (supremum) and intersection (infimum) in Theorem 18.2 cannot be removed, if considering a more strict definition of t-almost measurability, namely, when setting

$$\mathcal{L}(\mathcal{R},\Pi,t)=\{A\subset\Omega: d_{\mathcal{R},\Pi}(A)<t\}. \tag{18.31}$$

Indeed, in this case equality holds in (18.18), but (18.17) is violated and only the first inclusion in (18.20) can be proved.

Theorem 18.3 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a semi-Boolean complete lattice satisfying the following conditions:*

$$s \wedge (t_1 \vee t_2) = (s \wedge t_1) \vee (s \wedge t_2) \tag{18.32}$$

and

$$s \vee \left(\bigwedge S\right) = \bigwedge_{t \in S} (s \vee t) \tag{18.33}$$

for each $s, t_1, t_2 \in T$ and each $S \subset T$ (let us note that the inequalities $s \wedge (t_1 \vee t_2) \geq (s \wedge t_1) \vee (s \wedge t_2)$ and $s \vee (\bigwedge S) \leq \bigwedge_{t \in S}(s \vee t)$ are valid in general). Let $\mathcal{R} \subset \mathcal{P}(\Omega)$, $\Omega \neq \emptyset$, be such that $\{\emptyset, \Omega\} \subset \mathcal{R}$ and $\mathcal{R}$ is closed with respect to unions, i.e., if $A, B \in \mathcal{R}$, then also $A \cup B \in \mathcal{R}$. Let Π be a partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$, let $t \in T$, let $\mathcal{L}(\mathcal{R}, \Pi, t)$ be the system of all t-almost measurable subsets of Ω. Then $\mathcal{L}(\mathcal{R}, \Pi, t)$ is closed with respect to unions in the same sense as $\mathcal{R}$, hence, if $A, B \in \mathcal{L}(\mathcal{R}, \Pi, t)$ then $A \cup B \in \mathcal{L}(\mathcal{R}, \Pi, t)$ follows.

Proof. Omitting for the sake of simplicity, the parameters $\mathcal{R}$ and Π in $d_{\mathcal{R},\Pi}(\cdot)$, taking $A, B \subset \Omega$ and $A_1, A_2, B_1, B_2 \in \mathcal{R}$ such that the inclusions $A_1 \subset A \subset A_2$ and $B_1 \subset B \subset B_2$ are valid, we obtain that also $A_1 \cup B_1 \subset A \cup B \subset A_2 \cup B_2$ holds, hence, we obtain that

$$\begin{aligned} d(A \cup B) &\leq (\Pi(A_2 \cup B_2)) \wedge (\Pi(A_1 \cup B_1))^C = (\Pi(A_2) \vee \Pi(B_2)) \wedge (\Pi(A_1) \vee (B_1))^C = \\ &= [\Pi(A_2) \wedge (\Pi(A_1) \vee \Pi(B_1))^C] \vee [\Pi(B_2) \wedge (\Pi(A_1) \vee \Pi(B_1))^C] \end{aligned} \tag{18.34}$$

holds due to the assumptions imposed on $\mathcal{T}, \mathcal{R}$, and Π. For every $t_1, t_2 \in T$, $t_1 \leq t_2$ implies that $t_1^C \geq t_2^C$, hence

$$(\Pi(A_1) \vee \Pi(B_1))^C \leq (\Pi(A_1))^C, \ (\Pi(A_1) \vee \Pi(B_1)) \leq (\Pi(B_1))^C \tag{18.35}$$

immediately follows. Combining (18.34) with (18.35), we obtain that

$$d(A \cup B) \leq [\Pi(A_2) \wedge (\Pi(A_1))^C] \leq [\Pi(B_2) \wedge (\Pi(B_1))^C]. \tag{18.36}$$

Consequently,

$$\begin{aligned} d(A \cup B) &\leq \bigwedge_{A_1, A_2 \in \mathcal{R},\ A_1 \subset A \subset A_2} \left\{[\Pi(A_2) \wedge (\Pi(A_1))^C] \vee [\Pi(B_2) \wedge (\Pi(B_1))^C]\right\} = \\ &= \left\{\bigwedge_{A_1, A_2 \in \mathcal{R},\ A_1 \subset A \subset A_2} [\Pi(A_2) \wedge (\Pi(A_1))^C]\right\} \vee [\Pi(B_2) \wedge (\Pi(B_1))^C] \end{aligned} \tag{18.37}$$

due to the conditions (18.32) and (18.33) imposed on $\mathcal{T}$. Applying this operation once more, we obtain that the inequality

$$\begin{aligned} d(A \cup B) &\leq \left\{\bigwedge_{A_1, A_2 \in \mathcal{R},\ A_1 \subset A \subset A_2} [\Pi(A_2) \wedge (\Pi(A_1))^C]\right\} \vee \left\{\bigwedge_{B_1, B_2 \in \mathcal{R},\ B_1 \subset B \subset B_2} [\Pi(B_2) \wedge (\Pi(B_1))^C]\right\} = \\ &= d(A) \vee d(B) \leq t \vee t = t \end{aligned} \tag{18.38}$$

also holds. Hence, $A \cup B \in \mathcal{L}(\mathcal{R}, \Pi, t)$ follows and the assertion is proved. □

Theorem 18.4 *Let $\mathcal{T} - \langle T, \leq \rangle$ be a semi-Boolean distributive complete lattice, let $\mathcal{R} \subset \mathcal{P}(\Omega)$ be a nested system which contains $\emptyset$ and Ω, let Π be a partial $\mathcal{T}$-monotone measure on $\mathcal{R}$. Then, for each $t \in T$, the system $\mathcal{L}(\mathcal{R}, \Pi, t)$ of t-almost measurable subsets of Ω is closed with respect to unions and intersections, i.e., for each $A, B \in \mathcal{L}(\Pi, \mathcal{R}, t)$, $A \cup B$ and $A \cap B$ are also in $\mathcal{L}(\Pi, \mathcal{R}, t)$.*

Proof. Let $A, B \subset \Omega$, let $A_1, A_2, B_1, B_2 \in \mathcal{R}$ be such that the inclusions $A_1 \subset A \subset A_2$ and $B_1 \subset B \subset B_2$ hold. For $A \cup B$ we obtain that the inclusions $A_1 \cup B_1 \subset A \cup B \subset A_2 \cup B_2$ and $A_1 \cap B_1 \subset A \cap B \subset A_2 \cap B_2$ are also valid. As $\mathcal{T}$ is semi-Boolean, (18.18) implies the inequalities

$$d(A \cup B) \leq \Pi(A_2 \cup B_2) \wedge (\Pi(A_1 \cup B_1))^C, \tag{18.39}$$

$$d(A \cap B) \leq \Pi(A_2 \cap B_2) \wedge (\Pi(A_1 \cap B_1))^C, \tag{18.40}$$

omitting, again, the parameters $\mathcal{R}$ and Π in $d_{\mathcal{R},\Pi}(\cdot)$. As $\mathcal{R}$ is nested, just the following four particular cases of the mutual relations between A_1, B_1 and between A_2, B_2 are possible.

(i) $A_1 \subset B_1$ and $A_2 \subset B_2$. Then $A_1 \cup B_1 = B_1,\ A_2 \cup B_2 = B_2,\ A_1 \cap B_1 = A_1$ and $A_2 \cap B_2 = A_2$, so that (18.39) and (18.40) reduce to

$$d(A \cup B) \leq \Pi(B_2) \wedge (\Pi(B_1))^C,\ d(A \cap B) \leq \Pi(A_2) \wedge (\Pi(A_1))^C. \tag{18.41}$$

(ii) $A_1 \supset B_1,\ A_2 \supset B_2$ - the same case as (i), just with the roles of A_1 and B_1 interchanged, so that the inequalities

$$d(A \cup B) \leq \Pi(A_2) \wedge (\Pi(A_1))^C,\ d(A \cap B) \leq \Pi(B_2) \wedge (\Pi(B_1))^C \tag{18.42}$$

follow.

(iii) $A_1 \subset B_1,\ A_2 \supset B_2$. In this case, $A_1 \cup B_1 = B_1,\ A_2 \cup B_2 = A_2,\ A_1 \cap B_1 = A_1$ and $A_2 \cap B_2 = B_2$, so that (18.39) and (18.40) yield

$$d(A \cup B) \leq \Pi(A_2) \wedge (\Pi(B_1))^C \leq \Pi(A_2) \wedge (\Pi(A_1))^C., \tag{18.43}$$

$$d(A \cap B) \leq \Pi(B_2) \wedge (\Pi(A_1))^C \leq \Pi(A_2) \wedge (\Pi(A_1))^C, \tag{18.44}$$

as $\Pi(A_2) \geq \Pi(B_2)$ holds. Finally, if

(iv) $A_1 \supset B_1,\ A_2 \supset B_2$, the situation is the same as in (iii), just with the roles of A_i and B_i interchanged. Hence, (18.43) and (18.44) hold again, just with A_i replaced by B_i for both $i = 1, 2$.

Combining all these particular cases together, we obtain that in every case the inequality

$$d(A \cap B) \leq [\Pi(A_2) \wedge (\Pi(A_1))^C] \vee [\Pi(B_2) \wedge (\Pi(B_1))^C] \tag{18.45}$$

as well as the same inequality for $d(A \cap B)$ are valid, no matter which $A_1, A_2, B_1, B_2 \in \mathcal{R}$ satisfying the given inclusions may be. Hence, as the complete lattice $\mathcal{T}$ is supposed to be distributive, we obtain that

$$\begin{aligned} d(A \cup B) \quad &\leq \bigwedge_{A_1 \subset A \subset A_2,\ A_1, A_2 \in \mathcal{R}} \left(\bigwedge_{B_1 \subset B_\subset B_2,\ B_1, B_2 \in \mathcal{R}} ([\Pi(A_2) \wedge (\Pi(A_1))^C] \vee [\Pi(B_2) \wedge (\Pi(B_1))^C]) \right) = \\ &= \left(\bigwedge_{A_1 \subset A \subset A_2,\ A_1, A_2 \in \mathcal{R}} [\Pi(A_2) \wedge (\Pi(A_1))^C] \right) \wedge \left(\bigwedge_{B_1 \subset B_\subset B_2,\ B_1, B_2 \in \mathcal{R}} [\Pi(B_2) \wedge \Pi(B_1)^C] \right) = \\ &= d(A) \vee d(B), \end{aligned} \tag{18.46}$$

and similarly we obtain that

$$d(A \cap B) \leq d(A) \vee d(B) \tag{18.47}$$

holds. So, if $A, B \in \mathcal{L}(\mathcal{R}, \Pi, t)$, then $d(A) \leq t$, $d(B) \leq t$ holds, hence $d(A \cup B) \leq t$ and $d(A \cap B) \leq t$ is valid. Consequently, $A \cup B$ and $A \cap B$ are in $\mathcal{L}(\mathcal{R}, \Pi, t)$ and the assertion is proved. □

Let us analyze, in more detail, the conditions under which complements of t-almost measurable subsets of the basic space Ω are also t-almost measurable. As done already several times above, we take an inspiration from the most simple case, when identify mapping on $\mathcal{P}(\Omega)$ is understood as $\mathcal{T}_0$-valued possibilistic measure, where $\mathcal{T}_0 = \langle \mathcal{P}(\Omega), \subset \rangle$. Hence, $\mathcal{R} = \mathcal{P}(\Omega)$ and $\Pi : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ is defined simply by $\Pi(A) = A$ for every $A \subset \Omega$. Then pseudo-complement A^C agrees with the standard set-theoretic complement $\Omega - A$ and the $\mathcal{T}_0$-valued metric ϱ on $\mathcal{P}(\Omega)$ is defined by the symmetric difference, so that

$$\varrho(A, B) = A \div B = (A - B) \cup (B - A) \tag{18.48}$$

for every $A, B \subset \Omega$. As can be easily checked, $\varrho(A, B) = \varrho(\Omega - A, \Omega - B)$ holds in general. Indeed,

$$\begin{aligned} \varrho(\Omega - A, \Omega - B) &= (\Omega - A) \div (\Omega - B) = \\ &= [(\Omega - A) \cap (\Omega - (\Omega - B))] \cup [(\Omega - B) \cap (\Omega - (\Omega - A))] = \\ &= ((\Omega - A) \cap B) \cup ((\Omega - B) \cap A) = (B - A) \cup (A - B) = \varrho(A, B). \end{aligned} \tag{18.49}$$

In particular, if $A \subset B \subset \Omega$ holds, then

$$\varrho(A, B) = \varrho(\Omega - A,\ \Omega - B) = B - A. \tag{18.50}$$

So, if $\mathcal{R} \subset \mathcal{P}(\Omega)$ is closed with respect to complements, if $C \subset \Omega$ is given, and if a subset $A \subset \Omega$ is called C-almost measurable supposing that $\varrho(A_\star, A^\star) \subset C$ is the case, we can conclude immediately that the set $\Omega - A$ is also C-almost measurable. Here

$$A_\star = \bigcup \{B \subset A,\ B \in \mathcal{R}\},\ A^\star = \bigcap \{B \supset A,\ B \in \mathcal{R}\}. \tag{18.51}$$

In what follows, our aim will be to find conditions to be imposed on the complete lattice $\mathcal{T} = \langle T, \leq \rangle$, and on the partial $\mathcal{T}$-monotone or $\mathcal{T}$-possibilistic measure Π on $\mathcal{R}$, in order to ensure the validity of a relation like (18.50).

Lemma 18.3 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a Boolean-like complete distributive lattice, so that $t \wedge t^C = \mathbf{0}_\mathcal{T}$ and $t \vee t^C = \mathbf{1}_\mathcal{T}$ holds for each $t \in T$. Then $(t^C)^C = t$ for every $t \in T$.*

Proof. By definition

$$(t^C)^C = \bigvee \{s \in T : s \wedge t^C = \mathbf{0}_\mathcal{T}\}. \tag{18.52}$$

The condition $t \wedge t^C = \mathbf{0}_\mathcal{T}$ implies that t is among those s, for which $s \wedge t^C = \mathbf{0}_\mathcal{T}$ holds, hence, the inequality $(t^C)^C \geq t$ immediately follows. The condition $t \vee t^C = \mathbf{1}_\mathcal{T}$, valid for each $t \in T$, together with the distributivity of $\mathcal{T}$, yields that

$$(t^C)^C = (t^C)^C \wedge \mathbf{1}_\mathcal{T} = ((t^C)^C) \wedge (t^C \vee t) = ((t^C)^C \wedge t^C) \vee ((t^C)^C \wedge t) = (t^C)^C \wedge t, \tag{18.53}$$

as $(t^C)^C \wedge t^C = \mathbf{0}_\mathcal{T}$. Consequently, $(t^C)^C \leq t$ and $(t^C)^C = t$ follow. □

Remark 18.1 *As a matter of fact, for complementary distributive complete lattice $\mathcal{T}$ the equality $t \wedge t^C = \mathbf{0}_\mathcal{T}$ holds in general. Indeed, setting $S_t = \{s \in T : s \wedge t = \mathbf{0}_\mathcal{T}\}$, we obtain that*

$$t \wedge t^C = t \wedge \bigvee S_t = \bigvee_{s \in S_t} (s \wedge t) = \bigvee_{s \in S_t} \mathbf{0}_\mathcal{T} = \mathbf{0}_\mathcal{T}. \tag{18.54}$$

Theorem 18.5 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a distributive semi-Boolean complete lattice, let Ω be a nonempty set and $\mathcal{R}$ a system of subsets of Ω containing $\emptyset$ and closed with respect to complements, i.e., $\Omega - A \in \mathcal{R}$ for each $A \in \mathcal{R}$ (so that $\Omega \in \mathcal{R}$ follows). Let $\Pi : \mathcal{R} \to T$ be a partial T-monotone measure on $\mathcal{R}$ such that*

$$\Pi(B) \wedge \Pi(\Omega - B) = \mathbf{0}_{\mathcal{T}}, \ \Pi(B) \vee \Pi(\Omega - B) = \mathbf{1}_{\mathcal{T}} \tag{18.55}$$

holds for every $B \in \mathcal{R}$. Then, for each $t \in T$ and each $A \subset \Omega$, if A is t-almost measurable, its complement $\Omega - A$ is t-almost measurable as well. In other terms, the system $\mathcal{L}(\mathcal{R}, \Pi, t)$ of all t-almost measurable subsets of Ω is closed with respect to complements.

Proof. Let us prove, first of all, that under the conditions imposed, the relation $\Pi(\Omega - B) = (\Pi(B))^C$ holds for every $B \in \mathcal{R}$. Indeed, $\Pi(B) \wedge \Pi(\Omega - B) = \mathbf{0}_{\mathcal{T}}$ yields that $\Pi(\Omega - B) \leq (\Pi(B))^C$ holds. On the other side, however

$$\begin{aligned}
(\Pi(B))^C &= (\Pi(B))^C \wedge \mathbf{1}_{\mathcal{T}} = (\Pi(B))^C \wedge (\Pi(B) \vee \Pi(\Omega - B)) = \\
&= [(\Pi(B))^C \wedge \Pi(B)] \vee [(\Pi(B))^C \wedge \Pi(\Omega - B)] \\
&= (\Pi(B))^C \wedge \Pi(\Omega - B),
\end{aligned} \tag{18.56}$$

as $t \wedge t^C = \mathbf{0}_{\mathcal{T}}$ holds for each $t \in T$ due to the assumption that $\mathcal{T}$ is semi-Boolean. Hence, (18.56) yields the inequality $(\Pi(B))^C \leq \Pi(\Omega - B)$, so that the inequality $\Pi(\Omega - B) = (\Pi(B))^C$ holds for every $B \in \mathcal{R}$.

Let $A \subset \Omega$, let $A_1, A_2 \in \mathcal{R}$ be such that $A_1 \subset A \subset A_2$ holds. The conditions of Lemma 18.2 being satisfied, (18.7) holds, hence

$$d_{\mathcal{R},\Pi}(A) \leq \varrho(A_1, A_2) = \Pi(A_2) \wedge (\Pi(A_1))^C \tag{18.57}$$

follows. In this case, however, $\Omega - A_1$ and $\Omega - A_2$ are in $\mathcal{R}$ and the inclusion $\Omega - A_2 \subset \Omega - A \subset \Omega - A_1$ is valid, so that the relation

$$d_{\mathcal{R},\Pi}(\Omega - A) \leq \varrho(\Omega - A_2, \ \Omega - A_1) = \Pi(\Omega - A_1) \wedge (\Pi(\Omega - A_2))^C = (\Pi(A_1))^C \wedge \Pi(A_2) = \varrho(A_1, A_2) \tag{18.58}$$

is valid as well. Consequently,

$$d_{\mathcal{R},\Pi}(\Omega - A) \leq \bigwedge_{A_1, A_2 \in \mathcal{R}, \ A_1 \subset A \subset A_2} \varrho(A_1, A_2) = d_{\mathcal{R},\Pi}(A) \tag{18.59}$$

follows. As the roles of sets from $\mathcal{R}$ and their respective complements in our reasoning are completely dual, as a matter of fact the equality $d_{\mathcal{R},\Pi}(A) = d_{\mathcal{R},\Pi}(\Omega - A)$ results. Hence, if $d_{\mathcal{R},\Pi}(A) \leq t$ is the case, $d_{\mathcal{R},\Pi}(\Omega - A) \leq t$ holds as well and the assertion is proved. □

Corollary 18.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a semi-Boolean complete lattice which is distributive in the sense that (18.32) and (18.33) hold. Let Ω be a nonempty set and let $\mathcal{R}$ be a system of subsets of Ω which contains $\emptyset$ and Ω and which is closed with respect to set-theoretic operations of union and complement so that, for each $A, B \in \mathcal{R}$, also the sets $A \cup B$ and $\Omega - A$ are in $\mathcal{R}$. Let $\Pi : \mathcal{R} \to T$ be a complete partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$ which is orthogonal in the sense that (18.55) holds for every $B \in \mathcal{R}$. Then, for each $t \in T$, the system $\mathcal{L}(\mathcal{R}, \Pi, t)$ of all t-almost measurable subsets of Ω is closed with respect to unions, intersections and complements, hence, $\mathcal{L}(\mathcal{R}, \Pi, t)$ is a Boolean algebra containing $\mathcal{R}$.*

Proof. The conditions imposed on $\mathcal{T}, \mathcal{R}$, and Π combine those introduced in Theorems 18.3 and 18.5, so that $\mathcal{L}(\mathcal{R}, \Pi, t)$ is closed with respect to unions (Theorem 18.3) and complements (Theorem 18.5). However, the elementary de Morgan rules yield that $A \cap B = \Omega - ((\Omega - A) \cup (\Omega - B))$, so that for each $A, B \in \mathcal{L}(\mathcal{R}, \Pi, t)$ we obtain that

$$\begin{aligned} d(A\cap B) &= \bigvee_{C_1,C_2\in\mathcal{R},\ C_1\subset A\cap B\subset C_2} (\Pi(C_2)\wedge(\Pi(C_1))^C) \\ &= d(\Omega-((\Omega-A)\cup(\Omega-B)))= \\ &= d(\Omega-A)\cup(\Omega-B))\ \ \text{(by (18.59))} \\ &= d(\Omega-A)\vee d(\Omega-B))\ \ \text{(by (18.46))} \\ &= d(A)\vee d(B)\ \ \text{(by (18.59) again)} \\ &= t\vee t=t_1, \end{aligned} \tag{18.60}$$

applying the definition of $\mathcal{L}(\mathcal{R},\Pi,t)$. Hence, $A\cap B\in\mathcal{L}(\mathcal{R},\Pi,t)$ and the assertion is proved. □

19 A Strengthened Version of Inner and Outer Measures

As a matter of fact, the idea of inner and outer measures applied above and borrowed from the standard measure theory, cf. [20], e.g., is very simple and intuitive. Each subset $A\subset\Omega$ is approximated by two subsets $B_1,B_2\in\mathcal{R}$ such that $B_1\subset A\subset B_2$ holds, hence, the values of a monotone or possibilistic $\mathcal{T}$-valued measure Π for both B_1,B_2 are defined. The choice of B_1 and B_2 is then optimized in the sense that the distance $\varrho(B_1,B_2)$ $(=\Pi(B_2)\wedge(\Pi(B_1))^C$ under some simplifying conditions introduced and analyzed in the foregoing chapter) should be as small as possible. In this chapter we will try to modify this idea in such a way that not only sets from $\mathcal{R}$, but also intersections and unions of such sets can play the role of inner and outer approximations of the set A in question, even if these intersections and unions themselves do not belong to $\mathcal{R}$. To simplify our reasoning, we will suppose that $\mathcal{T}=\langle T,\leq\rangle$ is a complete lattice, so that all the suprema and infima occurring below will be defined, perhaps applying the conventions $\bigvee\emptyset=\mathbf{0}_{\mathcal{T}}$ and $\bigwedge\emptyset=\mathbf{1}_{\mathcal{T}}$ for the empty subset of T. Moreover, we will suppose that Π is a $\mathcal{T}$-monotone measure on $\mathcal{R}$, so assuring its most elementary relation to the set-theoretic inclusion.

Definition 19.1 *Let $\mathcal{T}=\langle T,\leq\rangle$ be a complete lattice, let Ω be a nonempty set, let $\{\emptyset,\Omega\}\subset\mathcal{R}\subset\mathcal{P}(\Omega)$ be a system of subsets of Ω, let $\Pi:\mathcal{R}\to T$ be a $\mathcal{T}$-monotone measure on $\mathcal{R}$. The* strong inner (or lower) measure Π_+ *and the* strong outer (or upper) measure Π^+ *induced by Π on $\mathcal{P}(\Omega)$ are mappings ascribing to each $A\subset\Omega$ the values*

$$\Pi_+(A)=\bigvee_{\mathcal{B}\subset\mathcal{R},\ \bigcap\mathcal{B}\subset A}\left(\bigwedge_{B\in\mathcal{B}}\Pi(B)\right) \tag{19.1}$$

$$\Pi^+(A)=\bigwedge_{\mathcal{B}\subset\mathcal{R},\ \bigcup\mathcal{B}\supset A}\left(\bigvee_{B\in\mathcal{B}}\Pi(B)\right). \tag{19.2}$$

Here $\bigcup\mathcal{B}$ ($\bigcap\mathcal{B}$, resp.) denotes the sets $\bigcup_{B\in\mathcal{B}}B$ ($\bigcap_{B\in\mathcal{B}}B$, resp.). Let us emphasize the fact that the sets $\bigcup\mathcal{B}$ and $\bigcap\mathcal{B}$, occurring in (19.1) and (19.2) need not be in $\mathcal{R}$, so that the values $\Pi_+(A)$ and/or $\Pi^+(A)$ need not be the values ascribed by Π to some sets from $\mathcal{R}$, or at least limits of such values, as it is the case for $\Pi_\star(A)$ and $\Pi^\star(A)$.

Lemma 19.1 *Under the notations and conditions of Definition 7.1, the inequalities*

$$\Pi_\star(A)\leq\Pi_+(A),\ \Pi^+(A)\leq\Pi^\star(A) \tag{19.3}$$

are valid for each $A\subset\Omega$.

Proof. Restricting ourselves to such systems $\mathcal{B}\subset\mathcal{R}$ which contain only one set, say $C\in\mathcal{R}$, we obtain easily that $\bigcup\mathcal{B}=\bigcap\mathcal{B}=C$ and

$$\Pi_+(A)\geq\bigvee_{\mathcal{B}\subset\mathcal{R},\ \mathcal{B}=\{C\},\ \bigcap\mathcal{B}\subset A}\left(\bigwedge_{B\in\mathcal{B}}\Pi(B)\right)=\bigvee_{C\in\mathcal{R},\ C\subset A}\Pi(C)=\Pi_\star(C). \tag{19.4}$$

Dually,

$$\Pi^+(A) \leq \bigwedge_{\mathcal{B}\subset\mathcal{R},\ \mathcal{B}=\{C\},\ \bigcup\mathcal{B}\supset A} \left(\bigvee_{B\in\mathcal{B}} \Pi(B)\right) = \bigwedge_{C\in\mathcal{R},\ C\supset A} \Pi(C) = \Pi^\star(C). \tag{19.5}$$

□

However, the intuitive inequality $\Pi_\star(A) \leq \Pi^\star(A)$, valid for every $A \subset \Omega$ as Π is supposed to be a partial $\mathcal{T}$-monotone measure on $\mathcal{R}$, does not hold true, in general, for Π_+ and Π^+, as the following simple example demonstrates. Take $\mathcal{R} = \{\emptyset, B_1, B_2, B_3, \Omega\}$, where B_1, B_2, B_3 are such nonempty subsets of Ω that B_1 is a proper subset of $B_2 \cup B_3 \neq \Omega$, but neither $B_1 \subset B_2$ nor $B_1 \subset B_3$ holds, moreover, neither $B_2 \subset B_3$ nor $B_3 \subset B_2$ is the case, consequently, $\emptyset \neq B_2 \cup B_3 \notin \mathcal{R}$ follows. Let $\mathcal{T} = \langle T, \leq\rangle$ be a complete lattice, let $\Pi : \mathcal{R} \to T$ be such that $\Pi(\emptyset) = \mathbf{0}_\mathcal{T} < t_2 = \Pi(B_2) = \Pi(B_3) < t_1 = \Pi(B_1) < \mathbf{1}_\mathcal{T} = \Pi(\Omega)$ holds for some $t_1, t_2 \in T$. As can be easily checked, Π is a partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$; the only unions of sets from $\mathcal{R}$ which are also in $\mathcal{R}$ are the trivial ones $\emptyset \cup B_i$ and $B_i \cup \Omega,\ i = 1, 2, 3$.

For the set $B_2 \cup B_3$ we compute easily that

$$\Pi_\star(B_2 \cup B_3) = \Pi(B_1) = t_1 < \mathbf{1}_\mathcal{T} = \Pi(\Omega) = \Pi^\star(B_1 \cup B_2). \tag{19.6}$$

The only subsystems of $\mathcal{R}$ the unions of which cover $B_2 \cup B_3$ are $\mathcal{B}_1 = \{B_2, B_3\}$ and every $\mathcal{B}_2$ containing Ω, so that

$$\begin{aligned} \Pi^+(B_1 \cup B_2) &= \left(\bigvee_{B\in\mathcal{B}_1} \Pi(B)\right) \wedge \left(\bigvee_{B\in\mathcal{B}_2} \Pi(B)\right) = \\ &= (\Pi(B_2) \vee \Pi(B_3)) \wedge \Pi(\Omega) = (t_2 \vee t_2) \wedge \mathbf{1}_\mathcal{T} = t_2 < t_1 = \\ &= \Pi_\star(B_2 \cup B_3) \leq \Pi_+(B_2 \cup B_2), \end{aligned} \tag{19.7}$$

the last inequality follows from (19.3). The result just obtained obviously follows from the assumption that the possibility degree ascribed to B_1 is greater than that ascribed to B_2 and B_3, even if these sets, joined together, cover $B_!$. This assumption may be felt rather counter-intuitive, but it does not violate our definitions, at least at the high degree of generality adopted here.

Let us consider the following way how to strengthen the demand of monotonicity imposed by the definition of $\mathcal{T}$-monotone measure on the mapping $\Pi : \mathcal{R} \to T$. Again, we limit ourselves to the case when $\mathcal{T} = \langle T, \leq\rangle$ is a complete lattice.

Definition 19.2 *Let $\mathcal{T}, \Omega$ and $\mathcal{R}$ be as in Definition 19.1. A mapping $\Pi : \mathcal{R} \to T$ is called a* strong partial $\mathcal{T}$-monotone measure on $\mathcal{R}$, *if $\Pi(\emptyset) = \mathbf{0}_\mathcal{T},\ \Pi(\Omega) = \mathbf{1}_\mathcal{T}$ and if, for every nonempty systems $\mathcal{A}, \mathcal{B} \in \mathcal{R}$ such that the inclusion $\bigcap\mathcal{A} \subset \bigcup\mathcal{B}$ holds, the relation*

$$\bigwedge\{\Pi(A) : A \in \mathcal{A}\} \leq \bigvee\{\Pi(B) : B \in \mathcal{B}\} \tag{19.8}$$

is valid.

Obviously, if $\mathcal{A}$ and $\mathcal{B}$ are singletons, i.e., if $\mathcal{A} = \{A\}$ and $\mathcal{B} = \{B\}$ for some $A, B \in \mathcal{R}$, then $\bigcap\mathcal{A} = A$, $\bigcup\mathcal{B} = B$ and the condition imposed on Π reduces to: if $A \subset B$, then $\Pi(A) \leq \Pi(B)$, what is just the definition of $\mathcal{T}$-monotone measure on $\mathcal{R}$. Also evident is the fact that the mapping Π defined just before (19.6) is not a strong partial $\mathcal{T}$-monotone measure on the system $\mathcal{R}$ in question.

Theorem 19.1 *Let the notations and conditions of Definition 19.1 hold, let Π be a strong partial $\mathcal{T}$-monotone measure on $\mathcal{R}$. Then, for every $A \subset \Omega$, the inequality*

$$\Pi_\star(A) \leq \Pi_+(A) \leq \Pi^+(A) \leq \Pi^\star(A) \tag{19.9}$$

is valid.

Proof. Given $A \subset \Omega$, let $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{R}$ be such systems that the inclusions $\bigcap \mathcal{B}_1 \subset A \subset \bigcup \mathcal{B}_2$ holds. Hence, (19.8) yields that

$$\bigwedge\{\Pi(B) : B \in \mathcal{B}_1\} \leq \bigvee\{\Pi(B) : B \in \mathcal{B}_2\} \tag{19.10}$$

holds for each such $\mathcal{B}_1$ and $\mathcal{B}_2$ so that

$$\Pi_+(A) = \bigvee_{\mathcal{B}_1 \subset \mathcal{R},\ \bigcap \mathcal{B}_1 \subset A} \left(\bigwedge_{B \in \mathcal{B}_1} \Pi(B) \right) \leq \bigwedge_{\mathcal{B}_2 \subset \mathcal{R},\ \bigcup \mathcal{B}_2 \supset A} \left(\bigvee_{B \in \mathcal{B}_2} \Pi(B) \right) = \Pi^+(A) \tag{19.11}$$

immediately results. Lemma 19.1 then completes the proof. □

The conditions imposed on strong monotone measures on $\mathcal{R}$ are rather restrictive. E.g., the equalities

$$\Pi\left(\bigcap \mathcal{A}\right) = \bigwedge\{\Pi(A) : A \in \mathcal{A}\},\ \Pi\left(\bigcup \mathcal{B}\right) = \bigvee\{\Pi(B) : B \in \mathcal{B}\} \tag{19.12}$$

must be valid for each $\mathcal{A}, \mathcal{B} \subset \mathcal{R}$ such that $\bigcap \mathcal{A}, \bigcup \mathcal{B} \in \mathcal{R}$. In particular, $\bigwedge\{\Pi(A) : A \in \mathcal{A}\} = \mathbf{0}_{\mathcal{T}}$ for each $\mathcal{A} \subset \mathcal{R}$ such that $\bigcap \mathcal{A} = \emptyset$, let us recall that we suppose that $\emptyset \in \mathcal{R}$. Indeed, each strong monotone measure Π on $\mathcal{R}$ is a monotone measure on $\mathcal{R}$, so that $\bigcap \mathcal{A} \in \mathcal{R}$ and $\bigcap \mathcal{A} \subset A$ for each $A \in \mathcal{A}$ yields that $\Pi(\bigcap \mathcal{A}) \leq \Pi(A)$, so that $\Pi(\bigcap \mathcal{A}) \leq \bigwedge\{\Pi(A) : A \in \mathcal{A}\}$ holds. On the other side, when taking $\mathcal{B} = \{\bigcap \mathcal{B}\} \subset \mathcal{R}$, we obtain that $\bigcap \mathcal{A} \subset \bigcup \mathcal{B} = \bigcap A$ is valid, hence, as Π is a strong monotone measure on $\mathcal{R}$, the inequality $\bigwedge\{\Pi(A) : A \in \mathcal{A}\} \leq \bigvee\{\Pi(B) : B \in \mathcal{B}\}$ and, consequently, the first equality from (19.12) follows. Dually, we obtain that $\Pi(B) \leq \Pi(\bigcup \mathcal{B})$ for each $B \in \mathcal{B}$ and $\bigvee\{\Pi(B) : B \in \mathcal{B}\} \leq \Pi(\bigcup \mathcal{B})$ hold. Setting $\mathcal{A} = \{\bigcup \mathcal{B}\}$ we obtain that $\bigcap \mathcal{A} = \bigcup \mathcal{B} \subset \bigcup \mathcal{B}$ holds, so that

$$\bigwedge\{\Pi(A) : A \in \mathcal{A}\} = \Pi\left(\bigcup \mathcal{B}\right) \leq \bigvee\{\Pi(B) : B \in \mathcal{B}\} \tag{19.13}$$

follows. Hence, the other equality in (19.12)) is also proved.

The most trivial example of a strong monotone measure is the identity on the power-set $\mathcal{P}(\Omega)$, hence, $\mathcal{T} = \langle T, \leq \rangle = \langle \mathcal{P}(\Omega), C \rangle$ and $\Pi(A) = A$ for every $A \subset \Omega$. Indeed, if $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\Omega)$ are such that $\bigcap \mathcal{A} \subset \bigcup \mathcal{B}$ holds, then

$$\bigwedge\{\Pi(A) : A \in \mathcal{A}\} = \bigcap\{A : A \in \mathcal{A}\} = \bigcap \mathcal{A} \subset \bigcup \mathcal{B} = \bigvee\{\Pi(B) : B \in \mathcal{B}\} \tag{19.14}$$

trivially follows. As a matter of fact, however, the system of all strong monotone measures on $\mathcal{P}(\Omega)$ is restricted just to isomorphisms between Boolean algebra $\langle \mathcal{P}(\Omega), \bigcup, \bigcap, \Omega - . \rangle$ and the subset $\Pi(\mathcal{P}(\Omega)) = \{\Pi(A) : A \subset \Omega\} \subset T$, as the next assertion proves.

Theorem 19.2 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a Boolean-like complete lattice, let Π be a strong $\mathcal{T}$-monotone measure on $\mathcal{P}(\Omega)$. Then the quadruple $\langle T_0, \vee, \wedge, (\cdot)^C \rangle$, where $T_0 = \Pi(\mathcal{P}(\Omega))$, $\vee$ and $\wedge$ are the supremum and infimum operations induced by $\leq$ in T, and $(\cdot)^C$ is the pseudo-complement operation in $\mathcal{T} \upharpoonright T_0$ (hence, $(t)^C = \bigvee\{s \in T_0 : s \wedge t = \mathbf{0}_{\mathcal{T}}\}$), defines a Boolean algebra and Π is an isomorphism between the Boolean algebras $\langle \mathcal{P}(\Omega), \cup, \cap, \Omega - \cdot \rangle$ and $\langle T_0, \vee, \wedge, (\cdot)^C \rangle$.*

Proof. Let $t_1.t_2, t_3 \in T_0$. Then there exist $A, B, C \subset \Omega$ such that $t_1 = \Pi(A)$, $t_2 = \Pi(B)$ and $t_3 = \Pi(C)$. As Π is a strong monotone measure on $\mathcal{P}(\Omega)$, the relations

$$\begin{aligned} t_1 \wedge t_2 &= \Pi(A) \wedge \Pi(B) = \Pi(A \cap B) = t_2 \wedge t_1, \\ t_1 \vee t_2 &= \Pi(A) \vee \Pi(B) = \Pi(A \cup B) = t_2 \vee t_1, \\ (t_1 \wedge t_2) \wedge t_3 &= (\Pi(A) \wedge \Pi(B)) \wedge \Pi(C) = \Pi(A \cap B \cap C) = \\ &= \Pi(A) \wedge (\Pi(B) \wedge \Pi(C)) = t_1 \wedge (t_2 \wedge t_3), \end{aligned} \tag{19.15}$$

and

$$
\begin{aligned}
(t_1 \vee t_2) \vee t_3 &= (\Pi(A) \vee \Pi(B)) \vee \Pi(C) = \Pi(A \cup B \cup C) = \\
&= \Pi(A) \vee (\Pi(B) \vee \Pi(C)) = t_1 \vee (t_2 \vee t_3)
\end{aligned} \tag{19.16}
$$

easily follow from (19.12). Hence, the suprema and infima of elements from T_0 are in T_0 and both the operations are commutative and associative. We can also easily deduce that

$$
\begin{aligned}
(t_1 \wedge t_2) \vee t_2 &= (\Pi(A) \wedge \Pi(B)) \vee \Pi(B) = \Pi(A \cap B) \vee \Pi(B) = \\
&= \Pi((A \cap B) \cup B) = \Pi(B) = t_2
\end{aligned} \tag{19.17}
$$

and, dually

$$
\begin{aligned}
(t_1 \vee t_2) \wedge t_2 &= (\Pi(A) \vee \Pi(B)) \wedge \Pi(B)) = \Pi(A \cup B) \wedge \Pi(B) = \\
&= \Pi((A \cup B) \cap B) = \Pi(B) = t_2.
\end{aligned} \tag{19.18}
$$

Moreover,

$$
\begin{aligned}
t_1 \wedge (t_2 \vee t_3) &= \Pi(A) \wedge (\Pi(B) \vee \Pi(C)) = \Pi(A) \wedge \Pi(B \cup C) = \\
&= \Pi((A \cap (B \cup C)) = \Pi((A \cap B) \cup (A \cap C)) = \\
&= \Pi(A \cap B) \vee \Pi(A \cap C) = \\
&= (\Pi(A) \wedge \Pi(B)) \vee (\Pi(A) \wedge \Pi(C)) = (t_1 \wedge t_2) \vee (t_1 \wedge t_3)
\end{aligned} \tag{19.19}
$$

and

$$
\begin{aligned}
t_1 \vee (t_2 \wedge t_3) &= \Pi(A) \vee (\Pi(B) \wedge \Pi(C)) = \Pi(A \cup (B \cap C)) = \\
&= \Pi((A \cup B) \cap (A \cup C)) = \Pi((A \cup B) \wedge \Pi(A \cup C) = \\
&= (\Pi(A) \vee \Pi(B)) \wedge (\Pi(A) \vee \Pi(C)) = (t_1 \vee t_2) \wedge (t_1 \vee t_3).
\end{aligned} \tag{19.20}
$$

As $\mathcal{T} = \langle T, \leq \rangle$ is Boolean-like, $t \wedge (t)^C = \mathbf{0}_{\mathcal{T}}$ and $t \vee (t)^C = \mathbf{1}_{\mathcal{T}}$ holds for every $t \in T$, so that

$$
(t_1 \wedge (t_1)^C) \vee t_2 = \mathbf{0}_{\mathcal{T}} \vee t_2 = t_2 = t_2 \wedge \mathbf{1}_{\mathcal{T}} = t_2 \wedge (t_1 \vee (t_1)^C) \tag{19.21}
$$

easily follows. All the axions of Boolean algebras (cf. [42], e.g.) are satisfied and Π defines an isomorphism between the Boolean algebras $\langle \mathcal{P}(\Omega), \cup, \cap, \Omega - \cdot \rangle$ and $\langle T_0, \vee, \wedge, (\cdot)^C \rangle$. Indeed, the inequality

$$
(t)^C = \bigvee \{s \in T_0 : s \wedge t = \mathbf{0}_{\mathcal{T}}\} \leq \bigvee \{s \in T : s \wedge t = \mathbf{0}_{\mathcal{T}}\} = t^C \tag{19.22}
$$

obviously holds, on the other side, if $t_1 = \Pi(A)$, then $\Pi(A) \wedge \Pi(\Omega - A) = \Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}$, so that $t_1)^C \geq \Pi(\Omega - A)$ and $t_1 \vee (t_1)^C = \Pi(A) \vee \Pi(\Omega - A) = \Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$ follows. The assertion is proved. □

An intuitive example of strong monotone measure defined on the whole power-set $\mathcal{P}(\Omega)$ may read as follows. Let $\omega_0 \in \Omega$ be fixed, let $\mathcal{T} = \langle \{0, 1\}, \leq \rangle$ be the most simple binary set of values, let $\Pi : \mathcal{P}(\Omega) \to \{0, 1\}$ be such that $\Pi(A) = 1$, if $\omega_0 \in A$, $\Pi(A) = 0$ otherwise, i.e., if $\omega_0 \in \Omega - A$, $A \subset \Omega$. Let $\mathcal{A}, \mathcal{B}$ be such systems of subsets of Ω, that $\bigcap \mathcal{A} \subset \bigcup \mathcal{B}$ holds. If there exists $B_0 \in \mathcal{B}$ such that $\omega_0 \in B_0$, so that $\Pi(B_0) = 1$, then $\bigvee \{\Pi(B) : B \in \mathcal{B}_0\} = 1$ as well and the inequality $\bigwedge \{\Pi(A) : A \in \mathcal{A}\} \leq \bigvee \{\Pi(B) : B \in \mathcal{B}\}$ holds trivially. If $\omega_0 \in B$ is not the case no matter which $B \in \mathcal{B}$ is taken, then $\omega_0 \in \bigcup \mathcal{B}$ and, consequently, $\omega_0 \in \bigcap \mathcal{A}$ do not hold. Hence, there exists $A_0 \in \mathcal{A}$ which does not contain ω_0, so that $\Pi(A_0) = 0$ and the inequality $0 = \bigwedge \{\Pi(A) : A \in \mathcal{A}\} \leq \bigvee \{\Pi(B) : B \in \mathcal{B}\}$ again trivially follows, so that Π is a strong monotone measure on $\mathcal{P}(\Omega)$. As can be easily verified, the same is the case when Π is defined in a slightly generalized way, namely, if $\Pi(A) = 1$ iff $A_\star \subset A$ holds, where $A_\star$ is a proper nonempty subset of Ω (not just a singleton, as above), hence $\Pi(A) = 0$ iff $A_\star \cap (\Omega - A) \neq \emptyset$.

On the other side, if Π is defined in such a way that $\Pi(A) = 1$ iff $A \cap A_\star \neq \emptyset$, then in general Π is not a strong monotone measure on $\mathcal{P}(\Omega)$, as there exists $A \subset \Omega$ such that $A \cap A_\star \neq \emptyset$ and $(\Omega - A) \cap A_\star \neq \emptyset$ (supposing that $A_\star$ is not a singleton), so that $\Pi(A) = \Pi(\Omega - A) = 1 > \Pi(A \cap (\Omega - A)) = \Pi(\emptyset) = 0$.

Contrary to this example, let us consider the following modification of the most simple set-valued strong monotone measure defined by the identity on $\mathcal{P}(\Omega)$ (as introduced above). Again, fix $\omega_0 \in \Omega$, take $\mathcal{T} = \langle T, \leq \rangle = \langle \mathcal{P}(\Omega), \subset \rangle$, and set $\Pi(A) = A$, if $\omega_0 \in A$, $\Pi(A) = 0$ otherwise, i.e., if $\omega_0 \in \Omega - A$, $A \subset \Omega$. Obviously, Π defines a monotone measure on $\mathcal{P}(\Omega)$. Indeed, if $A \subset B \subset \Omega$ and $\Pi(A) = 0$, the inequality $\Pi(A) \leq \Pi(B)$ holds trivially. If $\Pi(A) = 1$, then $\omega_0 \in A$ and, consequently, $\omega_0 \in B$ and $\Pi(B) = 1$ follows, so that, again, $\Pi(A) \leq \Pi(B)$ holds. However, in general, Π is not a strong monotone measure on $\mathcal{P}(\Omega)$. Let $\mathcal{B} \subset \mathcal{P}(\Omega)$ be such that there exists $\omega_1 \in \Omega$, $\omega_1 \neq \omega_0$, and $B_1 \in \mathcal{B}$ with this property: $\omega_0 \in B$ and $\omega_1 \in \Omega - B$ for every $B \in \mathcal{B}$, $B \neq B_1$, and $\omega_0 \in \Omega - B_1$. Take $\mathcal{A} = \{\bigcup \mathcal{B}\} \subset \mathcal{P}(\Omega)$, so that $\bigcap \mathcal{A} = \bigcap \mathcal{B} \subset \bigcup \mathcal{B}$ is trivially satisfied. In this case, $\bigwedge\{\Pi(A) : A \in \mathcal{A}\} = \Pi(\bigcup \mathcal{B}) = \bigcup \mathcal{B}$, as $\omega_0 \in \bigcup \mathcal{B}$ (we suppose that B_1 is not the only set in $\mathcal{B}$), and

$$\bigvee\{\Pi(B) : B \in \mathcal{B}\} = \bigvee\{\Pi(B) : B \in \mathcal{B} - \{B_1\}\} = \bigcup\{B : B \in \mathcal{B} - \{B_1\}\} = \bigcup \mathcal{B} - \{B_1\}, \quad (19.23)$$

as $\Pi(B_1) = \emptyset$ and $\Pi(B) = B$ for every $B \in \mathcal{B}$, $B \neq B_1$. However, $\omega_1 \in \bigcup \mathcal{B} = \Pi(\bigcup \mathcal{B}) = \bigwedge\{\Pi(A) : A \in \mathcal{A}\}$ holds, but $\omega_1 \in \bigcup(\mathcal{B} - \{B_1\}) = \bigvee\{\Pi(B) : B \in \mathcal{B}\}$ does not hold. Hence, the inclusion $\bigwedge\{\Pi(A) : A \in \mathcal{A}\} \subset \bigvee\{\Pi(B) : B \in \mathcal{B}\}$ is not valid, so that Π is not a strong monotone measure on $\mathcal{P}(\Omega)$.

Theorem 19.3 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\mathcal{R}$ be a system of subsets of a nonempty set Ω such that $\{\emptyset, \Omega\} \subset \mathcal{R}$, let $\Pi : \mathcal{R} \to T$ be a partial $\mathcal{T}$-monotone measure on $\mathcal{R}$ such that the set function $\Pi^+ : \mathcal{P}(\Omega) \to T$, defined by*

$$\Pi^+(A) = \bigwedge_{\mathcal{B} \subset \mathcal{R},\ \bigcup \mathcal{B} \supset A} \left(\bigvee_{B \in \mathcal{B}} \Pi(B) \right) \quad (19.24)$$

for every $A \subset \Omega$, conservatively extends Π from $\mathcal{R}$ to $\mathcal{P}(\Omega)$, hence, $\Pi^+(A) = \Pi(A)$ for every $A \in \mathcal{R}$. Then Π is a complete partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$. Let $\mathcal{R}_0 = \{\bigcup \mathcal{S} : \mathcal{S} \subset \mathcal{R}\}$ be the system of all unions of sets from $\mathcal{R}$, let $\Pi_0 : \mathcal{R}_0 \to T$ be the mapping defined for every $A \in \mathcal{R}_0$, $A = \bigcup \mathcal{S}, \mathcal{S} \subset \mathcal{R}$, by

$$\Pi_0(A) = \bigvee\{\Pi(B) : B \in \mathcal{S}\}. \quad (19.25)$$

Then Π_0 is a uniquely defined complete partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}_0$, extending conservatively Π from $\mathcal{R}$ to $\mathcal{R}_0$.

Proof. Let the conditions of Theorem 19.3 hold, let $A \subset \Omega$, $A \in \mathcal{R}$. Take $\mathcal{B}_0 = \{A\} \subset \mathcal{R}$, then $\bigvee\{\Pi(B) : B \in \mathcal{B}_0\} = \Pi(A)$, so that

$$\Pi(A) = \Pi^+(A) = \bigwedge_{\mathcal{B} \subset \mathcal{R},\ \bigcup \mathcal{B} \supset A} \left(\bigvee\{\Pi(B) : B \in \mathcal{B}\} \right) \leq \bigvee\{\Pi(B) : B \in \mathcal{B}_0\} = \Pi(A) \quad (19.26)$$

follows. Consequently, the inequality

$$\bigvee\{\Pi(B) : B \in \mathcal{B}\} \geq \Pi(A) \quad (19.27)$$

holds for each $\mathcal{B} \subset \mathcal{R}$ such that $\bigcup \mathcal{B} \supset A$. Let $\mathcal{B} \subset \mathcal{R}$ be such that $\bigcup \mathcal{B} \in \mathcal{R}$. Setting $A = \bigcup \mathcal{B}$ in (19.27) we obtain that $\bigvee\{\Pi(B) : B \in \mathcal{B}\} \geq \Pi(\bigcup \mathcal{B})$ holds. However, $B \subset \bigcup \mathcal{B}$ is valid for each $B \in \mathcal{B}$ and Π is a partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$, so that the inequalities $\Pi(B) \leq \Pi(\bigcup \mathcal{B})$, $\bigvee\{\Pi(B) : B \in \mathcal{B}\} \leq \Pi(\bigcup \mathcal{B})$ and, consequently, the equality $\Pi(\bigcup \mathcal{B}) = \bigvee\{\Pi(B) : B \in \mathcal{B}\}$ follow. Hence, Π is a complete partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$.

Considering $\mathcal{R}_0$ and $\Pi_0 : \mathcal{R}_0 \to T$ defined as above, we have to prove, first of all, that the definition (19.25) is correct, i.e., that the relation $\bigvee\{\Pi(B) : B \in \mathcal{S}_1\} = \bigvee\{\Pi(C) : C \in \mathcal{S}_2\}$ is valid for every

$\mathcal{S}_1, S_2 \subset \mathcal{R}$ such that $\bigcup \mathcal{S}_1 = \bigcup \mathcal{S}_2$. However, this equality immediately follows from (19.27), as both the inclusions $\bigcup \mathcal{S}_1 \supset \bigcup \mathcal{S}_2$ and $\bigcup \mathcal{S}_2 \supset \bigcup \mathcal{S}_1$ hold simultaneously. Given $A \in \mathcal{R}$, the system $\mathcal{S}_A = \{A\}$ is one of those for which $\bigcup \mathcal{S}_A = A$, so that

$$\Pi^+(A) = \bigvee \{\Pi(B) : B \in \mathcal{S}_A\} = \Pi(A), \tag{19.28}$$

hence, Π^+ conservatively extends Π from $\mathcal{R}$ to $\mathcal{R}_0$.

Let $\mathcal{S}_0 \subset \mathcal{R}_0$ be a system of subsets of Ω such that each of them is a union of sets from $\mathcal{R}$. Given $A \in \mathcal{S}_0$, let $\mathcal{S}_A \subset \mathcal{R}$ be a system of subsets from $\mathcal{R}$ such that $\bigcup \mathcal{S}_A = A$. Let $\mathcal{S}^\star \subset \mathcal{R}$ be the system of subsets from $\mathcal{R}$ containing just the sets occurring in some $\mathcal{S}_A$, hence, $\mathcal{S}^\star = \bigcup \{\mathcal{S}_A : A \in \mathcal{S}_0\}$. Obviously

$$\bigcup \mathcal{S}^\star = \bigcup_{B \in \mathcal{S}^\star} B = \bigcup_{A \in \mathcal{S}_0} \left(\bigcup_{B \in \mathcal{S}_A} B \right) \in \mathcal{R}_0, \tag{19.29}$$

so that, due to (7.28), the relation

$$\Pi^\star \left(\bigcup \mathcal{S}^\star \right) = \bigvee \{\Pi(B) : B \in \mathcal{S}^\star\} = \bigvee_{A \in \mathcal{S}_0} \left(\bigvee_{B \in \mathcal{S}_A} \Pi(B) \right) = \bigvee_{A \in \mathcal{S}_0} \Pi^+(A) \tag{19.30}$$

follows, so that Π^+ is a complete partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}_0$ and the assertion is proved. □

When summarizing our considerations concerning the set functions Π_+ and Π^+, induced by the partial $\mathcal{T}$-fuzzy or possibilistic measure Π, we may perhaps conclude that their role as possible alternatives to and improvements of the inner and the outer measures $\Pi_\star$ and $\Pi^\star$ is rather limited. It follows from the fact that the inequality $\Pi_\star(A) \leq \Pi_+(A) \leq \Pi^+(A) \leq \Pi^\star(A)$ for every $A \subset \Omega$, and the equality $\Pi_+(A) = \Pi^+(A) = \Pi(A)$ for every $A \in \mathcal{R}$, which should be intuitively valid in this case, are valid only under rather strong and far from being intuitive supplementary conditions. Moreover, the set function Π_+ satisfies the condition $\Pi_+(\bigcap \mathcal{A}) = \bigwedge \{\Pi_+(A),\ A \in \mathcal{A}\}$, $\mathcal{A} \subset \mathcal{P}(\Omega)$, which is too strong even for $\mathcal{T}$-partial possibilistic measures when just the inequality $\Pi(\bigcap \mathcal{A}) \leq \bigwedge \{\Pi(A) : A \in \mathcal{A}\}$ can be proved. Hence, it looks like quite reasonable to postpone a more detailed investigation of set functions Π_+ and Π^+ till the time when some new and qualitatively different interpretation of (or semantic for) them is suggested, proving the qualities of these set functions as good approximations and extensions of the original partial $\mathcal{T}$-monotone or possibilistic measures.

20 Extensions of Partial Lattice-Valued Possibilistic Measures from Nested Domains

Our goal, in what follows, will be to re-consider the problems presented in Chapter 5 for the case of lattice-valued possibilistic measure. In more detail, we will consider a partial lattice-valued possibilistic measure defined on a nested system of subsets of the universe of discourse, aiming to extend this measure to the whole power-set over this universe. To this end, we will apply the idea on which outer measures rely, covering each subset of this universe, as tightly as possible in the sense of set inclusion, by a set from the definition domain of the partial lattice-valued possibilistic measure under consideration. Various ways how to combine such approaches when having at hand more partial possibilistic measures of this kind will be also considered and analyzed.

For the reader's convenience, let us recall some notions to be of use in what follows. Let $\mathcal{T} = \langle T, \leq_\mathcal{T} \rangle$ be a partially ordered (p.o.) set, let $\wedge_\mathcal{T}$ and $\vee_\mathcal{T}$ denote the infimum and the supremum operations induced by $\leq_\mathcal{T}$ on T. Set $\mathbf{0}_\mathcal{T} = \bigwedge_\mathcal{T} T$ and $\mathbf{1}_\mathcal{T} = \bigvee_\mathcal{T} T$ (and call them the *zero* and the *unit* elements of $\mathcal{T}$) supposing that they are defined in $\mathcal{T}$.

A partially ordered set $\mathcal{I} = \langle I, \leq_\mathcal{I} \rangle$ is called a *lower semilattice*, if for each $\alpha, \beta \in I$ the infimum $\alpha \wedge_\mathcal{I} \beta$ is defined in $\mathcal{I}$. A lower semilattice $\mathcal{I} = \langle I, \leq_\mathcal{I} \rangle$ is called *complete*, if for each $\emptyset \neq J \subset I$ the infimum $\wedge_\mathcal{I} J$ is defined. Consequently, in each complete lower semilattice $\mathcal{I}$ the zero element $\mathbf{1}_\mathcal{T}$ is defined.

Let Ω be a nonempty set, let $\mathcal{R}$ be a nonempty system of subsets of Ω, let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be a p.o.set. A mapping $\varphi : \mathcal{R} \to T$ is called a *$\mathcal{T}$-valued monotone measure on $\mathcal{R}$*, if

(i) $\varphi(\emptyset) = \mathbf{0}_{\mathcal{T}}$ supposing that $\phi \in \mathcal{R}$ and $\mathbf{0}_{\mathcal{T}}$ is defined,

(ii) $\varphi(\Omega) = \mathbf{1}_{\mathcal{T}}$ supposing that $\Omega \in \mathcal{R}$ and $\mathbf{1}_{\mathcal{T}}$ is defined,

(iii) $\varphi(A) \leq_{\mathcal{T}} \varphi(B)$ holds for each $A \subset B \subset \Omega,\ A, B \in \mathcal{R}$.

In order to simplify our further reasoning and notation we will suppose, in this and in the next chapter, that ϕ and Ω are always in $\mathcal{R}$ and that $\mathbf{0}_{\mathcal{T}}$ as well as $\mathbf{1}_{\mathcal{T}}$ are defined in $\mathcal{T} = \langle T, \leq \rangle$. The triple $\langle \Omega, \mathcal{R}, \varphi \rangle$ will be called a *monotone space* analogously to the notions of probability space and possibility (or possibilistic) space.

Definition 20.1 *Let $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ be a complete lower semilattice, let $\langle \Omega, \mathcal{R}, \varphi \rangle$ be a monotone space. A system $\mathcal{S}$ of subsets of Ω is called a* classification system over $\langle \Omega, \mathcal{R}, \varphi \rangle$ indexed by $\mathcal{I}$ *(*$\mathcal{I}$-classification system over $\langle \Omega, \mathcal{R}, \varphi \rangle$*, abbreviately), if $\mathcal{S} \subset \mathcal{R}$, $\mathcal{S} = \{S_\alpha : \alpha \in I\}$, $S_{\mathbf{0}_{\mathcal{I}}} = \emptyset$ (let us recall that $\mathbf{0}_{\mathcal{I}} = \bigwedge_{\mathcal{I}} I$), there exists $\alpha \in I$ such that $S_\alpha = \Omega$, and for each $\alpha, \beta \in I$, if $\alpha \leq_{\mathcal{I}} \beta$ holds, then the inclusion $S_\alpha \subset S_\beta$ is valid.*

Let $\mathcal{S}$ be an $\mathcal{I}$-classification system over $\langle \Omega, \mathcal{R}, \varphi \rangle$. Given $A \subset \Omega$, set

$$\alpha(A, \mathcal{S}) = \bigwedge_{\mathcal{I}} \{\beta \in I : A \subset S_\beta\}. \tag{20.1}$$

As $\Omega \in \mathcal{S}$ holds, the set $\{\beta \in I : A \subset S_\beta\}$ is nonempty, hence, its infimum with respect to $\leq_{\mathcal{I}}$, i.e., the value $\alpha(A, \mathcal{S})$ is always defined and belongs to I. Hence, the set $S_{\alpha(A,\mathcal{S})}$ is defined and is in $\mathcal{R}$, consequently, also the value $F(A) = \varphi(S_{\alpha(A,\mathcal{S})}) \in T$ is defined. So, F is a mapping which takes $\mathcal{P}(\Omega)$ into $\langle T, \leq_{\mathcal{T}} \rangle$.

An $\mathcal{I}$-classification system $\mathcal{S}$ over $\langle \Omega, \mathcal{R}, \varphi \rangle$ is called *conservative*, if $S_{\alpha(A,\mathcal{S})} = A$ holds for each $A \in \mathcal{S}$. In this case, $F(A) = \varphi(S_{\alpha(A,\mathcal{S})}) = \varphi(A)$ for each $A \in \mathcal{S}$, hence, F extends conservatively φ from $\mathcal{S}$ (but not from $\mathcal{R}$, in general), to $\mathcal{P}(\Omega)$.

Lemma 20.1 *Let $\langle \Omega, \mathcal{R}, \varphi \rangle$ be a monotone space, let $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ be a complete lower semi-lattice, let $\mathcal{S}$ be an $\mathcal{I}$-classification system over $\langle \Omega, \mathcal{R}, \varphi \rangle$ and, for each $A \subset \Omega$, let $\alpha(A, \mathcal{S} \rangle$ be defined by (20.1) and $F(A)$ by*

$$F(A) = \varphi(S_{\alpha(A,\mathcal{S})}). \tag{20.2}$$

If $F(\Omega) = \mathbf{1}_{\mathcal{T}}$, then F is a $\mathcal{T}$-monotone measure on $\mathcal{P}(\Omega)$. In particular, if $\mathcal{I}$ is conservative, then F extends φ conservatively from $\mathcal{S}$ to $\mathcal{P}(\Omega)$.

Proof. As $\{\emptyset, \Omega\} \subset \mathcal{R}$ holds, $\varphi(\emptyset) = \mathbf{0}_{\mathcal{T}}$, so that

$$\alpha(\emptyset, \mathcal{S}) = \bigwedge \{\beta \in I : \phi \subset S_\beta\} = \bigwedge_{\mathcal{I}} I = \mathbf{0}_{\mathcal{I}}, \tag{20.3}$$

so that $S_{\alpha(\emptyset,\mathcal{S})} = S_{\mathbf{0}_{\mathcal{I}}} = \emptyset$ by the definition of $\mathcal{I}$-classification system, hence

$$F(\emptyset) = \varphi(S_{\alpha(\emptyset,\mathcal{S})}) = \varphi(\emptyset) = \mathbf{0}_{\mathcal{T}} \tag{20.4}$$

follows. Let $A \subset B \subset \Omega$, then for each $C \in \mathcal{S}$ such that $B \subset C$ holds, $A \subset C$ holds as well, so that the inclusion

$$\{\beta \in I : A \subset S_\beta\} \supset \{\beta \in I : B \subset S_\beta\} \tag{20.5}$$

and, consequently, the inequality

$$\alpha(A, \mathcal{S}) = \bigwedge_{\mathcal{I}} \{\beta \in I : A \subset S_\beta\} \leq_{\mathcal{I}} \bigwedge_{\mathcal{I}} \{B \in I : B \subset S_\beta\} = \alpha(B, \mathcal{S}) \tag{20.6}$$

follows. So, $S_{\alpha(A,\mathcal{S})} \subset S_{\alpha(B,\mathcal{S})}$ and

$$F(A) = \varphi(S_{\alpha(A,\mathcal{S})}) \leq_{\mathcal{T}} \varphi(S_{\alpha(B,\mathcal{S})}) = F(B) \tag{20.7}$$

hold and the assertion is proved. □

Example 20.1

Let $\Omega \neq \emptyset$, let $\{\emptyset, \Omega\} \subset \mathcal{R} \subset \mathcal{P}(\Omega)$, let $\langle T, \leq_{\mathcal{T}} \rangle = \langle \mathcal{P}(\Omega), \subset \rangle$, let φ_{id} be the identity mapping on $\mathcal{R}$, so that $\varphi_{id}(A) = A \in T$ for each $A \in \mathcal{R}$. Let $\{\emptyset, \Omega\} \subset \mathcal{S} \subset \mathcal{R}$ be such that $\mathcal{S}$ is completely closed with respect to intersections, i.e., $\bigcap \mathcal{S}_0 = \bigcap_{A \in \mathcal{S}_0} A$ is in $\mathcal{S}$ for each $\emptyset \neq \mathcal{S}_0 \subset \mathcal{S}$, let α be the identity mapping on $\mathcal{S}$, so that $S_A = A$ for every $A \in \mathcal{S}$. Then $\mathcal{I} = \langle \mathcal{S}, \subset \rangle$ is a complete lower semilattice and $\mathcal{S}$ is an $\mathcal{I}$-classification system over the monotone space $\langle \Omega, \mathcal{R}, \varphi_{id} \rangle$. If $A_0 \subset \Omega$, then

$$\alpha(A_0, \mathcal{S}) = \bigwedge_{\mathcal{I}} \{\beta \in I : A \subset S_\beta\} = \bigcap \{C \in \mathcal{S} : A \subset S_C\} = \bigcap \{C \in \mathcal{S} : A \subset C\}, \tag{20.8}$$

let us denote the last intersection by $A^{\mathcal{S}}$. As $\mathcal{I}$ is a complete semilattice, $A^{\mathcal{S}}$ is in $\mathcal{S}$, hence, $A^{\mathcal{S}}$ is in $\mathcal{R}$ and the value

$$F(A) = \varphi_{id}(A^{\mathcal{S}}) = A^{\mathcal{S}} \tag{20.9}$$

is defined. If $\mathcal{S} = \mathcal{P}(\Omega)$, then obviously $\bigcap \{C : A \subset C \subset \Omega\} = A$, so that F is the identity on $\mathcal{P}(\Omega)$.

Lemma 20.2 *Let $\langle \Omega, \mathcal{R}, \varphi \rangle$, $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ and $\mathcal{S} = \{S_\beta : \beta \in I\}$ be as in Lemma 20.1, let the relation*

$$S_{\bigwedge_{\mathcal{I}} J} = \bigcap \{S_\alpha : \alpha \in J\} \tag{20.10}$$

hold for each $\emptyset \neq J \subset I$. Then the $\mathcal{I}$-classification system $\mathcal{S}$ is conservative. Conversely, if $\mathcal{S}$ is conservative, then

$$S_{\alpha(S_\beta, \mathcal{S})} = \bigcap \{S_\gamma : S_\beta \subset S_\gamma\} \tag{20.11}$$

holds for each $\beta \in I$.

Proof. Take $\beta \in I,\ S_\beta \in \mathcal{S}$. By definition

$$\alpha(S_\beta, \mathcal{S}) = \bigwedge_{\mathcal{I}} \{\gamma \in I : S_\beta \subset S_\gamma\} \tag{20.12}$$

hence, (20.10) yields that

$$S_{\alpha(S_\beta, \mathcal{S})} = \bigcap \{S_\delta : \delta \in \{\delta \in I : S_\beta \subset S_\gamma\}\} = \bigcap \{S_\delta : S_\beta \subset S_\delta\} = S_\beta, \tag{20.13}$$

so that $\mathcal{S}$ is conservative. On the other side, let $\mathcal{S}$ be conservative, so that

$$S_{\alpha(S_\beta, \mathcal{S})} = S_\beta \tag{20.14}$$

for each $\beta \in I$. By (20.12), $\alpha(S_\beta, \mathcal{S}) \leq \gamma$ holds for each $\gamma \in I$ such that the inclusion $S_\beta \subset S_\gamma$ is valid. Consequently, the inclusions $S_{\alpha(S_\beta, \mathcal{S})} \subset S_\gamma$ and $S_{\alpha(S_\beta, \mathcal{S})} \subset \bigcap \{S_\gamma : S_\beta \subset S_\gamma\}$ easily follow from the definition of $\mathcal{I}$-classification systems. Combining this last inclusion with (20.14) and taking into consideration that β is among the values of γ for which $S_\beta \subset S_\gamma$ holds, we obtain that $S_\beta \subset \bigcap \{S_\gamma : S_\beta \subset S_\gamma\} \subset S_\beta$ so that the equality

$$S_{\alpha(S_\beta, \mathcal{S})} = S_\beta = \bigcap \{S_\gamma : S_\beta \subset S_\gamma\} \tag{20.15}$$

follows. The assertion is proved.

□

As mentioned already above at several occasions, monotone measures taking their values in partially ordered sets can be taken as perhaps the most general and still non-trivial mathematical formalization of an intuitive idea of size of subsets of a universe of discourse. When specifying monotone measures by imposing some more conditions which these set functions should fulfil, we will follow the main pattern applied in this work focusing our attention to lattice-valued possibilistic measures. For the reader's convenience and aiming to introduce this notion at the level of generality appropriate just in the present context, let us recall a very general definition of this notion as follows.

Definition 20.2 *Let $\Omega \neq \emptyset$, let $\{\emptyset, \Omega\} \subset \mathcal{R} \subset \mathcal{P}(\Omega)$, let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be a p.o. set such that $\mathbf{0}_{\mathcal{T}} = \bigwedge_{\mathcal{T}} T$ and $\mathbf{1}_{\mathcal{T}} = \bigvee_{\mathcal{T}} T$ are defined. A mapping $\Pi : \mathcal{R} \to T$ is called a $\mathcal{T}$-*(valued) possibilistic measure *on $\mathcal{R}$, if $\Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}$, $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$, and $\Pi(A \cup B) = \Pi(A) \vee_{\mathcal{T}} \Pi(B)$ for each $A, B, A \cup B \in \mathcal{R}$ such that $\Pi(A) \vee_{\mathcal{T}} \Pi(B)$ is defined. The $\mathcal{T}$-possibilistic measure Π on $\mathcal{R}$ is *complete*, if $\Pi(\bigcup \mathcal{R}_0) = \bigvee_{\mathcal{T}} \{\Pi(A) : A \in \mathcal{R}_0\}$ for each $\emptyset \neq \mathcal{R}_0 \subset \mathcal{R}$ such that $\bigcup \mathcal{R}_0 = \bigcup_{A \in \mathcal{R}_0} A$ is in $\mathcal{R}$ and the supremum $\bigvee_{\mathcal{T}} \{\Pi(A) : A \in \mathcal{R}_0\}$ is defined. The triple $\langle \Omega, \mathcal{R}, \Pi \rangle$ will be called the *possibilistic space*.*

Obviously, each $\mathcal{T}$-possibilistic measure on $\mathcal{R}$ is also $\mathcal{T}$-monotone measure on $\mathcal{R}$. As above, in order to simplify our notation and reasoning we will suppose that $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ is a complete lattice, so that for each $\emptyset \neq A$ the values $\bigwedge_{\mathcal{T}} A$ and $\bigvee_{\mathcal{T}} A$ are defined (for $A = \emptyset$ the well-known conventions are applied). Hence, each $\mathcal{T}$-possibilistic space $\langle \Omega, \mathcal{R}, \Pi \rangle$ is a particular case of $\mathcal{T}$-monotone space so that, given a complete lower semilattice $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ and an $\mathcal{I}$-classification system $\mathcal{S}$ over $\langle \Omega, \mathcal{R}, \Pi \rangle$, the values $\alpha(A, \mathcal{S})$ and $F(A)$ can be defined, for each $A \subset \Omega$, by (20.1) and (20.2). However, even if Π is a $\mathcal{T}$-possibilistic measure on $\mathcal{R}$, F need not be, in general, a $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$ unless some more conditions are imposed on the classification system $\mathcal{S}$ under consideration.

Definition 20.3 *Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be a partially ordered set, let $\langle \Omega, \mathcal{R}, \varphi \rangle$ be a $\mathcal{T}$-monotone space, let $\mathcal{I} = \langle I, \leq_{\mathcal{T}} \rangle$ be a complete lattice. An $\mathcal{I}$-classification system $\mathcal{S}$ over $\langle \Omega, \mathcal{R}, \varphi \rangle$ is called *continuous*, if for each $\phi \neq J \subset I$ the identities*

$$S_{\bigwedge_{\mathcal{I}} J} = \bigcap \{S_\alpha : \alpha \in J\}, \ S_{\bigvee_{\mathcal{I}} J} = \bigcup \{S_\alpha : \alpha \in J\} \tag{20.16}$$

are valid. Let us recall that $\bigwedge_{\mathcal{I}} J$ denotes the infimum and $\bigvee_{\mathcal{I}} J$ the supremum of J with respect to the partial ordering $\leq_{\mathcal{T}}$ on I.

Theorem 20.1 *Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ and $\mathcal{I} = \langle I, \leq_{\mathcal{T}} \rangle$ be complete lattices, let $\langle \Omega, \mathcal{R}, \Pi \rangle$ be a $\mathcal{T}$-possibilistic space with a complete $\mathcal{T}$-possibilistic measure Π on $\mathcal{R}$, let $\mathcal{S}$ be a continuous $\mathcal{I}$-classification system over $\langle \Omega, \mathcal{R}, \Pi \rangle$. For each $A \subset \Omega$, let $\alpha(A, \mathcal{S})$ be defined by (20.1) and $F(A)$ by (20.2) with φ replaced by Π. Then F is a complete $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$.*

Proof. As the conditions of Lemma 20.1 are satisfied, the identities $F(\emptyset) = \mathbf{0}_{\mathcal{T}}$ and $F(\Omega) = \mathbf{1}_{\mathcal{T}}$ immediately follow. Let $\phi \neq \mathcal{A}_0 \subset \mathcal{P}(\Omega)$, let $\bigcup \mathcal{A}_0$ denote $\bigcup_{A \in \mathcal{A}_0} A$. For every $S_\beta \in \mathcal{S}$, if $\bigcup \mathcal{A}_0 \subset S_\beta$, then $A \subset S_\beta$ holds for each $A \in \mathcal{A}_0$, so that the inequalities

$$\alpha(A, \mathcal{S}) = \bigwedge_{\mathcal{I}} \{\beta \in I : A \subset S_\beta\} \leq \bigwedge_{\mathcal{I}} \left\{\beta \in I : \bigcup \mathcal{A}_0 \subset S_\beta\right\} = \alpha\left(\bigcup \mathcal{A}_0, \mathcal{S}\right) \tag{20.17}$$

and, consequently,

$$\bigvee_{A \in \mathcal{A}_0} \alpha(A, \mathcal{S}) \leq \alpha\left(\bigcup \mathcal{A}_0, \mathcal{S}\right) \tag{20.18}$$

are valid. Hence, due to the supposed continuity property of $\mathcal{S}$, the relation

$$S_{\bigvee_{\mathcal{I}} \{\alpha(A,\mathcal{S}) : A \in \mathcal{A}_0\}} = \bigcup_{A \in \mathcal{A}_0} S_{\alpha(A,\mathcal{S})} \subset S_{\alpha(\bigcup \mathcal{A}_0, \mathcal{S})} = \bigcap \left\{S_\beta : \bigcup \mathcal{A}_0 \subset S_\beta\right\} \tag{20.19}$$

follows.

Let us prove the inverse inclusion $\bigcup_{A \in \mathcal{S}_0} S_{\alpha(A,\mathcal{S})} \supset S_{\alpha(\bigcup \mathcal{A}_0, \mathcal{S})}$. Due to (20.19) it is sufficient to find $\beta \in I$ such that

$$\bigcup \mathcal{A}_0 \subset S_\beta \subset \bigcup_{A \in \mathcal{A}_0} S_{\alpha(A,\mathcal{S})} \tag{20.20}$$

holds. As $\mathcal{S}$ is supposed to be continuous, the relation (20.16) applied to $J = \{\gamma \in I : A \subset S_\gamma\}$ yields that

$$S_{\alpha(A,\mathcal{S})} = \bigcap \{S_\gamma : \gamma \in I,\ A \in S_\gamma\} \tag{20.21}$$

hence, $A \subset S_{\alpha(A,\mathcal{S})}$ holds for every $A \subset \Omega$, so that the inclusion

$$\bigcup \mathcal{A}_0 \subset \bigcup_{A \in \mathcal{A}_0} S_{\alpha(A,\mathcal{S})} \tag{20.22}$$

follows. So, (20.20) is satisfied for $\beta = \bigvee_{A \in \mathcal{A}_0} \alpha(A,\mathcal{S})$ and this β is defined in $\mathcal{I}$, as $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ is supposed to be a complete lattice. So, (20.20) is proved and the identity

$$\bigcup_{A \in \mathcal{A}_0} S_{\alpha(A,\mathcal{S})} = S_{\alpha(\bigcup \mathcal{A}_0,\mathcal{S})} \tag{20.23}$$

follows, Consequently, as Π is a complete $\mathcal{T}$-possibilistic measure on $\mathcal{R}$ and $\mathcal{S} \subset \mathcal{R}$ holds, we obtain that

$$F\left(\bigcup \mathcal{A}_0\right) = \Pi\left(S_{\alpha(\bigcup \mathcal{A}_0,\mathcal{S})}\right) = \Pi\left(\bigcup_{A \in \mathcal{A}_0} S_{\alpha(A,\mathcal{S})}\right) = \bigvee_{A \in \mathcal{A}_0} \Pi(S_{\alpha(A,\mathcal{S})}) = \bigvee_{A \in \mathcal{A}_0} F(A), \tag{20.24}$$

hence, F is a complete $\mathcal{T}$-possibilistic measure on $P(\Omega)$ and the assertion is proved. □

The condition of continuity, introduced in Definition 20.3 and imposed to the $\mathcal{I}$-classification system $\mathcal{S}$ investigated in Theorem 20.1, deserves a more detailed analysis, as it represents a strong condition, more or less equivalent to the condition that $\leq_{\mathcal{I}}$ defines a linear ordering on I. The first part of the condition of continuity is nothing else than (20.10), i.e., that of conservativity, so that Definition 20.3 could be re-phrased saying that $\mathcal{S}$ is continuous, if it is conservative and if the condition dual to that of conservativity, i.e., the relation

$$S_{\bigvee_{\mathcal{I}} J} = \bigcup \{S_\alpha : \alpha \in J\} \tag{20.25}$$

holds for each $\emptyset \neq J \subset I$. If I is finite, the mutual comparability of each α, β with respect to $\leq_{\mathcal{I}}$ can be proved to imply the continuity of $\mathcal{S}$, as the following assertion demonstrates.

Lemma 20.3 *Let $\mathcal{T} = \langle T, \leq_{\mathcal{I}} \rangle$ be a complete lattice, let $\langle \Omega, \mathcal{R}, \varphi \rangle$ be a $\mathcal{T}$-monotone space, let $J = \langle I, \leq_{\mathcal{I}} \rangle$ be a finite lattice such that $\alpha \leq_{\mathcal{I}} \beta$ or $\beta \leq_{\mathcal{I}} \alpha$ holds for each $\alpha, \beta \in I$, let $\mathcal{S}$ be an $\mathcal{I}$-classification system over $\langle \Omega, \mathcal{R}, \varphi \rangle$. Then the system $\mathcal{S}$ is continuous.*

Proof. As $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ is finite (hence, trivially complete) lattice in which either $\alpha \leq_{\mathcal{I}} \beta$ or $\beta \leq_{\mathcal{I}} \alpha$ holds for each $\alpha, \beta \in I$, we obtain easily that for each $\emptyset \neq J \subset I$ there exist $\alpha, \beta \in J$ such that $\bigwedge_{\mathcal{I}} J = \alpha$ and $\bigvee_I J = \beta$. So, for every $\gamma \in I$, the relation

$$S_{\bigvee J} = S_\beta \supset S_\gamma \supset S_\alpha = S_{\bigwedge_{\mathcal{I}} J} \tag{20.26}$$

is valid, so that the inclusion

$$S_{\bigwedge_{\mathcal{I}} J} = \bigcap \{S_\gamma : \gamma \in J\} \subset \bigcup \{S_\gamma : \gamma \in J\} = S_{\bigvee_{\mathcal{I}} J} \tag{20.27}$$

easily follows and $\mathcal{S}$ is continuous. The assertion is proved. □

When investigating classification systems and real-valued possibilistic measures defined over a probability space $\langle \Omega, \mathcal{A}, P \rangle$ [36], we focused our attention to classification systems induced by real-valued random variables, i.e., by measurable mappings which take the probability space $\langle \Omega, \mathcal{A}, P \rangle$ into the Borel line $\langle R, \mathcal{B} \rangle$, $R = (-\infty, \infty)$. In this case, the possibilistic measure on $\mathcal{P}(\Omega)$ defined by such random variable can be described and processed using the distribution function of the random variable in question. As there were rather the qualitative and comparative than the quantitative properties and relations of the probability distributions involved what mattered in our constructions, reasoning and calculations, a great portion of these ideas and constructions can be applied also to the case when real-valued probability measure P on a σ-field $\mathcal{A}$ of subsets of Ω is replaced by a lattice-valued possibilistic measure Π on an appropriate nonempty system $\mathcal{R}$ of subsets of Ω. A mathematical formalization of the informal description above may read as follows.

Definition 20.4 *Let Ω and L be nonempty sets, let $\mathcal{R} \subset \mathcal{P}(\Omega)$ and $\mathcal{L} \subset \mathcal{P}(L)$ be nonempty systems of subsets of these sets. A mapping X which takes Ω into L is called* $(\mathcal{R}, \mathcal{L})$-measurable, *if the inverse image of each set in $\mathcal{L}$ belongs to $\mathcal{R}$, in symbols, if the inclusion*

$$\{\{\omega \in \Omega : X(\omega) \in A\} : A \in \mathcal{L}\} \subset \mathcal{R} \tag{20.28}$$

holds. In particular, if $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ is a complete lower semilattice and if $\mathcal{L}_{\mathcal{I}} = \{\{\alpha \in I : \alpha \leq_{\mathcal{I}} \beta\} : \beta \in I\}$ is the system of all initial segments in I with respect to $\leq_{\mathcal{I}}$, then an $(\mathcal{R}, \mathcal{L})$-measurable mapping $X : \Omega \to I$ is called $\mathcal{I}$-measurable *supposing that the system $\mathcal{R} \subset \mathcal{P}(\Omega)$ is fixed in the context under consideration.*

Consequently, the inclusion

$$\{\{\omega \in \Omega : X(\omega) \leq_{\mathcal{I}} \beta\} : \beta \in I\} \subset \mathcal{R} \tag{20.29}$$

is valid for each $\mathcal{I}$-measurable mapping X on Ω.

Lemma 20.4 *Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be a complete lattice, let $\langle \Omega, \mathcal{R}, \varphi \rangle$ be a $\mathcal{T}$-monotone space such that $\{\emptyset, \Omega\} \subset \mathcal{R}$, let $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ be a complete lower semilattice, let $X : \Omega \to I$ be an $\mathcal{I}$-measurable mapping such that there exists $\omega_0 \in \Omega$ with the property that $\mathbf{0}_{\mathcal{I}} < X(\omega) \leq X(\omega_0)$ holds for each $\omega \in \Omega$. Then the system $\mathcal{S} = \{S_\alpha : \alpha \in I\}$, where $S_\alpha = \{\omega \in \Omega : X(\omega) \leq \alpha\}$, is a conservative $\mathcal{I}$-classification system over $\langle \Omega, \mathcal{R}, \varphi \rangle$.*

Proof. We can easily observe that

$$S_{\mathbf{0}_{\mathcal{I}}} = \{\omega \in \Omega : X(\omega) \leq \mathbf{0}_{\mathcal{I}}\} = \emptyset \tag{20.30}$$

and, for $\alpha_0 = X(\omega_0)$,

$$S_{\alpha_0} = \{\omega \in \Omega : X(\omega) \leq \alpha_0\} = \{\omega \in \Omega : X(\omega) \leq X(\omega_0)\} = \Omega. \tag{20.31}$$

If $\alpha, \beta \in I,\ \alpha \leq_{\mathcal{I}} \beta$ is the case, then the inclusion

$$S_\alpha = \{\omega \in \Omega : X(\omega) \leq \alpha\} \subset \{\omega \in \Omega : X(\omega) \leq_{\mathcal{I}} \beta\} = S_\beta \tag{20.32}$$

is obvious. Let J be a nonempty subset of I. For each $\omega \in \Omega$, the inequality $X(\omega) \leq_{\mathcal{I}} \alpha$ holds simultaneously for each $\alpha \in J$ if and only if the inequality $X(\omega) \leq_{\mathcal{I}} \bigwedge_{\mathcal{I}} J$ holds, hence,

$$\{\omega \in \Omega : X(\omega) \leq \bigwedge_{\mathcal{I}} J\} = \bigcap_{\alpha \in J} \{\omega \in \Omega : X(\omega) \leq \alpha\}. \tag{20.33}$$

However, this is nothing else than the equality

$$S_{\bigwedge_{\mathcal{I}} J} = \bigcap \{S_\alpha : \alpha \in J\}, \tag{20.34}$$

i.e., (20.10). So, due to Lemma 20.2, the classification system $\mathcal{S}$ is conservative and the assertion is proved. □

Lemma 20.5 *Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ and $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ be complete lattices, let $\langle \Omega, \mathcal{R}, \varphi \rangle$ be a $\mathcal{T}$-monotone space such that $\{\emptyset, \Omega\} \subset \mathcal{R}$, let $X : \Omega \to I$ be an $\mathcal{I}$-measurable mapping such that $X(\omega) > \mathbf{0}_{\mathcal{I}}$ holds for each $\omega \in \Omega$, let $\mathcal{S} = \{\{\omega \in \Omega : X(\omega) \leq \alpha\} : \alpha \in I\}$. Set, for each $A \subset \Omega$,*

$$\bigvee(A, X) = \bigvee\{X(\omega) : \omega \in A\}. \tag{20.35}$$

Then, for each $A \subset \Omega$,

$$\alpha(A, \mathcal{S}) = \bigvee(A, X), \tag{20.36}$$

so that

$$S_{\alpha(A,\mathcal{S})} = \left\{\omega \in \Omega : X(\omega) \leq \bigvee(A, X)\right\} \tag{20.37}$$

and

$$F(A) = \varphi\left(S_{\bigvee(A,X)}\right) \tag{20.38}$$

Proof. As $S_{\mathbf{0}_{\mathcal{I}}} = \emptyset$ and $S_{\alpha_0} = \Omega$ for $\alpha_0 = \bigvee(\Omega, X) \in I$, the system $\mathcal{S} = \{S_\alpha : \alpha \in I\}$ with $S_\alpha = \{\omega \in \Omega : X(\omega) \leq \alpha\}$ is obviously an $\mathcal{I}$-classification system over $\langle \Omega, \mathcal{R}, \varphi \rangle$, as $\mathcal{I}$ is complete lattice and the inverse image of each set $\{\beta : \beta \leq \alpha\} \subset I$ with respect to X is in $\mathcal{R}$. Due to Lemma 20.4 the classification system $\mathcal{S}$ is conservative.

The inclusion

$$A \subset S_{\bigvee(A,X)} = \{\omega \in \Omega : X(\omega) \leq \bigvee(A, X)\} \tag{20.39}$$

and the inequality

$$\alpha(A, \mathcal{S}) \leq \bigvee(A, X) \tag{20.40}$$

are obviously valid for every $A \subset \Omega$. Suppose, in order to arrive at a contradiction, that the strict inequality holds in (20.40). As

$$S_{\alpha(A,\mathcal{S})} = \bigcap\{S_\beta : A \subset S_\beta\} \supset A \tag{20.41}$$

holds due to the fact that $\mathcal{S}$ is conservative, we obtain that $A \subset S_{\alpha(A,\mathcal{S})}$, hence, for each $\omega \in A$ the inequality $X(\omega) \leq \alpha(A, \mathcal{S}) < \bigvee(A, X)$ holds, but this contradicts the definition of the supremum $\bigvee(A, X)$. So, the equality holds in (20.40) and (20.36) is proved. (20.37) and (20.38) immediately follow from the definition of the mapping $F : \mathcal{P}(\Omega) \to T$. The assertion is proved. □

21 Combinations of Classification Systems and Higher-Order Lattice-Valued Monotone and Possibilistic Measures

Till now, we have considered and analyzed the idea to approximate effectively inaccessible (in the sense of pointwise decidability) sets of elementary random events by their coverings defined by sets of possibly favorable elementary random events, these sets being defined by certain $\mathcal{I}$-classification systems. Like as in the case of classification systems defined by real-valued probability measures and, in particular, by distribution functions of real-valued random variables [35], [36], an immediately arising idea reads to repeat this approximation step at a higher level, so arriving at sets of elementary random events which are possibly the members of the sets of elementary random events possibly favorable to the random event in question. In what follows, we will introduce and analyze this approach in more detail, restricting ourselves, for the sake of simplicity, to $\mathcal{I}$-classification systems induced by $\mathcal{I}$-measurable mappings.

Lemma 21.1 *Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$, $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ and $\langle \Omega, \mathcal{R}, \varphi \rangle$ be as in Lemma 20.5, let $X, Y : \Omega \to I$ be $\mathcal{I}$-measurable mappings such that, for both $Z = X, Y$ and for each $\omega \in \Omega$, $Z(\omega) >_{\mathcal{I}} \mathbf{0}_{\mathcal{I}}$ holds, let*

$$\mathcal{S}_Z = \{S_{Z,\alpha} : \alpha \in I\} = \{\{\omega \in \Omega : Z(\omega) \leq_{\mathcal{I}} \alpha\} : \alpha \in I\} \tag{21.1}$$

for both $Z = X, Y$. For each $A \subset \Omega$ and for both $Z = X, Y$ set $A^Z = S_{\alpha(A, \mathcal{S}_Z)}$. Set

$$F_{XY}(A) = \varphi((A^X)^Y) = \varphi(S_{\alpha(A^X, \mathcal{S}_Y)}). \tag{21.2}$$

Then $F_{XY} : \mathcal{P}(\Omega) \to T$ is a $\mathcal{T}$-monotone measure on $\mathcal{P}(\Omega)$.

Proof. The conditions of Lemma 20.5 are satisfied for both $Z = X, Y$, hence, for each $A \subset \Omega$,

$$\alpha(A, \mathcal{S}_X) = \bigvee(A, X), \tag{21.3}$$

and

$$A^X = S_{\alpha(A, \mathcal{S}_X)} = S_{\bigvee(A,X)} = \left\{\omega \in \Omega : X(\omega) \leq_{\mathcal{I}} \bigvee(A, X)\right\}. \tag{21.4}$$

So, $\emptyset^X = S_{\bigvee(\emptyset, X)} = S_{\mathbf{0}_{\mathcal{I}}} = \emptyset$, and $\Omega^X = S_{\bigvee(\Omega, X)} = \{\omega \in \Omega : X(\omega) \leq_{\mathcal{I}} \bigvee(\Omega, X)\} = \Omega$, the same being valid when replacing X by Y. Given $A \subset \Omega$ we obtain that

$$(A^X)^Y = (S_{\alpha(A, \mathcal{S}_X)})^Y = \left(\left\{\omega \in \Omega : X(\omega) \leq_{\mathcal{I}} \bigvee(A, X)\right\}\right)^Y = \left\{\omega \in \Omega : Y(\omega) \leq_{\mathcal{I}} \bigvee(A^X, Y)\right\} \tag{21.5}$$

so that

$$F_{XY}(A) = \varphi((A^X)^Y). \tag{21.6}$$

Obviously, $(\emptyset^X)^Y = \emptyset^Y = \emptyset$, $(\Omega^X)^Y = \Omega^Y = \Omega$, so that $F_{XY}(\emptyset) = \mathbf{0}_{\mathcal{T}}$ and $F_{XY}(\Omega) = \mathbf{1}_{\mathcal{T}}$. If $A \subset B \subset \Omega$, then $\bigvee(A, X) \leq_{\mathcal{I}} \bigvee(B, X)$ obviously holds, hence, $A^X \subset B^X$, the inequality $\bigvee(A^X, Y) \leq_{\mathcal{I}} \bigvee(B^X, Y)$, the inclusion $(A^X)^Y \subset (B^X)^Y$ and, finally, the inequality

$$F_{XY}(A) = \varphi((A^X)^Y) \leq_{\mathcal{T}} \varphi((B^X)^Y) = F_{XY}(B) \tag{21.7}$$

result. The assertion is proved. □

Lemma 21.2 *Let the notations and conditions of Lemma 25.1 hold. Then, for each $A \subset \Omega$, the relations*

$$F_{XX}(A) = F_X(A),\ F_{XY}(A) \geq F_X(A) \bigvee_{\mathcal{T}} F_Y(A) \tag{21.8}$$

are valid.

Proof. For each $A \subset \Omega$,

$$\bigvee(A^X, X) = \bigvee\{X(\omega) : \omega \in A^X\} = \bigvee\{X(\omega) : \omega \in A\} = \bigvee(A, X), \tag{21.9}$$

so that

$$\begin{aligned} F_{XX}(A) &= \varphi((A^X)^X) = \varphi\left(\left\{\omega \in \Omega : X(\omega) \leq_{\mathcal{I}} \bigvee(A^X, X)\right\}\right) = \\ &= \varphi\left(\left\{\omega \in \Omega : X(\omega) \leq_{\mathcal{I}} \bigvee(A, X\right\}\right) = \varphi(A^X) = F_X(A). \end{aligned} \tag{21.10}$$

For each $A \subset \Omega$ and both $Z = X, Y$, the inclusions $A \subset A^Z$, hence, also $A^X \subset (A^X)^Y$ and $A^Y \subset (A^Y)^X$ are valid, so that the inequality

$$F_{XY}(A) = \varphi((A^X)^Y) \geq_{\mathcal{T}} \varphi(A^X) = F_X(A) \tag{21.11}$$

follows. The inclusion $A \subset A^X$ also yields that the inequality $\bigvee(A^X, Y) \geq_{\mathcal{I}} \bigvee(A, Y)$ holds, so that the inclusion

$$(A^X)^Y = \left\{\omega \in \Omega : Y(\omega) \leq_{\mathcal{I}} \bigvee(A^X, Y)\right\} \supset \left\{\omega \in \Omega : Y(\omega) \leq_{\mathcal{I}} \bigvee(A, Y)\right\} = \mathcal{A}^Y \tag{21.12}$$

and the inequality

$$F_{XY}(A) = \varphi((A^X)^Y) \geq_{\mathcal{T}} \varphi(A^Y) = F_Y(A) \tag{21.13}$$

easily follow. The assertion is proved. □

As a matter of fact, the mapping F_{XY} is not, in general, commutative in X and Y, i.e., $F_{XY}(A)$ and $F_{YX}(A)$ may differ for some $A \subset \Omega$. Let us consider the following example.

Let $\Omega = [0,1] \times [0,1]$ so that, for each $\omega \in \Omega$, $\omega = \langle \omega_1, \omega_2 \rangle$, $\omega_1, \omega_2 \in [0,1]$. Let $\mathcal{T} = \langle \mathcal{P}(\Omega), \subset \rangle$ and let φ_{id} be the identity mapping on $\mathcal{P}(\Omega)$, so that $\langle \Omega, \mathcal{P}(\Omega), \varphi_{id} \rangle$ is a simple $\mathcal{T}$-monotone (and, as a matter of fact, $\mathcal{T}$-possibilistic) space where the size of each subset of Ω is simply this set itself. Let $X, Y : \Omega \rightarrow [0,1]$ be mappings defined by $X(\omega) = (\frac{1}{2})(\omega_1 + \omega_2)$ and $Y(\omega) = \omega_2$ for each $\omega \in \Omega$. For each $\alpha \in R = (-\infty, \infty)$, set

$$S_{X,\alpha} = \{\omega \in \Omega : X(\omega) \leq \alpha\}, \; S_{Y,\alpha} = \{\omega \in \Omega : Y(\omega) \leq \alpha\}, \tag{21.14}$$

so that $S_{X,\alpha} = S_{Y,\alpha} = \emptyset$ for $\alpha < 0$, $S_{X,\alpha} = S_{Y,\alpha} = \Omega$ for $\alpha \geq 1$, and both the systems $\mathcal{S}_X = \{S_{X,\alpha} : \alpha \in R\}$ and $\mathcal{S}_Y = \{S_{Y,\alpha} : \alpha \in R\}$ are obviously nested with respect to the standard linear ordering on R. Hence, both $\mathcal{S}_X$ and $\mathcal{S}_Y$ are $\mathcal{T}$-classification systems over $\langle \Omega, \mathcal{P}(\Omega), \varphi_{id} \rangle$.

Take $A = S_{X,\frac{1}{4}} = \{\omega \in \Omega : (\frac{1}{2})(\omega_1 + \omega_2) \leq \frac{1}{4}\}$, in other terms. A is the left-bottom corner triangle in Ω defined by the corner points $\langle 0,0 \rangle$, $\langle \frac{1}{2}, 0 \rangle$ and $\langle 0, \frac{1}{2} \rangle$. Now,

$$A^X = \left\{\omega \in \Omega : X(\omega) \leq \bigvee(A, X)\right\} = \{\omega \in \Omega : X(\omega) \leq \frac{1}{4}\} = A, \tag{21.15}$$

so that

$$\begin{aligned} (A^X)^Y = A^Y &= \left\{\omega \in \Omega : Y(\omega) \leq \bigvee(A, Y)\right\} = \\ &= \left\{\omega \in \Omega : \omega_2 \leq \bigvee\{\omega_2 : \omega_2 \in A\}\right\} = \{\omega \in \Omega : \omega_2 \leq \frac{1}{2}\} = \\ &= S_{Y,\frac{1}{2}} = [0,1] \times [0, \frac{1}{2}], \end{aligned} \tag{21.16}$$

i.e., $(A^X)^Y$ defines the bottom half of the square $[0,1] \times [0,1]$. However,

$$\begin{aligned} (A^Y)^X &= \left\{\omega \in \Omega : X(\omega) \leq \bigvee\{X(\omega) : \omega \in A^Y\}\right\} = \\ &= \left\{\omega \in \Omega : (\frac{1}{2})(\omega_1 + \omega_2) \leq \bigvee\{\frac{1}{2}(\omega_1 + \omega_2) : \omega_2 \leq \frac{1}{2}\}\right\} = \\ &= \{\omega_2 : \frac{1}{2}(\omega_1 + \omega_2) \leq \frac{3}{4}\} = S_{X,(\frac{3}{4})} \neq (A^X)^Y. \end{aligned} \tag{21.17}$$

As can be easily seen, $S_{X,\frac{3}{4}}$ is just the square $[0,1] \times [0,1]$ without the right-upper corner triangle defined by the corner points $\langle \frac{1}{2}, 1 \rangle, \langle 1, \frac{1}{2} \rangle$, and $\langle 1, 1 \rangle$. As φ_{id} is the identity mapping on $\mathcal{P}(\Omega)$, the inequality

$$\begin{aligned} F_{XY}(A) &= \varphi_{id}((A^X)^Y) = (A^X)^Y = [0,1] \times [0, \frac{1}{2}] \neq S_{X,(\frac{3}{4})} = \\ &= \varphi_{id}(S_{X,(\frac{3}{4})}) = \varphi_{id}((A^Y)^X) = F_{YX}(A) \end{aligned} \tag{21.18}$$

is obvious.

It is perhaps worth introducing explicitly, that the real-valued nature of the mappings X and Y is not substantial in our context and serves just to an easy to understood and imagine description of certain subsets of the unit square $[0,1]\times[0,1]$. Indeed, our example can be re-phrased in non-numerical terms just supposing that all the sets entering our considerations and constructions are correctly taken and identified.

Hence, let $\Omega = [0,1]\times[0,1]$ as above, let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle = \mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle = \langle \mathcal{P}(\Omega), \subset \rangle$, let φ_{id} and $\langle \Omega, \mathcal{P}(\Omega), \varphi_{id}\rangle$ be as above. Let the mappings $X, Y : \Omega \to I(= \mathcal{P}(\Omega))$ be defined as follows: $X(\omega) = S_{X,(\frac{1}{4})}$, if $\omega \in S_{X,(\frac{1}{4})}$, $X(\omega) = S_{X,(\frac{3}{4})}$ if $\omega \in S_{X,(\frac{3}{4})} - S_{X,(\frac{1}{4})}$, and $X(\omega) = \Omega$, if $\omega \in \Omega - S_{X,(\frac{3}{4})}$. For Y, $Y(\omega) = S_{Y,(\frac{1}{2})}(= [0,1]\times[0,\frac{1}{2}])$, if $\omega \in S_{Y,(\frac{1}{2})}$, and $Y(\omega) = \Omega$, if $\omega \in \Omega - S_{Y,(\frac{1}{2})}(= [0,1]\times[(\frac{1}{2},1])$. Taken $A = S_{X,(\frac{1}{4})}$ as above, we obtain again, that $A^X = A$ and $(A^X)^Y = A^Y = S_{Y,(\frac{1}{2})} = [0,1]\times[0,\frac{1}{2}]$, but $(A^Y)^X = ([0,1]\times[0,\frac{1}{2}])^X = S_{X,(\frac{3}{4})}$ as above. As φ_{id} is the identity mapping on $\mathcal{P}(\Omega)$, the inequality (21.18) also results. Let us note that, when taking $A = S_{Y,(\frac{1}{2})} = [0,1]\times[0,\frac{1}{2}]$, we obtain that $(A^Y)^X = A^X = S_{X,(\frac{3}{4})}$, but $(A^X)^Y = (S_{X,(\frac{3}{4})})^Y = \Omega$, so that the inequality $(A^X)^Y \neq (A^Y)^X$ also follows.

Given an $\mathcal{I}$-classification system $\mathcal{S} = \{S_\alpha : \alpha \in I\}$ and a subset A of the universe Ω of elementary random events, if the inclusion $A \subset S_\alpha$ holds for some $\alpha \in I$, we can state that for an elementary random event $\omega_0 \in \Omega$ the condition that ω_0 is in S_α is necessary, but in general not sufficient, to be allowed to claim that the elementary random event ω_0 is favorable with respect to A. Supposing that the inclusion $A \subset S_\alpha$ is the only characterization of the set A being at our disposal, it is easy to understand that our aim will be to have the covering S_α of A as narrow or tight (close to A) as possible. Consequently, having at hand two $\mathcal{I}$-classification systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_i = \{S_{i,\alpha} : \alpha \in I\}$ for both $i = 1,2$ and obtaining that $A \subset S_{1,\alpha}$ and $A \subset S_{2,\beta}$ holds when applying these classification systems separately, our intuitive conclusion would be to take the inclusion $A \subset S_{1,\alpha} \cap S_{2,\beta}$ as the best necessary condition (i.e., the narrowest covering of A) obtainable in the situation under consideration. In general, however, the system $\{S_{1,\alpha} \cap S_{2,\beta} : \alpha, \beta \in I\}$ of subsets of Ω is not nested in the sense that for each $\alpha_1, \alpha_2, \beta_1, \beta_2 \in I$ either $S_{1,\alpha_1} \cap S_{2,\beta_1} \subset S_{l,\alpha_2} \cap S_{2,\beta_2}$ or the inverse inclusion is valid. Hence, this system does not define an $\mathcal{I}$-classification system with respect to a *linear* ordering $\leq_{\mathcal{I}}$ on I. Happy enough, under the weakened condition introduced and analyzed above, according to which $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ defines just a complete lower semilattice, the combination of S_α and S_β into $S_\alpha \cap S_\beta$ can be done within the framework of $\mathcal{I}$-classification systems when replacing the structure $\mathcal{I} = \langle I, \leq_{\mathcal{I}} \rangle$ by its Cartesian product (Cartesian square, in particular) defined on the set $I \times I$ of pairs $\langle \alpha, \beta \rangle, \alpha, \beta \in I$.

Let $\mathcal{I}\langle I, \leq_{\mathcal{I}} \rangle$ be a complete lower semilattice and let us introduce the binary relation $\leq_{\mathcal{I}\times\mathcal{I}}$ on $I \times I$ as the pointwise combination of $\leq_{\mathcal{I}}$ on I. So, for every $\langle \alpha_1, \beta_1 \rangle$, $\langle \alpha_2, \beta_2 \rangle \in I \times I$, $\langle \alpha_1, \beta_1 \rangle \leq_{\mathcal{I}\times\mathcal{I}} \langle \alpha_2, \beta_2 \rangle$ hold if and only if $\alpha_1 \leq_{\mathcal{I}} \alpha_2$ and $\beta_1 \leq_{\mathcal{I}} \beta_2$ hold together. As can be easily checked, $\leq_{\mathcal{I}\times\mathcal{I}}$ defines a partial ordering on $I \times I$. Also the infimum operation on $I \times I$ induced by $\leq_{\mathcal{I}\times\mathcal{I}}$ can be given as the pointwise infima taken by both the dimensions in $I \times I$. Indeed, as can be also easily checked, for each $\langle \alpha_1, \beta_1 \rangle$, $\langle \alpha_2, \beta_2 \rangle \in I \times I$ and for each $\emptyset \neq A_0 \subset I \times I$ we obtain that

$$\langle \alpha_1, \beta_1 \rangle \bigwedge_{\mathcal{I}\times\mathcal{I}} \langle \alpha_2, \beta_2 \rangle = \left\langle \alpha_1 \bigwedge_{\mathcal{I}} \alpha_2,\ \beta_1 \bigwedge_{\mathcal{I}} \beta_2 \right\rangle, \tag{21.19}$$

and

$$\begin{aligned} \bigwedge_{\mathcal{I}\times\mathcal{I}} A_0 &= \bigwedge_{\mathcal{I}\times\mathcal{I}} \{\langle \alpha, \beta \rangle : \langle \alpha, \beta \rangle \in A_0\} = \\ &= \left\langle \bigwedge_{\mathcal{I}} \{\alpha : \langle \alpha, \beta \rangle \in A_0\},\ \bigwedge_{\mathcal{I}} \{\beta : \langle \alpha, \beta \rangle \in A_0\} \right\rangle \end{aligned} \tag{21.20}$$

so that $\mathcal{I} \times \mathcal{I} = \langle I \times I, \leq_{\mathcal{I}\times\mathcal{I}} \rangle$ is a complete lower semilattice.

Lemma 21.3 *Let Ω be a nonempty universe, let $\mathcal{I}\langle I, \leq_{\mathcal{I}} \rangle$ be a complete lower semilattice, let $\mathcal{S}_i = \{S_{i,\alpha} : \alpha \in I\} \subset \mathcal{P}(\Omega)$, $i = j, 2$, be two $\mathcal{I}$-classification systems over Ω. Then the system $\mathcal{S}_{\langle 1,2 \rangle} =$*

$\{S_{\langle 1,2\rangle,\langle\alpha,\beta\rangle} : \langle\alpha,\beta\rangle \in I \times I\}$, *where* $S_{\langle 1,2\rangle,\langle\alpha,\beta\rangle} = S_{1,\alpha} \cap S_{2,\beta}$ *for each* $\langle\alpha,\beta\rangle \in I \times I$ *defines an* $\mathcal{I} \times \mathcal{I}$*-classification system over* Ω.

Proof. Obviously

$$\mathbf{0}_{\mathcal{I}\times\mathcal{I}} = \bigwedge_{\mathcal{I}\times\mathcal{I}} \{\langle\alpha,\beta\rangle : \alpha,\beta \in I\} = \left\langle \bigwedge_{\alpha\in I} \alpha,\ \bigwedge_{\beta\in I} \beta \right\rangle = \langle \mathbf{0}_{\mathcal{I}}, \mathbf{0}_{\mathcal{I}} \rangle \tag{21.21}$$

and

$$S_{\langle 1,2\rangle,\langle \mathbf{0}_{\mathcal{I}},\mathbf{0}_{\mathcal{I}}\rangle} = S_{1,\mathbf{0}_{\mathcal{I}}} \cap S_{2,\mathbf{0}_{\mathcal{I}}} = \emptyset \cap \emptyset = \emptyset. \tag{21.22}$$

According to the definition of $\mathcal{I}$-classification systems there exist $\alpha_0, \beta_0 \in I$, such that $S_{1,\alpha_0} = S_{2,\beta_0} = \Omega$, so that $S_{\langle 1,2\rangle\langle\alpha_0,\beta_0\rangle} = S_{1,\alpha_0} \cap S_{2,\beta_0} = \Omega$. Finally, if $\langle\alpha_1,\beta_1\rangle \leq_{\mathcal{I}\times\mathcal{I}} \langle\alpha_2,\beta_2\rangle$ holds, then by definition $\alpha_1 \leq_{\mathcal{I}} \alpha_2$ and $\beta_1 \leq_{\mathcal{I}} \beta_2$ hold as well, so that $S_{1,\alpha_1} \subset S_{1,\alpha_2}$ and $S_{2,\beta_1} \subset S_{2,\beta_2}$ follows. So, we obtain the relation

$$S_{\langle 1,2\rangle\langle\alpha_1,\beta_1\rangle} = S_{1,\alpha_1} \cap S_{2,\beta_1} \subset S_{1,\alpha_2} \cap S_{2,\beta_2} = S_{\langle 1,2\rangle\langle\alpha_2,\beta_2\rangle} \tag{21.23}$$

and the assertion is proved. □

As can be easily seen, for the sets $S_{1,\alpha_1}, S_{1,\alpha_2}, S_{2,\beta_1}$ and S_{2,β_2}, occuring in (21.23) also the inclusion

$$S_{1,\alpha_1} \cup S_{2,\beta_1} \subset S_{1,\alpha_2} \cup S_{2,\beta_2} \tag{21.24}$$

is valid. Hence, the system $\mathcal{S}^{\star}_{\langle 1,2\rangle} = \{S^{\star}_{\langle 1,2\rangle,\langle\alpha,\beta\rangle} : \alpha,\beta \in I\}$, where $S^{\star}_{\langle 1,2\rangle,\langle\alpha,\beta\rangle} = S_{1,\alpha} \cup S_{2,\beta}$, defines also an $\mathcal{I} \times \mathcal{I}$-classification system over Ω. The intuition behind can be dual to that taken as inspiration when combining $\mathcal{I}$-classification systems $\mathcal{S}_1$ and $\mathcal{S}_2$ into $\mathcal{S}_{\langle 1,2\rangle}$ as defined above. Indeed, when considering the case in which the only specification being at our disposal and concerning a subset $A \subset \Omega$ reads that *either* $A \subset S_\alpha$ *or* $A \subset S_\beta$ holds, e.g., either $X(\omega) \leq \alpha$ or $Y(\omega) \leq \beta$ holds for each $\omega \in A$, given $\mathcal{I}$-measurable mappings $X, Y : \Omega \to I$, we arrive just at the $\mathcal{I} \times \mathcal{I}$-classification system $\mathcal{S}^{\star}_{\langle 1,2\rangle}$. Hence, also this kind of combination of particular pieces of knowledge concerning the subset A of Ω can be described and processed within the framework of $\mathcal{I} \times \mathcal{I}$-classification systems.

A common feature of both the (mutually dual) constructions leading from classification systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are taken to their combinations $\mathcal{S}_{\langle 1,2\rangle}$ or $\mathcal{S}^{\star}_{\langle 1,2\rangle}$ consists in the fact that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are taken as equivalently important and with the same degree of influence imposed on the resulting $\mathcal{I} \times \mathcal{I}$-classification systems over Ω. An alternative approach may be inspired by the well-known idea of the so called *lexicographical ordering*, then the dominant role in ordering pairs of elements is given to the first members of each pair, the second one entering the scene only when these first members are identical. A rather general mathematical formalization of this idea may read as follows.

For both $i = 1, 2$, consider a nonempty set I_i and a binary relation $\leq_i$ on I_i, and define the following binary relation $\leq_{1,2,L}$ on the Cartesian product $I_1 \times I_2$. For each $\langle\alpha_1,\beta_1\rangle,\ \langle\alpha_2,\beta_2\rangle \in I_1 \times I_2$

$$\langle\alpha_1,\beta_1\rangle \leq_{1,2,L} \langle\alpha_2,\beta_2\rangle \Leftrightarrow_{df} (\alpha_1 <_1 \alpha_2) \text{ or } ((\alpha_1 = \alpha_2) \text{ and } (\beta_1 \leq_2 \beta_2)), \tag{21.25}$$

here $\alpha_1 <_1 \alpha_2$ means that $\alpha_1 \leq_1 \alpha_2$ and $\alpha_1 \neq \alpha_2$.

Lemma 21.4 *If both* $\mathcal{I}_i = \langle I_i, \leq_i \rangle$, $i = 1, 2$, *are partially ordered sets, then* $\leq_{1,2,L}$ *defines a partial ordering on* $I_1 \times I_2$. *If* $\leq_i$ *is a linear ordering on* I_i *for both* $i = 1, 2$, *then* $\leq_{1,2,L}$ *defines a linear ordering on* $I_1 \times I_2$.

Proof. Taking both $I_i = \langle I_i, \leq_i \rangle,\ i = 1, 2$ as fixed, we write simply $\leq_L$ instead of $\leq_{1,2,L}$. For each $\alpha, \beta \in I_1 \times I_2$ the relation $\langle\alpha,\beta\rangle \leq_L \langle\alpha,\beta\rangle$ is evident and if the inequalities $\langle\alpha_1,\beta_1\rangle \leq_L \langle\alpha_2,\beta_2\rangle$ and $\langle\alpha_2,\beta_2\rangle \leq_L \langle\alpha_1,\beta_1\rangle$ are simultaneously valid, then the identities $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, hence, also $\langle\alpha_1,\beta_1\rangle = \langle\alpha_2,\beta_2\rangle$ easily follow. Let $\langle\alpha_1,\beta_1\rangle \leq_L \langle\alpha_2,\beta_2\rangle$ and $\langle\alpha_2,\beta_2\rangle \leq_L \langle\alpha_3,\beta_3\rangle$ hold simultaneously, so that the inequalities $\alpha_2 \leq_1 \alpha_2$ and $\alpha_2 \leq_1 \alpha_3$ follow. If $\alpha_1 <_1 \alpha_2$ or $\alpha_2 <_1 \alpha_3$ holds, then the inequalities $\alpha_1 <_1 \alpha_3$ and, consequently, also $\langle\alpha_1,\beta_1\rangle \leq_L \langle\alpha_3,\beta_3\rangle$ are evidently valid. If $\alpha_1 = \alpha_2 = \alpha_3$ is the case, then $\beta_1 \leq_2 \beta_2 \leq_2 \beta_3$ must hold, so that $\langle\alpha_1,\beta_1\rangle \leq_L \langle\alpha_3,\beta_3\rangle$ follows as well. Hence,

$\leq_L$ defines a partial ordering on $I_1 \times I_2$. If both $\leq_i$ are linear orderings on $I_i,\ i = 1, 2$, then either $\alpha_1 <_1 \alpha_2$, or $\alpha_1 = \alpha_2$, or $\alpha_2 \leq_1 \alpha_1$, as well as $\beta_1 <_2 \beta_2$, or $\beta_1 = \beta_2$, or $\beta_2 \leq_2 \beta_1$ is the case, so that either $\langle \alpha_1, \beta_1 \rangle \leq_L \langle \alpha_2, \beta_2 \rangle$ or $\langle \alpha_2, \beta_2 \rangle \leq_L \langle \alpha_1, \beta_1 \rangle$ (or both) is valid. The assertion is proved. □

Lemma 21.5 *If both $\mathcal{I}_i = \langle I_i, \leq_i \rangle$, $i = 1, 2$, are complete lower semilattices, then their lexicographical Cartesian product $(\mathcal{I}_1 \times \mathcal{I}_2)_L = \langle I_1 \times I_2, \leq_{1,2,L} \rangle$ is a complete lower semilattice supposing that $\bigvee_{\mathcal{I}_2} I_2$ is defined.*

Proof. Let A be a nonempty subset of $I_1 \times I_2$, denote by A^1, A^2 the corresponding projections of A on I_1 and I_2, i.e.,

$$A^1 = \{\alpha_1 \in I_1 : \langle \alpha_1, \beta_1 \rangle \in A \text{ for some } \beta_1 \in I_2\}, \tag{21.26}$$

$$A^2 = \{\beta_2 \in I_2 : \langle \alpha_2, \beta_2 \rangle \in A \text{ for some } \alpha_2 \in I_1\}, \tag{21.27}$$

As $\mathcal{I}_1 = \langle I_1, \leq_1 \rangle$ is a complete lower semilattice, the infimum $\bigwedge_{\mathcal{I}_1} A^1$ is defined. Let

$$A_0^2 = \{\beta \in A^2 : \left\langle \bigwedge_{\mathcal{I}_1} A^1, \beta \right\rangle \in A\}, \tag{21.28}$$

hence $A_0^2 = \emptyset$, if there is no pair $\langle \bigwedge_{\mathcal{I}_1} A^1, \beta \rangle$ in A, what is the case when $\bigwedge_{\mathcal{I}_1} A^1 \leq_1 \alpha$ holds for every $\alpha \in A_1$. Obviously, the value $\bigwedge_{\mathcal{I}_2} A_0^2$ is defined in $\mathcal{I}_2 = \langle I_2, \leq_2 \rangle$, applying the convention $\bigwedge_{\mathcal{I}_2} A_0^2 = \bigvee_{\mathcal{I}_2} I_2 (= \mathbf{1}_{\mathcal{I}_2}$, the unit element of $\mathcal{I}_2)$, if $A_0^2 = \emptyset$. Let us prove that the pair $\langle \bigwedge_{\mathcal{I}} A^1, \bigwedge_{\mathcal{I}_2} A_0^2 \rangle \in I_1 \times I_2$ defines the infimum of the subset A of $I_1 \times I_2$ with respect to the lexicographical ordering $\leq_{1,2,L}$ on $I_1 \times I_2$, i.e., in symbols, that

$$\bigwedge_{(\mathcal{I}_1 \times \mathcal{I}_2)_L} A = \left\langle \bigwedge_{\mathcal{I}_1} A^1,\ \bigwedge_{\mathcal{I}_2} A_0^2 \right\rangle. \tag{21.29}$$

Let us omit the index $\mathcal{I}_1 \times \mathcal{I}_2$ in $\bigwedge_{(\mathcal{I}_1 \times \mathcal{I}_2)_L}$, writing simply $\bigwedge_L$, if no misunderstanding menaces. Let $\langle \alpha, \beta \rangle \in A$. If $\bigwedge_{\mathcal{I}_1} A^1 <_1 \alpha$ holds, then the inequality $\langle \bigwedge_{\mathcal{I}_1} A^1,\ \bigwedge_{\mathcal{I}_2} A_0^2 \rangle \leq_{1,2,L} \langle \alpha, \beta \rangle$ is obvious, if $\bigwedge_{\mathcal{I}_1} A^1 = \alpha$ is the case, the same inequality follows from the inequality $\bigwedge_{\mathcal{I}_2} A_0^2 \leq \beta$, as $\beta \in A_0^2$ holds due to the definition of A_0^2. If $\langle \alpha_0, \beta_0 \rangle \in I_1 \times I_2$ is such that $\langle \alpha_0, \beta_0 \rangle \leq_{1,2,L} \langle \alpha, \beta \rangle$ holds for each $\langle \alpha, \beta \rangle \in A$, then $\alpha_0 \leq \alpha$ for each $\alpha \in A^1$ and, consequently, $\alpha_0 \leq \bigwedge_{\mathcal{I}_1} A^1$ follows. If $\alpha_0 = \bigwedge_{\mathcal{I}_1} A^1$, then $\beta_0 \leq_2 \beta$ holds for each β such that $\langle \bigwedge_{\mathcal{I}_1} A^1, \beta \rangle$ is in A, hence, for each $\beta \in A_0^2$. Consequently, $\beta_0 \leq_2 \bigwedge_{\mathcal{I}_2} A_0^2$ holds as well, so that the relation $\langle \alpha_0, \beta_0 \rangle \leq_{1,2,L} \langle \bigwedge_{\mathcal{I}_1} A^1,\ \bigwedge_{\mathcal{I}_2} A_0^2 \rangle$ is valid. To conclude, $\langle \bigwedge_{\mathcal{I}_1} A^1,\ \bigwedge_{\mathcal{I}_2} A_0^2 \rangle$ indeed defines the infimum of A with respect to the lexicographical ordering $\leq_{1,2,L}$ on $\mathcal{I}_1 \times I_2$, so that the structure $(\mathcal{I}_1 \times \mathcal{I}_2)_L = \langle I_1 \times I_2, \leq_{1,2,L} \rangle$ is a complete lower semilattice. The assertion is proved. □

Theorem 21.1 *Let Ω be a nonempty set, let $\mathcal{I}_i = \langle I_i, \leq_i \rangle$, $i = 1, 2$, be complete lower semilattices with $\mathbf{1}_{\mathcal{I}_2} = \bigvee_{\mathcal{I}_2} I_2$ defined, let $(\mathcal{I}_1 \times \mathcal{I}_2)_L = \langle I_1 \times I_2, \leq_{1,2,L} \rangle$ be their Cartesian product with respect to the lexicographical ordering $\leq_{1,2,L}$. Let $X : \Omega \to I_1,\ Y : \Omega \to I_2$ be two mappings such that there exist $\alpha_0 \in I_1,\ \beta_0 \in I_2$, with this property: for each $\omega \in \Omega$ the relations*

$$\mathbf{0}_{\mathcal{I}_1} (= \bigwedge_{\mathcal{I}_1} I_1) <_1 X(\omega) \leq \alpha_0 \tag{21.30}$$

and

$$\mathbf{0}_{\mathcal{I}_2} (= \bigwedge_{\mathcal{I}_2} I_2) <_2 Y(\omega) \leq \beta_0 \tag{21.31}$$

are valid. Set, for each $\langle \alpha, \beta \rangle \in I_1 \times I_2$,

$$S^L_{\langle X,Y \rangle, \langle \alpha, \beta \rangle} = \{\omega \in \Omega : \langle X(\omega), Y(\omega) \rangle \leq_{1,2,L} \langle \alpha, \beta \rangle\}. \tag{21.32}$$

Then the system

$$\mathcal{S}^L_{\langle X,Y\rangle} = \{S^L_{\langle X,Y\rangle,\langle\alpha,\beta\rangle} : \langle\alpha,\beta\rangle \in I_1 \times I_2\} \tag{21.33}$$

of subsets of Ω defines an $(\mathcal{I}_2 \times \mathcal{I}_2)_L$-classification system of Ω with $\mathcal{P}(\Omega)$ playing the role of $\mathcal{A}$ from the definition of such systems, so that the inclusion $\mathcal{S}^L_{\langle X,Y\rangle} \subset \mathcal{A}$ is trivial.

Proof. Due to Lemma 21.5, $(\mathcal{I}_1 \times \mathcal{I}_2)_L$ is a complete lower semilattice and, as can be easily checked, the pair $\langle \mathbf{0}_{\mathcal{I}_1}, \mathbf{0}_{\mathcal{I}_2}\rangle$ defines its zero element $\mathbf{0}_{(\mathcal{I}_1\times\mathcal{I}_2)_L} = \bigwedge_{(\mathcal{I}_1\times\mathcal{I}_2),L}\{\langle\alpha,\beta\rangle : \langle\alpha,\beta\rangle \in I_1 \times I_2\}$. Omitting, in what follows, the index $\langle X, X\rangle$ in $S^L_{\langle X,Y\rangle,\langle\alpha,\beta\rangle}$ we obtain that the constraints

$$\mathcal{S}^L_{\langle \mathbf{0}_{\mathcal{I}_1},\mathbf{0}_{\mathcal{I}_2}\rangle} = \{\omega \in \Omega : \langle X(\omega), Y(\omega)\rangle \leq_{1,2,L} \langle \mathbf{0}_{\mathcal{I}_1}, \mathbf{0}_{\mathcal{I}_2}\rangle\} = \emptyset \tag{21.34}$$

and

$$S^L_{\langle\alpha_0,\beta_0\rangle} = \{\omega \in \Omega : \langle X(\omega), Y(\omega)\rangle \leq_{1,2,L} \langle\alpha_0, \beta_0\rangle\} = \Omega \tag{21.35}$$

are valid. Hence, the only what we have still to prove is the inclusion $S^L_{\langle\alpha_1,\beta_1\rangle} \subset S^L_{\langle\alpha_2,\beta_2\rangle}$ for each $\langle\alpha_1,\beta_1\rangle \leq_{1,2,L} \langle\alpha_2,\beta_2\rangle$.

Take $\langle\alpha,\beta\rangle \in I_1 \times I_2$. If $X(\omega) <_1 \alpha$ holds, the inequality $\langle X(\omega), Y(\omega)\rangle \leq_{1,2,L} \langle\alpha,\beta\rangle$ is valid no matter which the value $Y(\omega)$ may be. The only remaining case when $\langle X(\omega), Y(\omega)\rangle \leq_{1,2,L} \langle\alpha,\beta\rangle$ also holds reads that $X(\omega) = \alpha$ and $Y(\omega) = \beta$. Hence

$$\begin{aligned} &\{\omega \in \Omega : \langle X(\omega),\ Y(\omega)\rangle \leq_{1,2,L} \langle\alpha,\beta\rangle\} = \\ =\ &\{\omega \in \Omega : X(\omega) <_1 \alpha\} \cup \{\omega \in \Omega : X(\omega) = \alpha,\ Y(\omega) \leq_2 \beta\} \end{aligned} \tag{21.36}$$

Let $\langle\alpha_1,\beta_1\rangle$, $\langle\alpha_1,\beta_2\rangle$ and $\omega \in \Omega$ be such that the relation

$$\langle X(\omega), Y(\omega)\rangle \leq_{1,2,L} \langle\alpha_1,\beta_1\rangle \leq_{1,2,L} \langle\alpha_2,\beta_2\rangle \tag{21.37}$$

is valid, so that $X(\omega) \leq \alpha_1$ holds. If $\alpha_1 <_1 \alpha_2$ is the case, then $X(\omega) <_1 \alpha_2$ follows, so that, due to (21.36), $\langle X(\omega), Y(\omega)\rangle \leq_{1,2,L} \langle\alpha_2,\beta_2\rangle$ holds. Consequently, the inclusion $S^L_{\langle\alpha_1,\beta_1\rangle} \subset S^L_{\langle\alpha_2,\beta_2\rangle}$ is valid in this case. If $\alpha_1 = \alpha_2$, the the identity

$$\{\omega \in \Omega : X(\omega) < \alpha_1\} = \{\omega \in \Omega : X(\omega) < \alpha_2\} \tag{21.38}$$

is trivial and the second inequality in (21.37) implies that $\beta_1 \leq_2 \beta_2$ holds. Consequently, the inclusion

$$\{\omega \in \Omega : Y(\omega) \leq_2 \beta_1\} \subset \{\omega \in \Omega : Y(\omega) \leq \beta_2\} \tag{21.39}$$

follows, so that, combining (21.38) and (21.39), we obtain, that the inclusion $S^L_{\langle\alpha_1,\beta_1\rangle} \subset S^L_{\langle\alpha_2,\beta_2\rangle}$ is valid also in this case. So, the system $\mathcal{S}^L_{\langle X,Y\rangle}$ of subsets of Ω defined by (21.33) is an $(\mathcal{I}_1 \times \mathcal{I}_2)$-classification system over Ω and the assertion is proved. □

Corollary 21.1 *Let the notations and conditions of Theorem 21.1 hold, let $\mathcal{S}_X$ be the $\mathcal{I}_1$-classification system induced on Ω by the mapping $X : \Omega \to I_1$, so that*

$$\mathcal{S}_X = \{\{\omega \in \Omega : X(\omega) \leq \alpha\} : \alpha \in I_1\}. \tag{21.40}$$

Then the $(\mathcal{I}_2 \times \mathcal{I}_2)_L$-classification system $\mathcal{S}^L_{\langle X,Y\rangle}$ is a refinement of the $\mathcal{I}_1$-classification system $\mathcal{S}^L_{\langle X,Y\rangle}$ in the sense that the inclusion $\mathcal{S}_X \subset \mathcal{S}^L_{\langle X,Y\rangle}$ holds.

Proof. For every $\alpha \in I_1$, we obtain that

$$\begin{aligned}
\mathcal{S}_{X,\alpha} &= \{\omega \in \Omega : X(\omega) \leq_1 \alpha\} = \{\omega \in \Omega : X(\omega) <_1 \alpha\} \cup \{\omega \in \Omega : X(\omega) = \alpha\} = \\
&= \{\omega \in \Omega : X(\omega) <_1 \alpha\} \cup (\{\omega \in \Omega : X(\omega) = \alpha\} \cap \Omega) = \\
&= \{\omega \in \Omega : X(\omega) <_1 \alpha\} \cup \{\omega \in \Omega : X(\omega) = \alpha,\ Y(\omega) \leq_2 \beta_0\} = S_{\langle X,Y\rangle\langle\alpha,\beta_0\rangle}
\end{aligned}$$

let us recall that $\beta_0 \in I_2$ satisfies the property that $Y(\omega) \leq_2 \beta_0$ holds for each $\omega \in \Omega$. So, the inclusion $\mathcal{S}_X \subset \mathcal{S}^L_{\langle X,Y\rangle}$ is proved. □

The refinement of the classification system $\mathcal{S}_X$ resulting when taking profit of the values of another variable Y as a secondary and auxiliary criterion can improve (i.e., reduce) the values $\Pi(A, \mathcal{S}_X)$ ascribed to some subsets A of Ω when using a $\mathcal{T}$-possibilistic measure Π defined on a subsystem $\mathcal{A}$ of $\mathcal{P}(\Omega)$. Indeed, let the notations of Theorem 21.1 hold, let $\mathbf{0}_{\mathcal{I}_1} <_1 \alpha_1 <_1 \alpha_2$ and $\mathbf{0}_{\mathcal{I}_2} <_2 \beta_1 <_2 \beta_2$ be such elements of I_1 and I_2 that, for every $\omega \in \Omega$, $X(\omega)$ is either α_1 or α_2 and $Y(\omega)$ is either β_1 or β_2. Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ be the minimal field (algebra) of subsets of Ω containing all the sets $\{\omega \in \Omega : Z(\omega) \leq \gamma\}$ and $\{\omega \in \Omega : Z(\omega) = \gamma\}$ for $Z = X, Y$ and $\gamma = \alpha_1, \alpha_2, \beta_1, \beta_2$. Let $\mathcal{T} = \langle T, \leq_{\mathcal{T}} \rangle$ be a complete lattice, let Π be the $\mathcal{T}$-possibilistic measure on $\mathcal{A}$ such that $\Pi(\{\omega \in \Omega : X(\omega) = \alpha_2,\ Y(\omega) = \beta_2\}) = \mathbf{1}_{\mathcal{T}}$, $\Pi(A_1) = \mathbf{0}_{\mathcal{T}}$ for each $A_1 \in \mathcal{A}$ such that $A_1 \cap (\{\omega \in \Omega : X(\omega) = \alpha_2,\ Y(\omega) = \beta_2\}) = \emptyset$. Take

$$A = \{\omega \in \Omega : X(\omega) <_1 \alpha_2\} \cup \{\omega \in \Omega : X(\omega) = \alpha_2,\ Y(\omega) \leq_2 \beta_1\}, \tag{21.41}$$

i.e., $A = S^L_{\langle X,Y\rangle,\langle\alpha_2,\beta_1\rangle}$ in our notation from above. We obtain that

$$\Pi(A, \mathcal{S}^L_{\langle X,Y\rangle}) = \Pi(A) = \Pi(\Omega - \{\omega \in \Omega : X(\omega) = \alpha_2,\ Y(\omega) = \beta_2\}) = \mathbf{0}_{\mathcal{T}} \tag{21.42}$$

due to the definition of Π on $\mathcal{A}$ and due to the fact that the mapping $\Pi(\cdot, \mathcal{S}^L_{\langle X,Y\rangle}) : \mathcal{P}(\Omega) \to T$ extends Π conservatively from $\mathcal{S}^L_{\langle X,Y\rangle}$ to $\mathcal{P}(\Omega)$ (it is a general property of extensions of $\mathcal{T}$-possibilistic measures induced by classification systems). However,

$$\Pi(A, \mathcal{S}_X) = \Pi(\{\omega \in \Omega : X(\omega) \leq \alpha_2\}) = \Pi(\Omega) = \mathbf{1}_{\mathcal{T}}. \tag{21.43}$$

22 Continuity of Lattice-Valued Possibilistic Measures

When investigating real-valued set functions in general, important properties to which our attention will be focused are those of continuity from below (from the bottom, lower continuity) and from above (from the top, upper continuity), finite additivity or maxitivity and countable variants of these properties, in particular σ-additivity, and the problem whether there exists a pointwise defined distribution function (e.g., probabilistic or possibilistic distribution), i.e., a function, which enables to define uniquely the set function under consideration.

In the case of probability measure the situation is rather simple, cf. [13] or [38], e.g. Both the continuity from below and from above are equivalent in the sense that each of them, combined with the assumption of finite additivity, yields σ-additive probability measures. In other terms, each σ-additive probability measure is continuous from below as well as from above, and if a finitely additive probability measure is continuous from below (from above, resp.), it is also σ-additive. A probability measure P defined on a σ-field $\mathcal{A}$ of subsets of a universe Ω possesses the probability distribution, if there exists a finite or countable subset $\Omega_0 \subset \Omega$ such that, for each $A \in \mathcal{A}$, $P(A) = P(\Omega_0 \cap A)$ and for each $\omega \in \Omega$ the singleton $\{\omega\}$ is in $\mathcal{A}$. Indeed, in this case $P(A) = \Sigma_{\omega \in A \cap \Omega_0} P(\{\omega\})$, so that $\{P(\{\omega\}) : \omega \in \Omega\}$ defines the probability distribution which induces P. Evidently, each such probability measure can be conservatively extended from $\mathcal{A}$ to the power-set $\mathcal{P}(\Omega)$.

In [47], inspired by some ideas and results from [1], [39] and [40], the authors analyzed, from the point of view just illustrated in the case of probability measures, also the real-valued possibilistic measures. First of all, they show that in this case the continuity from above defines a substantially stronger demand than the continuity from below and they analyze in more detail the conditions under which continuity from above implies the existence of the possibilistic distribution inducing the possibilistic measure in question. Some consequences concerning the relations of these results to the cardinality of the universe Ω under consideration are also introduced in [47].

However, when analyzing these results and their proofs, we find that they rely substantially on some specific properties of the unit interval of real numbers in which the investigated possibilistic measures take their values. Namely, what principially matters are the following facts.

(i) The linear ordering on [0,1], so that each two possibility degrees may be compared and ordered by the standard relation $\leq$ on [0,1].

(ii) The Archimedean structure of this ordering due to which, informally said, each value in [0,1] is "accessible" from another value in [0,1] by a finite number of equidistant steps of a no matter how small, but fixed and positive size.

(iii) For each nonempty subset A of [0,1] there exists a finite or countable subset $A_0 \subset A$ such that the supreme values of A and A_0 are identical. Hence, for each $\emptyset \neq A_0 \subset [0,1]$ the value *sup*A can be reached as the limit value of an increasing sequence of elements of A. Consequently, the supremum of any nonempty set of possibilistic degrees can be approximated, up to a no matter how small a priori given difference, by the supremum of a finite set of possibility degrees, hence, by the value ascribed by the (not necessarily complete) possibilistic measure under consideration to a finite union of subsets of the universe Ω.

As can be easily seen, these conditions are not satisfied, in general, when considering lattice-valued possibilistic measures. Indeed, take the most simple case of the identity mapping- Π_{id} which takes the power-set $\mathcal{P}(\Omega)$ onto itself, simply ascribing to each $A \subset \Omega$ the same set A as its value $\Pi_{id}(A)$; this mapping obviously defines a $\langle \mathcal{P}(\Omega), \subset \rangle$-valued possibilistic measure on $\mathcal{P}(\Omega)$. However, if Ω and A are uncountable sets, there is no finite or countable $A_0 \subset \Omega$ such that $A = \bigcup_{\omega \in A}\{\omega\} = \bigcup_{\omega \in A_0}\{\omega\} = A_0$ would hold; obviously, the set union defines the supremum in $\mathcal{P}(\Omega)$ with respect to the set inclusion as partial ordering. In the same way, when fixing a "small" $B \subset \Omega$ and defining a set $A_0 \subset \Omega$ as a "good enough" approximation of a subset $A \subset \Omega$ if and only if the symmetric difference $A_0 \doteq A$ is a subset of B, in general there need not exist a finite A_0 with this property.

To conclude the introductory reasoning of this chapter, we have to admit that in the case of lattice-valued possibilistic measures the mutual relations among the notions and properties like continuity of possibilistic measures from below and from above, existence of possibilistic distribution, completeness of the possibilistic measure in question, etc., are substantially different from the case of real-valued possibilistic measures and are far from being trivial. So, let us take this fact as a challenge and let us try to shed some light on these problems in what follows.

Definition 22.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be a nonempty set, let $\{\emptyset, \Omega\} \subset \mathcal{A} \subset \mathcal{P}(\Omega)$ be a system of subsets of Ω, let Π be a $\mathcal{T}$-possibilistic measure on $\mathcal{A}$. The $\mathcal{T}$-possibilistic measure Π is* continuous from above (upper continuous)*, if for every infinite sequence $A_1, A_2, \dots$ of subsets of Ω such that each A_i and the set $\bigcap_{i=1}^{\infty} A_i$ are in $\mathcal{A}$ and $A_i \supset A_{i+1}$ holds for each $i = 1, 2, \dots$, the relation*

$$\Pi\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigwedge_{i=1}^{\infty} \Pi(A_i) \tag{22.1}$$

is valid.

The $\mathcal{T}$-possibilistic measure Π on $\mathcal{A}$ is a continuous from below (lower continuous)*, if for every infinite sequence $A_1, A_2, \dots$ of subsets of Ω such that each A_i and the set $\bigcup_{i=1}^{\infty} A_i$ are in $\mathcal{A}$ and $A_i \subset A_{i+1}$ holds for each $i = 1, 2, \dots$, the relation*

$$\Pi\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigvee_{i=1}^{\infty} \Pi(A_i) \tag{22.2}$$

is valid.

For every complete lattice $\mathcal{T} = \langle T, \leq \rangle$, every $t \in T$ and every $\emptyset \neq S \subset T$, the inequality

$$\bigwedge_{s \in S} (t \vee s) \geq t \vee \left(\bigwedge S \right) \tag{22.3}$$

obviously holds, but the equality is not valid in general. The complete lattice $\mathcal{T}$ is called $\mathbf{0}_{\mathcal{T}}$-*distributive (distributive in* $\mathbf{0}_{\mathcal{T}}$*)*, if equality holds in (22.3) for each $t \in T$ and each $\emptyset \neq S \subset T$ such that $\bigwedge S = \mathbf{0}_{\mathcal{T}}$. Both the complete lattices $\langle \mathcal{P}(X), \subset \rangle$, $X \neq \emptyset$, and $\langle [0,1], \leq \rangle$ are $\mathbf{0}_{\mathcal{T}}$-distributive. Indeed, in the first case for every $A \subset X$ and every $\emptyset \neq \mathcal{S} \subset \mathcal{P}(X)$ the relation

$$\bigcap_{B \in \mathcal{S}} (A \cup B) = A \cup \left(\bigcap_{B \in \mathcal{S}} B \right) \tag{22.4}$$

is valid, hence, it is the case also when $\bigcap_{B \in \mathcal{S}} B = \emptyset$. In the case of $\langle [0,1], \leq \rangle$ we obtain, for every $x \in [0,1]$ and every $\emptyset \neq S \subset [0,1]$ such that $\bigwedge S = 0$, that $\bigwedge_{s \in S}(x \vee s) = x = x \vee (\bigwedge S)$ as demanded. However, not every complete lattice is $\mathbf{0}_{\mathcal{T}}$-distributive, as the following example demonstrates. Let $T = \{t_0, t_1, t_2, t_3, t_4\}$, let the partial ordering on T be such that $t_0 < t_i < t_4$ holds for each $i = 1,2,3$, but for no pair $t_i, t_j, i \neq j, i, j = 1,2,3$ the relation $t_i \leq t_j$ is defined. Hence, $\mathcal{T} = \langle T_0, \leq \rangle$ is a complete lattice with $\mathbf{0}_{\mathcal{T}} = t_0$, $\mathbf{1}_{\mathcal{T}} = t_2$, $t_i \vee t_j = \mathbf{1}_{\mathcal{T}}$ for each $i \neq j, i, j = 1,2,3$. Take $t_1 \in T$ and $\emptyset \neq S = \{t_2, t_3\} \subset T_0$, then

$$\begin{aligned} t_1 \vee \left(\bigwedge S \right) &= t_1 \vee (t_2 \wedge t_3) = t_1 \vee \mathbf{0}_{\mathcal{T}} = t_1 < \\ < \bigwedge_{s \in S} (t_1 \vee s) &= (t_1 \vee t_2) \wedge (t_1 \vee t_3) = \mathbf{1}_{\mathcal{T}} \wedge \mathbf{1}_{\mathcal{T}} = \mathbf{1}_{\mathcal{T}} \end{aligned} \tag{22.5}$$

so that $\mathcal{T} = \langle T_0, \leq \rangle$ is not $\mathbf{0}_{\mathcal{T}}$-distributive.

Theorem 22.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a $\mathbf{0}_{\mathcal{T}}$-distributive complete lattice, let $\mathcal{A}$ be a σ-field of subsets of a nonempty set Ω, let Π be a continuous from above $\mathcal{T}$-possibilistic measure on $\mathcal{A}$. Then Π is also continuous from below on $\mathcal{A}$.*

Proof. Let $A_1 \subset A_2 \subset \ldots$ be a nested sequence of subsets from $\mathcal{A}$. Due to the fact that Π is monotonous with respect to set inclusion, we obtain, for each $n = 1, 2, \ldots$, that

$$\Pi(A_n) = \Pi \left(\bigcup_{j=1}^{n} A_j \right) \leq \Pi \left(\bigcup_{j=1}^{\infty} A_j \right) \tag{22.6}$$

and, consequently

$$\bigvee_{j=1}^{n} \Pi(A_j) \leq \bigvee_{j=1}^{\infty} \Pi(A_j) \leq \Pi \left(\bigcup_{j=1}^{\infty} A_j \right) \tag{22.7}$$

hold. Let $B_i = (\bigcup_{j=1}^{\infty} A_j) - A_i$, $i = 1, 2, \ldots$, so that each B_i is in $\mathcal{A}, B_1 \supset B_2 \supset \ldots$ holds and $\bigcap_{j=1}^{\infty} B_j = \emptyset$. As Π is continuous from above on $\mathcal{A}$, the relation

$$\mathbf{0}_{\mathcal{T}} = \Pi(\emptyset) = \Pi \left(\bigcap_{j=1}^{\infty} B_j \right) = \bigwedge_{j=1}^{\infty} \Pi(B_j) \tag{22.8}$$

is valid. However, for every $n = 1, 2, \ldots$ the identity

$$A_n \cup B_n = A_n \cup \left(\left(\bigcup_{j=1}^{\infty} A_j \right) - A_n \right) = \bigcup_{j=1}^{\infty} A_j \tag{22.9}$$

is the case, so that

$$\Pi(A_n \cup B_n) = \Pi(A_n) \vee \Pi(B_n) = \Pi\left(\bigcup_{j=1}^{\infty} A_j\right) \tag{22.10}$$

holds. Consequently, the inequality

$$\Pi\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \left(\bigvee_{j=1}^{\infty} \Pi(A_j)\right) \vee \Pi(B_n) \tag{22.11}$$

is valid for every $n = 1, 2, \ldots$, hence, as $\mathcal{T}$ is $\mathbf{0}_{\mathcal{T}}$-distributive, we obtain that

$$\begin{aligned}
\Pi\left(\bigcup_{j=1}^{\infty} A_j\right) &\leq \bigwedge_{n=1}^{\infty}\left[\left(\bigvee_{j=1}^{\infty} \Pi(A_j)\right) \vee \Pi(B_n)\right] = \\
&= \left(\bigvee_{j=1}^{\infty} \Pi(A_j)\right) \vee \left(\bigwedge_{n=1}^{\infty} \Pi(B_n)\right) = \left(\bigvee_{j=1}^{\infty} \Pi(A_j)\right) \vee \mathbf{0}_{\mathcal{T}} = \\
&= \bigvee_{j=1}^{\infty} \Pi(A_j).
\end{aligned} \tag{22.12}$$

Combining (22.7) and (22.12) we obtain that

$$\Pi\left(\bigcup_{j=1}^{\infty} A_j\right) = \bigvee_{j=1}^{\infty} \Pi(A_j), \tag{22.13}$$

so that Π is continuous from below on $\mathcal{A}$ and the assertion is proved. □

The result just proved offers an inspiration to the following idea. $\Omega, \mathcal{A}$ and Π being as in Theorem 22.1, the $\mathcal{T}$-possibilistic measure Π on $\mathcal{A}$ is called *σ-complete (countably complete)*, if for each sequence $A_1, A_2, \ldots$ of sets from $\mathcal{A}$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ holds, the identity $\Pi(\bigcup_{i=1}^{\infty} A_i) = \bigvee_{i=1}^{\infty} \Pi(A_i)$ is valid. As for each sequence $A_1, A_2, \ldots A_i \in \mathcal{A}$, the set $\bigcup_{i=1}^{\infty} A_i$ can be written as $\bigcup_{j=1}^{\infty} B_j,\ B_j = \bigcup_{i=1}^{j} A_i \subset B_{j+1}$ for every j, the following corollary is self-evident.

Corollary 22.1 *Let $\mathcal{T}$ be a $\mathbf{0}_{\mathcal{T}}$-distributive complete lattice, let Π be a $\mathcal{T}$-possibilistic measure continuous from above on a σ-field of subsets of Ω. Then Π is*

The most simple example demonstrates that the implication inverse to that proved in Theorem 22.1 does not hold in general. Take $\mathcal{T} = \langle\{\mathbf{0}_{\mathcal{T}}, \mathbf{1}_{\mathcal{T}}\}, \mathbf{0}_{\mathcal{T}} \leq \mathbf{1}_{\mathcal{T}}\rangle$, take $\Omega = \{\omega_1, \omega_2, \ldots\}$, take $\Pi(\emptyset) = \mathbf{0}_{\mathcal{T}},\ \Pi(A) = \mathbf{1}_{\mathcal{T}}$ for every $\emptyset \neq A \subset \Omega$. Obviously, $\mathcal{T}$ is a $\mathbf{0}_{\mathcal{T}}$-distributive complete lattice and Π is a continuous from below $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$. Indeed, for every $A_1 \subset A_2 \subset \ldots \subset \Omega$, either $A_1 = A_2 = \ldots = \emptyset$, but then $\bigcup_{i=1}^{\infty} A_i = \emptyset$ and $\Pi(A_i) = \Pi(\bigcup_{i=1}^{\infty} A_i) = \mathbf{0}_{\mathcal{T}}$ for every $i = 1, 2, \ldots$, so that the relation $\bigvee_{i=1}^{\infty} \Pi(A_i) = \Pi(\bigcup_{i=1}^{\infty} A_i)$ is valid. Or, $A_{i_0} \neq \emptyset$ for some i_0 (and, consequently, for every $i \geq i_0$), so that $\Pi(A_{i_0}) = \mathbf{1}_{\mathcal{T}} = \Pi(\bigcup_{i=1}^{\infty} A_i)$ and the relation $\bigvee_{i=1}^{\infty} \Pi(A_i) = \Pi(\bigcup_{i=1}^{\infty} A_i)$ holds again. However, take $A_n = \{\omega_n, \omega_{n+1}, \ldots\},\ n = 1, 2, \ldots\}$, then $\Pi(A_n) = \mathbf{1}_{\mathcal{T}} = \bigwedge_{n=1}^{\infty} \Pi(A_n)$ for each n, but $\Pi(\bigcap_{n=1}^{\infty} A_n) = \Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}$, hence, Π is not continuous from above on $\mathcal{P}(\Omega)$.

The trivial $\mathcal{T}$-possibilistic measure Π introduced in this example obviously possesses the $\mathcal{T}$-distribution; it is the constant mapping which ascribes the value $\mathbf{1}_{\mathcal{T}}$ to every $\omega \in \Omega$. Hence, at least in the case when the complete lattice $\mathcal{T}$ in question is $\mathbf{0}_{\mathcal{T}}$-distributive, the fact that a $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$ possesses the distribution does not imply that this $\mathcal{T}$-possibilistic measure is continuous from above. However, it is the case when considering the continuity from below, as the next assertion proves.

Theorem 22.2 *Let $\mathcal{T} = \langle T, \leq\ \rangle$ be a complete lattice, let Π be a $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$ which possesses the $\mathcal{T}$-distribution. Then Π is continuous from below on $\mathcal{P}(\Omega)$.*

Proof. Under the conditions imposed on Π, this $\mathcal{T}$-possibilistic measure is obviously complete, so that

$$\Pi(A) = \bigvee_{\omega \in A} \Pi(\{\omega\}) \tag{22.14}$$

holds for every $\emptyset \neq A \subset \Omega$ and $\Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}$ by convention. Let $A_1 \subset A_2 \subset \ldots \subset \bigcup_{i=1}^{\infty} A_i \subset \Omega$. Then

$$\Pi\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigvee\left\{\Pi(\{\omega\}) : \omega \in \bigcup_{i=1}^{\infty} A_i\right\} = \bigvee_{i=1}^{\infty}\left(\bigvee_{\omega \in A_i} \Pi(\{\omega\})\right) = \bigvee_{i=1}^{\infty} \Pi(A_i), \tag{22.15}$$

so that Π is continuous from below on $\mathcal{P}(\Omega)$. □

Let us focus our attention to the particular case of $\mathcal{T}$-possibilistic measures over a countable universe of discourse Ω and with $\mathcal{A} = \mathcal{P}(\Omega)$, as in this case some relations introduced and analyzed above in general will become more simple.

Lemma 22.1 *Let $T = \langle T, \leq \rangle$ be a complete lattice, let Π be a $\mathcal{T}$-possibilistic measure defined on the power set $\mathcal{P}(\Omega)$ of a countable set Ω, let Π be σ-complete. Then Π is a complete $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$.*

Proof. For each $\omega \in \Omega$, $\Pi(\{\omega\})$ is defined and each $A \subset \Omega$ is countable (including the empty set and finite sets), hence, as Π is σ-complete, the relation (22.14) holds. For each nonempty system $\mathcal{A} \subset \mathcal{P}(\Omega)$,

$$\Pi\left(\bigcup \mathcal{A}\right) = \Pi\left(\bigcup_{A \in \mathcal{A}} A\right) = \bigvee\left\{\Pi(\{\omega\}) : \omega \in \bigcup \mathcal{A}\right\} = \bigvee_{A \in \mathcal{A}}\left(\bigvee_{\omega \in \mathcal{A}} \Pi(\{\omega\})\right) = \bigvee_{A \in \mathcal{A}} \Pi(A) \tag{22.16}$$

due to the elementary properties of the operation of supremum. Hence, Π is a complete $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$. □

Lemma 22.2 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a $\mathbf{0}_{\mathcal{T}}$-distributive complete lattice, let Π be a continuous from above $\mathcal{T}$-possibilistic measure defined on the power-set $\mathcal{P}(\Omega)$ of a countable set Ω. Then Π possesses the $\mathcal{T}$-possibilistic distribution, i.e., (22.14) holds for each $A \subset \Omega$.*

Proof. As $\mathcal{P}(\Omega)$ is trivially a σ-field of subsets of Ω, Corollary 22.1 yields that Π is σ-complete on $\mathcal{P}(\Omega)$, hence, due to Lemma 22.1, Π is also a complete $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$. Consequently, (22.14) holds for each $A \subset \Omega$, so that $\Pi(\{\omega\})$ defines the $\mathcal{T}$-possibilistic distribution on Ω which induces Π on $\mathcal{P}(\Omega)$. The assertion is proved. □

In the case of a countable universe Ω, even the weaker condition of continuity from below imposed on Π is sufficient to prove that Π possesses the distribution.

Lemma 22.3 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be countable, let Π be a $\mathcal{T}$-possibilistic measure continuous from below on $\Pi(\Omega)$. Then Π possesses the $\mathcal{T}$-possibilistic distribution, i.e., (22.14) holds for each $A \subset \Omega$.*

Proof. Being countable, Ω can be written as a sequence $\langle \omega_1, \omega_2, \ldots \rangle$ of all its elements (the choice of a particular ordering will be irrelevant in what follows), and each $A \subset \Omega$ is defined by a subsequence (finite or infinite) $\langle \omega_{i_1}, \omega_{i_2}, \ldots \rangle$ of $\langle \omega_1, \omega_2, \ldots \rangle$. Let $A_n = \langle \omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_n} \rangle$be the initial segment of the length n of $\langle \omega_{i_1}, \omega_{i_2}, \ldots \rangle$. As $\Pi(\{\omega\})$ is defined for every $\omega \in \Omega$ and Π is a $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$, we obtain that

$$\Pi(A_n) = \bigvee_{j=1}^{n} \Pi(\{\omega_i\}) = \bigvee_{\omega \in A_n} \Pi(\{\omega\}). \tag{22.17}$$

Moreover, if A is infinite,

$$A_n \subset A_{n+1} \subset A = \bigcup_{j=1}^{\infty} A_j \tag{22.18}$$

holds for each $n = 1, 2, \ldots$. Consequently, as Π is continuous from below we obtain that

$$\Pi(A) = \bigvee_{n=1}^{\infty} \Pi(A_n) = \bigvee_{n=1}^{\infty} \left(\bigvee_{j=1}^{n} \Pi(\{\omega_{i_j}\}) \right) = \bigvee_{j=1}^{\infty} \Pi(\{\omega_{i_j}\}) = \bigvee_{\omega \in A} \Pi(\{\omega\}) \tag{22.19}$$

due to the elementary properties of the operation of supremum. Hence, for each $A \subset \Omega$, (22.14) holds. The assertion is proved. □

Theorem 22.3 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Π be a $\mathcal{T}$-possibilistic measure defined on the power-set $\mathcal{P}(\Omega)$ of all subsets of a countable universe Ω. Then the following conditions are equivalent:*

(i) Π is continuous from below;

(ii) Π possesses the $\mathcal{T}$-possibilistic distribution;

(iii) Π is complete;

(iv) Π is σ-complete.

Proof. $(i) \Rightarrow (ii)$ by Lemma 22.3, $(ii) \Rightarrow (i)$ by Theorem 22.2, $(iv) \Rightarrow (iii)$ by Lemma 22.1, $(iii) \Rightarrow (iv)$ is obvious. The equivalence $(ii) \Leftrightarrow (iii)$ follows immediately from the definitions of the notions under consideration. The assertion is proved. □

It is perhaps worth re-calling explicitly that even in the specific case with countable universe Ω the property of continuity from above is stronger than any of the conditions (i) – (iv) listed in Theorem 22.3, as the example above (following Corollary 22.1) demonstrates. Hence, if $\mathcal{T}$ is a $\mathbf{0}_{\mathcal{T}}$-distributive complete lattice and Π is a $\mathcal{T}$-possibilistic measure continuous from above on $\mathcal{P}(\Omega)$ for a countable Ω, then Π satisfies (i), hence, also (ii) – (iv) of Theorem 22.3, due to Theorem 22.1, but not vice versa, in general.

Let us note that, in general, Lemma 22.3 does not hold for uncountable universes Ω. Indeed, take the most simple complete lattice $\mathcal{T} = \langle \{\mathbf{0}_{\mathcal{T}}, \mathbf{1}_{\mathcal{T}}\}, \ \mathbf{0}_{\mathcal{T}} < \mathbf{1}_{\mathcal{T}} \rangle$, an uncountable universe Ω, and the mapping $\Pi : \mathcal{P}(\Omega) \to \{\mathbf{0}_{\mathcal{T}}, \mathbf{1}_{\mathcal{T}}\}$ such that $\Pi(A) = \mathbf{0}_{\mathcal{T}}$, if A is empty, finite, or countable, and $\Pi(A) = \mathbf{1}_{\mathcal{T}}$ otherwise, i.e., if A is an uncountable subset of Ω. As can be easily checked, Π is a σ-complete $\mathcal{T}$-possibilistic measure. The relations $\Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}$ and $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$ are obvious, let $A_1, A_2, \ldots$ be a sequence of subsets of Ω. Either, each A_i is empty finite, or countable. Then their union $\bigcup_{i=1}^{\infty} A_i$ is also at most countable so that, for each $i = 1, 2, \ldots$, the relation

$$\mathbf{0}_{\mathcal{T}} = \Pi(A_{i_0}) = \bigvee_{i=1}^{\infty} \Pi(A_i) = \Pi\left(\bigcup_{i=1}^{\infty} A_i \right) \tag{22.20}$$

easily follows. Or, there exists i_0 such that A_{i_0} is uncountable. Then $\bigcup_{i=1}^{\infty} A_i$ is uncountable as well, so that the equality

$$\mathbf{1}_{\mathcal{T}} = \Pi(A_i) = \bigvee_{i=1}^{\infty} \Pi(A_i) = \Pi\left(\bigcup_{i=1}^{\infty} A_i \right) \tag{22.21}$$

holds again and the σ-completeness of Π is proved. Applying this result to the particular case of a nested sequence $A_1 \subset A_2 \subset \ldots \subset \Omega$, we obtain that Π is continuous from below on $\mathcal{P}(\Omega)$. Nevertheless, the relation $\Pi(A) = \bigvee_{\omega \in A} \Pi(\{\omega\})$ evidently does not hold for uncountable subsets of Ω, so that Π does not possess the $\mathcal{T}$-distribution on Ω.

As a matter of fact, the just considered two-valued possibilistic measure Π on the power-set of an uncountable universe Ω can be taken as the analogy of the possibilistic measure which ascribes the value $\mathbf{0}_{\mathcal{T}}$ to finite sets (including the empty one), and the value $\mathbf{1}_{\mathcal{T}}$ to infinite sets, just shifted by one level up with respect to the hierarchy of cardinalities of subsets of the universe under consideration. A reasoning proving that this possibilistic measure Π on $\mathcal{P}(\Omega)$ is not continuous from above may read as follows.

Let Ω be an uncountable universe of discourse, so that the cardinality of Ω is greater than or equal to that of continuum, in symbols, $\|\Omega\| \geq c$ holds. Hence, there exists a subset $\Omega_0 \subset \Omega$ the cardinality of which is just that of continuum, i.e., $\| = \Omega_0\| \geq c$ holds. Consequently, there exists a one-to-one mapping φ which takes the open interval (0,1) of real numbers onto the set Ω_0. For each $A \subset (0,1)$, let $\varphi(A) = \{\varphi(x) : x \in A\}$ be the image of A induced by φ in Ω_0. Let us consider the open intervals $(0, 1/n),\ n = 1, 2, \ldots$, and denote by $B_n \subset \Omega_0$ the set $\varphi((0, 1/n))$. As $\varphi : (0,1) \leftrightarrow \Omega_0$ is one-to-one, its restriction on $0, 1/n)$ is also a one-to-one mapping which takes $(0, 1/n)$ onto B_n. Hence, the cardinality of each B_n is also that of continuum, i.e., $\|B_n\| = c$, so that $\Pi(B_n) = \mathbf{1}_{\mathcal{T}}$ holds. The sequence $(0, 1/n),\ n = 1, 2, \ldots$, is a nested sequence of subsets of $(0,1)$ such that $(0, 1/n) \supset (0, 1/(n+1))$ holds for each $n = 1, 2, \ldots$, and $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. It follows that $B_n \supset B_{n+1}$ and $\bigcap_{n=1}^{\infty} B_n = \emptyset$ holds as well. Indeed, if there were $\omega_0 \in \bigcap_{n=1}^{\infty} B_n$, then its inverse image $\varphi^{-1}(\omega_0)$ must be in every $(0, 1/n),\ n = 1, 2, \ldots$, but this is impossible. Hence, we have a nested sequence $B_1 \supset B_2 \supset \ldots$ of subsets of $\Omega_0 \subset \Omega$ such that $\bigwedge_{n=1}^{\infty} \Pi(B_n) = \mathbf{1}_{\mathcal{T}}$, but $\Pi(\bigcap_{n=1}^{\infty} B_n) = \Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}$, so that Π is not continuous from above on $\mathcal{P}(\Omega)$.

23 Continuous from Above and Strongly Continuous Lattice-Valued Possibilistic Measures

Definition 23.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice. A $\mathcal{T}$-monotone measure Π defined on a nonempty system $\mathcal{R}$ of subsets of a nonempty space Ω is called* separable*, if there exists, for every $A \in \mathcal{R}$, a finite of countable subset $A^K \subset A,\ A^K \in \mathcal{R}$, such that $\Pi(A^K) = \Pi(A)$. Each A^K possessing this property is called a* kernel *of (the set) A.*

Obviously, if Ω is at most countable, then each $\mathcal{T}$-monotone measure on each $\emptyset \neq \mathcal{R} \subset \mathcal{P}(\Omega)$ is separable, as each $A \in \mathcal{R}$ can be taken as its own kernel. The adjective "separable" will be applied also to particular cases of T-monotone measures, namely to $\mathcal{T}$-possibilistic measures defined on $\emptyset \neq \mathcal{R} \subset \mathcal{P}(\Omega)$. In the still more particular case when Π is a complete $\mathcal{T}$possibilistic measure on $\mathcal{R}$, the condition of separability yields that, for every $A \in \mathcal{R}$, there exists a finite or countable $A^K \subset A,\ A^K \in \mathcal{R}$, such that

$$\bigvee_{\omega \in A} \Pi(\{\omega\}) = \bigvee_{\omega \in A^K} \Pi(\{\omega\}). \tag{23.1}$$

Hence, the property of separability imitates the well-known property of the space of real numbers equipped by their standard linear ordering, according to which there exists, for each nonempty set $A \subset R^+ = [-\infty, \infty]$, a sequence $\langle a_1, a_2, \ldots \rangle$ such that $a_i \in A$ for each $i = 1, 2, \ldots$, and $sup A = sup\{a_i : i = 1, 2, \ldots\}$.

Theorem 23.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Π be a $\mathcal{T}$-possibilistic measure on the power-set $\mathcal{P}(\Omega)$ of all subsets of a space Ω, the cardinality of Ω being that of the continuum. If Π is continuous from above, i.e., if the relation*

$$\Pi\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigwedge_{i=1}^{\infty} \Pi(A_i) \tag{23.2}$$

holds for each nested sequence $A_1 \supset A_2 \supset \ldots$ of subsets of Ω, then there exists, for each $\omega \in \Omega$, a sequence $\{\omega\}_1 \supset \{\omega\}_2 \supset \ldots$ of subsets of Ω such that $\bigcap_{n=1}^{\infty}\{\omega\}_n = \{\omega\}$, hence, $\bigwedge_{n=1}^{\infty} \Pi(\{\omega\}_n) = \Pi(\{\omega\})$, and for each $A \subset \Omega$ and each $n = 1, 2, \ldots$, the relations

$$\Pi\left(\bigcup_{\omega \in A} (\{\omega\}_n)\right) = \bigvee_{\omega \in A} \Pi(\{\omega\}_n) \tag{23.3}$$

and

$$\bigwedge_{n=1}^{\infty} \Pi\left(\bigcup_{\omega \in A}(\{\omega\}_n)\right) = \Pi(A) \tag{23.4}$$

are valid. If Π *is, moreover, separable, then*

$$\Pi(A) = \bigvee_{\omega \in A} \Pi(\{\omega\}) \tag{23.5}$$

holds for each $A \subset \Omega$*, hence, the* $\mathcal{T}$*-possibilistic measure* Π *is complete and possesses the* $\mathcal{T}$*-possibilistic distribution* $\{\Pi(\{\omega\}) : \omega \in \Omega\}$*.*

Proof. As the cardinality of Ω is supposed to be that of continuum, there exists a one-to-one mapping $c : \Omega \leftrightarrow \{0,1\}^{\infty}$, i.e., $c(\omega) = \langle c_1(\omega), c_2(\omega), \dots \rangle, c_i(\omega) \in \{0,1\}$, for each $\omega \in \Omega$ and $i = 1, 2, \dots$. To avoid the difficulties resulting from the twofold encoding of some real numbers, we purposely do not take $c(\omega)$ as a real number from $[0,1]$. Let us consider the following sequence $\Omega_{(0)}, \Omega_{(1)}, \Omega_{(2)}, \dots$ of finite disjoint decompositions of the universe Ω:

$$\begin{aligned} \Omega_0 &= \{\Omega\}, \Omega_{(1)} = \{\Omega_{1,0}, \Omega_{1,1}\}, \Omega_{(2)} = \{\Omega_{2,0}, \Omega_{2,1}, \Omega_{2,2}, \Omega_{2,3}\}, \dots, \Omega_{(n)} = \\ &= \{\Omega_{n,0}, \Omega_{n,1}, \dots, \Omega_{n,2^n-1}\}, \end{aligned} \tag{23.6}$$

where, for each $n = 1, 2, \dots$ and each $i = 0, 1, \dots, 2^n - 1$

$$\Omega_{n,i} = \{\omega \in \Omega : \Sigma_{j=1}^{n} c_j 2^{j-1} = i\}. \tag{23.7}$$

Informally, $\Omega_{n,i}$ is the set of all $\omega \in \Omega$ for which the initial segment $\langle c_1(\omega), c_2(\omega), \dots, c_n(\omega)\rangle$ of their encoding $c(\omega)$ defines the binary decomposition of the integer i. Obviously, for each $\omega \in \Omega$ and each $n \geq 1$ there exists uniquely defined integer $i_n(\omega)$, $0 \leq i_n(\omega) < 2^n$, such that ω is in $\Omega_{n,i_n(\omega)}$ and this $\Omega_{n,i_n(\omega)}$ will be denoted by $\{\omega\}_n$ in what follows. The following relations are obviously valid:

(i) $\{\omega\}_1 \supset \{\omega\}_2 \supset \{\omega\}_3 \supset \dots \{\omega\} = \bigcap_{n=1}^{\infty} \{\omega\}_n$ holds for each $\omega \in \Omega$,

(ii) for each $n = 1, 2, \dots$ and each $0 \leq i, j < 2^n$, $i \neq j$, the sets $\Omega_{n,i}$ and $\Omega_{n,j}$ are disjoint

(iii) for each $n = 1, 2, \dots$, $\bigcup_{i=0}^{2^n-1} \Omega_{n,i} = \Omega$.

Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Π be a $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$. For each $A \subset \Omega$ and each $n = 1, 2, \dots$, as the decomposition $\Omega_{(n)}$ is finite, there exists a finite number ($\leq 2^n$) of different subsets among the subsets $\{\omega\}_n, \omega \in A$, in other terms, there exists a finite subset $A^n \subset A$ such that

$$\bigcup_{\omega \in A} (\{\omega\}_n) = \bigcup_{\omega \in A^n} (\{\omega\}_n). \tag{23.8}$$

Hence, for each $\omega \in A$, the inequality

$$\Pi(\{\omega\}_n) \leq \Pi\left(\bigcup_{\omega \in A^n} (\{\omega\}_n)\right) = \bigvee_{\omega \in A^n} \Pi(\{\omega\}_n) \tag{23.9}$$

is valid, so that the relation

$$\bigvee_{\omega \in A} \Pi(\{\omega\}_n) \leq \bigvee_{\omega \in A^n} \Pi(\{\omega\}_n) \tag{23.10}$$

follows. The inverse inequality holds trivially due to the inclusion $A^n \subset A$. Hence, we obtain that

$$\Pi\left(\bigcup_{\omega \in A} (\{\omega\}_n)\right) = \Pi\left(\bigcup_{\omega \in A^n} (\{\omega\}_n)\right) = \bigvee_{\omega \in A^n} \Pi(\{\omega\}_n) = \bigvee_{\omega \in A} \Pi(\{\omega\}_n), \tag{23.11}$$

so that (23.3) is proved. If Π is continuous from above, we obtain that

$$\bigwedge_{n=1}^{\infty} \Pi(\{\omega\}) = \Pi\left(\bigcap_{n=1}^{\infty} \{\omega\}_n\right) = \Pi(\{\omega\}) \tag{23.12}$$

and, as

$$\bigcap_{n=1}^{\infty}\left(\bigcup_{\omega\in A}(\{\omega\}_n)\right) = A, \tag{23.13}$$

the relation (23.4) follows as well.

Let the $\mathcal{T}$-possibilistic measure Π on $\mathcal{P}(\Omega)$ under consideration be continuous from above and separable. Given $A \subset \Omega$ and a kernel $A^K \subset A$, the identity $\Pi(A) = \Pi(A^K)$ follows from the definition of kernel. Also the inequalities

$$\bigvee_{\omega\in A^K} \Pi(\{\omega\}) \leq \bigvee_{\omega\in A} = \Pi(\{\omega\}) \leq \Pi(A) \tag{23.14}$$

are obvious, the first one because of the inclusion $A^K \subset A$, the second one because of the fact that for each $\omega \in A$ the inclusion $\{\omega\} \subset A$ is valid, hence, also the inequalities

$$\Pi(\{\omega\}) \leq \Pi(A), \ \bigvee_{\omega\in A} \Pi(\{\omega\}) \leq \Pi(A) \tag{23.15}$$

easily follow.

Let us prove that for each finite or countable $A \subset \Omega$ the relation $\Pi(A) = \bigvee_{\omega\in A} \Pi(\{\omega\})$ is valid, in particular, that $\Pi(A^K) = \bigvee_{\omega\in A^K} \Pi(\{\omega\})$ holds for each $A \subset \Omega$ and each kernel A^K of A. If A is finite, this is obvious, so let us assume that $A = \{\omega_1, \omega_2, \ldots\}$ is an infinite countable subset of Ω.

Denote by A_n the set $\bigcup_{\omega\in A}(\{\omega\}_n)$ and by B_n the set $\{\omega_1, \omega_2, \ldots, \omega_n\} \subset A$. Evidently, for each $n = 1, 2, \ldots, (A_n - B_n) \cup B_n = A_n$, so that the identity

$$\Pi(A_n - B_n) \vee \Pi(B_n) = \Pi(A_n) \tag{23.16}$$

holds for every $n = 1, 2, \ldots$. Due to (23.13) we obtain that the inequality

$$\Pi(A_n - B_n) \vee \Pi(B_n) = \Pi(A_n - B_n) \vee \bigvee_{i=1}^{n} \Pi(\{\omega_i\}) \geq \Pi(A) \tag{23.17}$$

and, obviously, also the inequality

$$\Pi(A_n - B_n) \vee \bigvee_{i=1}^{\infty} \Pi(\{\omega_i\}) \geq \Pi(A) \tag{23.18}$$

are valid for each $n = 1, 2, \ldots$. Consequently, also

$$\left(\bigwedge_{n=1}^{\infty} \Pi(A_n - B_n)\right) \vee \bigvee_{i=1}^{\infty} \Pi(\{\omega_i\}) \geq \Pi(A) \tag{23.19}$$

holds, but

$$\bigwedge_{n=1}^{\infty} \Pi(A_n - B_n) = \Pi\left(\bigcap_{n=1}^{\infty} (A_n - B_n)\right) = \Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}, \tag{23.20}$$

as Π is supposed to be continuous from above on $\mathcal{P}(\Omega)$. Hence, we obtain the inequality $\bigvee_{i=1}^{\infty} \Pi(\{\omega_i\}) \geq \Pi(A)$. The inverse inequality is immediately implied by the obvious relation $\Pi(\{\omega_i\}) \leq \Pi(A)$, valid for every $\omega_i \in A$, hence, the equality

$$\Pi(A) = \bigvee_{i=1}^{\infty} \Pi(\{\omega_i\}) = \bigvee_{\omega \in A} \Pi(\{\omega\}) \tag{23.21}$$

follows. Combining all inequalities obtained so far we arrive at the relation

$$\Pi(A) = \Pi(A^K) = \bigvee_{\omega \in A^K} \Pi(\{\omega\}) \leq \bigvee_{\omega \in A} \Pi(\{\omega\}) \leq \Pi(A) \tag{23.22}$$

so that (23.5) is valid and the theorem is proved. □

Let us strengthen the notion of continuity from above as follows.

Definition 23.2 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be a nonempty set, let $\{\emptyset, \Omega\} \subset \mathcal{R} \subset \mathcal{P}(\Omega)$ be a system of subsets of Ω, let $\Pi : \mathcal{R} \to T$ be a $\mathcal{T}$-monotone measure on $\mathcal{R}$. Π is called* strongly continuous from above*, if for every $\emptyset \neq \mathcal{R}_0 \subset \mathcal{R}$ such that $card\mathcal{R}_0 \leq card\Omega$ and $\bigcap \mathcal{R}_0 = \bigcap_{A \in \mathcal{R}_0} A = \emptyset$ holds, the relation $\bigwedge_{A \in \mathcal{R}_0} \Pi(A) = \mathbf{0}_{\mathcal{T}}$ holds as well.*

Let us note that this definition of strong continuity from above does not demand the validity of the relation $\bigwedge_{A \in \mathcal{R}_0} \Pi(A) = \mathbf{0}_{\mathcal{T}}$ for *every* $\mathcal{R}_0 \subset R$ such that $\bigcap \mathcal{R}_0 = \emptyset$, as it does not touch subsystems $\mathcal{R}_0 \subset \mathcal{R}$ such that $card\mathcal{R}_0 > card\Omega$ holds, which may exist when $card\mathcal{R} = card\mathcal{P}(\Omega) = 2^{card\Omega}$. As above, the property "strongly continuous from above" will be ascribed also to mappings which take $\mathcal{R}$ into T and which are particular cases of $\mathcal{T}$-monotone measures, in particular to $\mathcal{T}$-possibilistic measures.

Lemma 23.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice distributive in $\mathbf{0}_{\mathcal{T}}$, let $\emptyset \neq \mathcal{R} \subset \mathcal{P}(\Omega)$ be a system of subsets of a nonempty set Ω, let $\Pi : \mathcal{R} \to T$ be a strongly continuous from above $\mathcal{T}$-possibilistic measure defined on $\mathcal{R}$, let $\emptyset \neq \mathcal{R}_0 \subset \mathcal{R}$ be such a subsystem of $\mathcal{R}$ that $\bigcap R_0 \in \mathcal{R}$ and $A - \bigcap \mathcal{R}_0 \in \mathcal{R}$ holds for every $A \in \mathcal{R}_0$. Then*

$$\Pi\left(\bigcap \mathcal{R}_0\right) = \bigwedge \{\Pi(A) : A \in \mathcal{R}_0\}. \tag{23.23}$$

Proof. For each $A \in \mathcal{R}_0$ the identity $(A - \bigcap \mathcal{R}_0) \cup \bigcap \mathcal{R}_0 = A$ is obvious, hence, the relation

$$\Pi\left(A - \bigcap \mathcal{R}_0\right) \vee \Pi\left(\bigcap \mathcal{R}_0\right) = \Pi(A) \tag{23.24}$$

holds for every $A \in \mathcal{R}_0$ and the equality

$$\begin{aligned} \bigwedge\left\{\Pi\left(A - \bigcap \mathcal{R}_0\right) \vee \Pi\left(\bigcap \mathcal{R}_0\right) : A \in \mathcal{R}_0\right\} &= \bigwedge\left\{\Pi\left(A - \bigcap \mathcal{R}_0\right) : A \in \mathcal{R}_0\right\} \vee \Pi\left(\bigcap \mathcal{R}_0\right) \\ &= \bigwedge\{\Pi(A) : A \in \mathcal{R}_0\} \end{aligned} \tag{23.25}$$

follows due to the supposed $\mathbf{0}_{\mathcal{T}}$-distributivity of $\mathcal{T}$. However, as Π is strongly continuous from above we obtain that

$$\bigwedge\left\{\Pi\left(A - \bigcap \mathcal{R}_0\right) : A \in \mathcal{R}_0\right\} = \Pi\left(\bigcap_{A \in \mathcal{R}_0}\left(A - \bigcap \mathcal{R}_0\right)\right) = \Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}, \tag{23.26}$$

so that (23.23) is proved. □

Theorem 23.2 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice distributive in $\mathbf{0}_{\mathcal{T}}$, let Π be a strongly continuous from above $\mathcal{T}$-possibilistic measure on the power-set $\mathcal{P}(\Omega)$ of all subsets of a nonempty set Ω. Then Π is defined by the $\mathcal{T}$-possibilistic distribution $\{\Pi(\{\omega\}) : \omega \in \Omega\}$, i.e., $\Pi(A) = \bigvee_{\omega \in A} \Pi(\{\omega\})$ holds for each $A \subset \Omega,\ A \neq \emptyset$.*

Proof. For every $\emptyset \neq A \subset \Omega$ and every $\omega \in A$ we obtain

$$\Pi(A - \{\omega\}) \vee \Pi(\{\omega\}) = \Pi(A), \tag{23.27}$$

so that the inequality

$$\Pi(A - \{\omega\}) \vee \bigvee_{\omega \in A} \Pi(\{\omega\}) \geq \Pi(A) \tag{23.28}$$

easily follows. As $\mathcal{T}$ is $\mathbf{0}_{\mathcal{T}}$-distributive, we obtain that

$$\begin{aligned} & \bigwedge\{\Pi(A - \{\omega\}) \vee \bigvee_{\omega \in A} \Pi(\{\omega\}) : \omega \in A\} \\ = \; & \bigwedge\{\Pi(A - \{\omega\}) : \omega \in A\} \vee \bigvee_{\omega \in A} \Pi(\{\omega\}) \geq \Pi(A) \end{aligned} \tag{23.29}$$

holds as well. But, as $cardA \leq card\Omega$ holds, the strong continuity from above of Π on $\mathcal{P}(\Omega)$ yields that

$$\bigwedge\{\Pi(A - \{\omega\}) : \omega \in A\} = \Pi\left(\bigcap_{\omega \in A} (A - \{\omega\})\right) = \Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}, \tag{23.30}$$

so that the inequality $\bigvee_{\omega \in A} \Pi(\{\omega\}) \geq \Pi(A)$ results. As showed above, the inverse inequality holds trivially, so that the assertion is proved. □

When applying Theorem 23.2 to countable spaces Ω, we obtain immediately that each continuous from above $\mathcal{T}$-possibilistic measure Π on $\mathcal{P}(\Omega)$ is defined by its $\mathcal{T}$-possibilistic distribution $\{\Pi(\{\omega\}) : \omega \in \Omega\}$. In the foregoing Chapter we obtained this result as a consequence of other assertions valid in the particular case of countable universes. As Theorem 23.2 shows, the cardinality of the system of subsets occurring in the definition of the generalized continuity from above must be, in general, the same as that of the universe Ω. The results of the first part of this chapter then prove that in the case of a countable universe Ω and separable $\mathcal{T}$-possibilistic measures on $\mathcal{P}(\Omega)$ the cardinality of set systems used in the definition of the continuity from above can be "exponentially" reduced from the countable cardinality of continuum to the countable cardinality, in symbols, from $2^{\aleph_0}$ to $\aleph_0$. It is a matter of further reasoning, whether the condition of separability can be generalized or modified also to the case of $\mathcal{T}$-possibilistic measures defined on the power-sets $\mathcal{P}(\Omega)$ of spaces of cardinality greater then that of continuum. Let us recall that in the proof of Theorem 23.1 we have substantially taken profit of the fact that the elements of a continuum Ω can be put into one-to-one correspondence with infinite binary sequences.

Indeed, if $card\Omega > 2^{\aleph_0}$ holds, then for no matter which infinite sequence of more and more fine binary splittings of Ω (like the sequence $\Omega_{(0)}, \Omega_{(1)}, \Omega_{(2)}, \ldots$ of decompositions used in the proof of Theorem 23.1) we obtain at most $2^{\aleph_0}$ different subsets of Ω of the type $\bigcap_{n=1}^{\infty} \{\omega\}_n$. Hence, the identity $\{\omega\} = \bigcap_{n=1}^{\infty} \{\omega\}_n$ cannot hold in general, i.e., for every $\omega \in \Omega$, moreover, at least one of the intersections $\bigcap_{n=1}^{\infty} \{\omega\}_n$ must be of the same cardinality as Ω. As a matter of fact, if the inequality $card \bigcap_{n=1}^{\infty} \{\omega\}_n < card\Omega$ held for each $\omega \in \Omega$, we would obtain the inequality

$$card\Omega = card\left(\bigcup_{\omega \in \Omega} \left(\bigcap_{n=1}^{\infty} \{\omega\}_n\right)\right) < 2^{\aleph_0} card\Omega = card\Omega, \tag{23.31}$$

as $card\Omega > 2^{\aleph_0}$ holds by assumption, hence, we have arrived at a contradiction. Consequently, to go on in the way applied when proving Theorem 23.1 for spaces with continuum cardinality, we would have to suppose that the lattice-valued possibilistic measure Π in question is complete at every set $\bigcap_{n=1}^{\infty} \{\omega\}_n$, i.e., that the equality

$$\Pi\left(\bigcap_{n=1}^{\infty} \{\omega\}_n\right) = \bigvee\left\{\Pi(\{\omega_0\}) : \omega_0 \in \bigcap_{n=1}^{\infty} \{\omega\}_n\right\} \tag{23.32}$$

holds for each $\omega \in \Omega$ including the cases when the cardinality of the intersection $\bigcap_{n=1}^{\infty}\{\omega\}_n$ is the same as that of Ω. However, such an assumption evidently does not solve our problem to prove an analogy of Theorem 23.1 also for universes of greater cardinalities.

24 Decision-Making Under Uncertainty — Motivation and General Preliminaries

When suggesting and analyzing possibilistic lattice-valued decision functions, we take an inspiration and motivation in the theory of statistical decision functions developed on the grounds of Kolmogorov axiomatic probability theory. Let us recall, very briefly, the elementary ideas on which statistical decision functions are based. The reader is supposed to be familiar with the notations used in the axiomatic probability theory as introduced, e.g., in [4], [13] or [38]. However, when developing a possibilistic and lattice-valued version of this model of decision-making under uncertainty, our explanation will aim to be self-explanatory to the most possible degree.

Consider a system $\mathcal{S}$ of no matter which nature, the actual internal state of which is s_0. This actual internal state is, up to the most trivial cases, neither known nor immediately observable by a subject, an expert, say, who controls the system $\mathcal{S}$ with respect to some reasonable and rational criteria of acceptability or optimality. The only fact being known apriori reads that s_0 belongs to a set S of possible actual internal states of the system $\mathcal{S}$. Another interpretation may read that $\mathcal{S}$ is a problem to be solved and S is the set of possible solutions (or candidates to solution), the only one $s_0 \in S$ being the optimal (the correct) one. A reformulation of this interpretation in the terms of hypothesis testing is obvious and easy to be done. In the language of medical science: the system $\mathcal{S}$ is a patient treated by a doctor who wants to identify the disease the patient is suffering from under the apriori knowledge that this diagnosis is just one element of a set S of diagnoses under consideration. In what follows, we will use the "engineering" terminology introduced above as the primary one, keeping in mind that our further rather abstract constructions and considerations can be easily and more or less routinely re-phrased also under each of the other semantical interpretations just sketched.

The subject confronted with the system $\mathcal{S}$ takes a decision from a fixed set D of decisions being at his/her disposal. E.g., in the case of a technical device possible decisions are given by various interventions into the system through its regulation devices, in the case of medical care to take a decision means to apply some medicaments and/or other therapies, or simply to declare one hypothesis (diagnosis, e.g.) as the true one. However, various decisions are, in general, not equivalent as far as their consequences are taken into consideration. The two following assumptions are accepted: the consequences resulting when taking a decision $d \in D$ depend only on this d and on the actual state s of the system under consideration, and they are quantified by a non-negative real number $\lambda(s,d)$ taken as the loss (financial, say) suffered by the subject when applying the decision d to $\mathcal{S}$ in the internal state s. In symbols, λ is a real-valued function which takes the Cartesian product $S \times D$ into $[0,\infty)$, hence, if $\lambda(s,d) = 0$, then d is an absolutely best solution with respect to s, as its application does not bring any loss. More generally, a solution d_0 is optimal with respect to $s \in S$, if the inequality $\lambda(s,d_0) \leq \lambda(s,d_1)$ holds for each $d_1 \in D$. If the set D of possible decisions is finite, such d_0 obviously always exists, if D is infinite, then for every $\varepsilon > 0$ there exists $d_0(\varepsilon) \in D$ such that the inequality $\lambda(s, d_0(\varepsilon)) < \lambda(s,d_1) + \varepsilon$ is valid for each $d_1 \in D$.

Supposing, that the actual state s_0 of $\mathcal{S}$ is known to the subject, it is a very simple matter (from the theoretical point of view, not taking into consideration the computational complexity and other problems possibly arising when processing the function λ) to obtain a decision optimal w.r. to s or a decision approximating the optimal one up to a given fixed $\varepsilon > 0$. However, up to the trivial cases the actual states are not known or directly observable, the only what is at the subject's disposal are the empirical data-results of various observations, treatments of experiments done by the subject. Let E denote the set (perhaps a vector space) of possible values of these empirical procedures, so that what is at the subject's disposal when choosing an appropriate decision is just a value $e \in E$. We suppose that the way in which the subject takes his/her decision can be described by a decision function δ taking the set E of empirical data (values) into the set D of decisions. Hence, first of all, the subject chooses a decision function δ (the way in which he/she does so is the key problem in the decision making under uncertainty and will be discussed in more detail below), and then the subject applies this decision function to the actual empirical value being at his/her disposal. Combining our notation together, if s is the actual internal state of the system $\mathcal{S}$, if e is the empirical value being at the subject's disposal and if δ is the decision function which he/she applies, the suffered loss is $\lambda(s, \delta(e)) \in [0, \infty)$. E.g., if the decision problem consists in the identification of the actual internal state s of $\mathcal{S}$ and if the most simple loss function applies, i.e., if $S = D$ and $\lambda(s, d) = 0$, if $s = d$, $\lambda(s, d) = 1$ otherwise, and if the decision function δ is used, then the loss $\lambda(s, \delta(e)) = 0$ iff $\delta(e) = s$, and $\lambda(s, \delta(e)) = 1$ otherwise.

A decision function $\delta_0 : E \to D$ is *uniformly optimal* (uniformly the best one) if for every $\delta : E \to D$, every $s \in S$, and every $e \in E$ the inequality

$$\lambda(s, \delta_0(e)) \leq \lambda(s, \delta(e)) \tag{24.1}$$

is valid. However, up to the most elementary cases when the actual internal state of the system under consideration can be identified from the empirical data being at the subject's disposal, such a uniformly optimal decision function does not exist. Indeed, let us consider, again, the example with $S = D$ and $\lambda(s, d) \in \{0, 1\}$ described above and suppose that some empirical value $e \in E$ is compatible with at least two states $s_1, s_2 \in S$, hence, observing e the subject is not able to decide whether s_1 or s_2 is the internal state of $\mathcal{S}$. Now, for each decision function $\delta : E \to D$, if $\delta(e) = s_1$ but s_2 is the internal state of $\mathcal{S}$, then

$$\lambda(s_2, \delta(e)) = 1 > 0 = \lambda(s_2, \delta_{s_2}(e)), \tag{24.2}$$

where δ_{s_2} is the constant decision function ascribing the value s_2 to each $e \in E$. Hence, the uniformly optimal decision function in the sense of (24.1) does not exist.

Let us leave aside numerous heuristics and partial solutions suggested and applied when choosing a reasonable decision function δ, either when introducing some more assumptions into our model or when focusing our attention to some particular cases worth being analyzed in more detail. In the rest of this chapter we will sketch, very briefly, the case when the phenomenon of uncertainty, taken as randomness and described and processed by the tools offered by the probability theory in its axiomatic setting, enters our model, so giving arise the notion of statistical decision functions. Some basic ideas of this approach will be used as an useful inspiration and motivation when aiming to develop and analyze a possibilistic alternative of the statistical model, cf. Section 25 below.

Let us begin with the notion of *probability space*, defined by a triple $\langle \Omega, \mathcal{A}, P \rangle$, where Ω is a nonempty space, $\mathcal{A}$ is a nonempty σ-field of subsets of Ω, and $P : \mathcal{A} \to [0, 1]$ is a normalized σ-additive measure on $\mathcal{A}$. Elements $\omega \in \Omega$ are called *elementary random events*, sets from $\mathcal{A}$ are called *random events*, and P is called the *probability (measure)* ascribing to each random event $A \in \mathcal{A}$ its probability $P(A)$. Let Z be a nonempty set, let $\mathcal{Z}$ be a nonempty σ-field of subsets of Z. A mapping $X = \Omega \to Z$ is called *Z-valued random variable*, if it is measurable with respect to the σ-fields $\mathcal{A}$ and $\mathcal{Z}$, i.e., if the inverse image of each $B \in \mathcal{Z}$ is in $\mathcal{A}$ or, what turns to be the same, if the inclusion

$$\{\{\omega \in \Omega : X(\omega) \in B\} : B \in \mathcal{Z}\} \subset \mathcal{A} \tag{24.3}$$

is valid. If the set Z is finite and if $\mathcal{Z}$ is not defined explicitly, we suppose that $\mathcal{Z} = \mathcal{P}(Z)$, i.e., the power-set of all subsets of Z.

Turing back to our general model for decision making introduced above, two inputs (parameters) entering this model can be supposed to be charged by uncertainty in the sense of randomness. We will suppose that the empirical value e, being at the subject's disposal, is the value taken by an E-valued random variable $\eta : \Omega \to E$. Accepting the so called bayesian approach, we will suppose that also the actual internal state s_0 of the system $\mathcal{S}$ is the value taken by an S-valued random variable $\sigma : \Omega \to S$. In the rest of this chapter we suppose, not to charge our explanation by mathematical technicalities, that both the sets E and S, as well as the set D of possible decisions, are finite, so that our convention concerning the σ-fields over E and $\mathcal{S}$ applies.

Under this setting, the decision function δ converts into a D-valued random variable $\delta^\star : \Omega \to D$ such $\delta^\star(\omega) = \delta(\eta(\omega))$ for every $\omega \in \Omega$, and the loss function λ converts into a real-valued random variable $\lambda^\star : \Omega \to R = (-\infty, \infty)$, setting for every $\omega \in \Omega$

$$\lambda^\star(\omega) = \lambda(\sigma(\omega),\ \delta(\eta(\omega))). \tag{24.4}$$

Taking into account the realistic assumption that the random variables σ and η describe the properties of the world in which the subject realizes his/her decision procedure, so that these random variables are taken as fixed, the only way in which the subject can optimize his/her decision making, i.e., minimize the loss, is to choose appropriately the decision function δ.

For the same reasons as discussed above we cannot expect that there exists a decision function $\delta_0 : E \to D$ minimizing the loss uniformly for every $\omega \in \Omega$, i.e., satisfying the inequality

$$\lambda(\sigma(\omega),\ \delta_0(\eta(\omega))) \leq \lambda(\sigma(\omega),\ \delta(\eta(\omega))) \tag{24.5}$$

for every $\delta : E \to D$ and every $\omega \in \Omega$. However, the bayesian approach enables to define, under some conditions, the expected value

$$E\lambda^\star = \int_{-\infty}^{\infty} \lambda^\star(\omega) dP \tag{24.6}$$

of the random variable $\lambda^\star$. Under our simplifying assumption of finiteness of the sets S and E, a sufficient condition for the expected value $E\lambda^\star$ to be defined and finite reads that the loss function λ is uniformly majorized by a constant value, i.e., that the inequality $\lambda(s,e) \leq K < \infty$ holds for each $\langle s,e\rangle \in S \times E$. A routine calculation yields that in this case the expected value $E\lambda^\star$ can be written in a more explicit and intuitive way, namely

$$\begin{aligned} E\lambda^\star &= E\lambda(\sigma(\cdot),\ \delta(\eta(\cdot))) \\ &= \sum_{\langle s,e\rangle \in S\times E} \lambda(s,e)P(\{\omega \in \Omega : \sigma(\omega) = s\} \cap \{\omega \in \Omega : \eta(\omega) = e\}). \end{aligned} \tag{24.7}$$

A decision function $\delta_0 : E \to D$ is *optimal* in the bayesian sense and with respect to the apriori random variable σ, if

$$E(\lambda(\sigma(\cdot),\ \delta_0(\eta(\cdot)))) = \inf\{E(\lambda(\sigma(\cdot),\ \delta(\eta(\cdot)))) : \delta : E \to D\}. \tag{24.8}$$

Given $\varepsilon > 0$, a decision function $\delta_{0,\varepsilon} : E \to D$ is ε-*optimal* in the bayesian sense and w.r. to σ, if for each $\delta : E \to D$ the inequality

$$E(\lambda(\sigma(\cdot),\ \delta_{0,\varepsilon}(\eta(\cdot)))) < E(\lambda(\sigma(\cdot),\ \delta(\eta(\cdot)))) + \varepsilon \tag{24.9}$$

holds. A decision function δ_0 satisfying (24.8) need not, in general, exist, but a decision function $\delta_{0,\varepsilon}$ satisfying (24.9) obviously exists for every $\varepsilon > 0$. Of course, our reasoning proves the existence of such a $\delta_{0,\varepsilon}$ only at the implicit level, to define $\delta_{0,\varepsilon}$ explicitly may be a very difficult task.

Still keeping ourselves within the framework of the statistical decision making under uncertainty as sketched in this chapter, the qualities of the decision process can be improved (in the sense that the expected loss is reduced) supposing that the subject can modify the space E of empirical values (enlarging it appropriately), and if he/she modifies also the random variable η taking its values in E. Let us demonstrate this idea by a simple example.

Consider a person who tosses a regular coin (probability 1/2 for H(ead) as well as for T(ail)). This person will be the system $\mathcal{S}$ and the result of the toss will be the actual internal state of $\mathcal{S}$. The subject is a colleague of the first person and the decision problem under consideration is to identify the actual intervnal state of $\mathcal{S}$, i.e., to identify the result of the toss made by $\mathcal{S}$. First, suppose that the subject has no empirical data in the sense of something coming from the world around independently of the subject's own activity. Hence, under our notation, $S = \{H,T\}$, $\sigma : \Omega \to S$ is such that

$$P(\{\omega \in \Omega : \sigma(\omega) = H\}) = P(\{\omega \in \Omega : \sigma(\omega) = T\}) = 1/2, \tag{24.10}$$

$D = \{H,T\}$, and $E = \{e\}$ (no empirical data and data taking only one constant value e turn to be the same within our model). Consequently, $\eta(\omega) = e$ for every $\omega \in \Omega$. Let $\lambda(s,d) = 0$, if

$s = d,\ \lambda(s,d) = 1$ elsewhere. Then the only decision functions which exist are the constant ones: $\delta_H(e) = H,\ \delta_T(e) = T$. Hence, for the expected loss of both these decision functions we obtain easily that

$$E(\lambda(\sigma(\cdot),\ \delta_H(\eta(\cdot)))) = E(\lambda(\sigma(\cdot), H)) = P(\{\omega \in \Omega : \sigma(\omega) = T\}) = 1/2, \qquad (24.11)$$

the same result being valid when δ_H replaced by δ_T.

However, the situation changes substantially, if the subject can do his/her own coin toss which is not statistically independent of that one made by $\mathcal{S}$, but brings some information about the result achieved by $\mathcal{S}$. To describe this case formally, let E be replaced by $E_0 = \{H, T\}(= S = D)$ and let $\eta_0 : \Omega \rightarrow E_0$ be such that

$$P(\{\omega \in \Omega : \eta_0(\omega) = H\}) = P(\{\omega \in \Omega : \eta_0(\omega) = T\}) = 1/2, \qquad (24.12)$$

but also

$$P(\{\omega \in \Omega : \sigma(\omega) = \eta_0(\omega)\}) > 1/2 \qquad (24.13)$$

hold together. Let $\delta_0 : E_0 \rightarrow D$ be the identity on E_0, so that $\delta_0(\eta_0(\omega)) = \eta_0(\omega)$ for every $\omega \in \Omega$. We obtain that

$$E(\lambda(\sigma(\cdot),\ \delta_0(\eta_0(\cdot)))) = P(\{\omega \in \Omega : \sigma(\omega) \neq \eta_0(\omega)\}) < 1/2 \qquad (24.14)$$

holds, hence, δ_0 is better than both δ_H and δ_T.

Obviously, the value $E\lambda^{\star}$ defined in (24.6), when taken as the degree of quality of the decision function δ, depends ultimately on the apriori probability distribution, i. e., on the random variable σ. Aiming to eliminate this dependence, we can apply the well-known minimax principle or the worst-case analysis principle. Set, for each $s \in S$,

$$E_{s,\delta}(\lambda) = E\lambda(s, \delta(\eta(\cdot))) = \int_{-\infty}^{\infty} \lambda(s, \delta(\eta(\omega)))dP \qquad (24.15)$$

and consider the supremum of these values for s ranging over S, i. e., set

$$E_{\delta}^{sup}(\lambda) = sup_{s \in S} E_{s,\delta}(\lambda). \qquad (24.16)$$

Taking $E_{\delta}^{sup}(\lambda)$ as a real-valued characteristic of the decision function δ, each two decision functions can be compared with respect to these values. A decision function δ_0 is optimal (the best) with respect to the minimax principle, if the relation

$$E_{\delta_0}^{sup}(\lambda) = inf\{E_{\delta}^{sup}(\lambda) : \delta : E \rightarrow D\} \qquad (24.17)$$

is valid. This value can be either reached or at least approximated up to a given $\varepsilon > 0$, at least in the implicite non-constructive sense. The problem with the minimax approach consists in the fact that the value $E_{\delta}^{sup}(\lambda)$ depends on one value $s \in S$, which may be very far from being a typical or at least more or less possible actual state of the system in question, when applying the decision function δ.

In the model outlined above, the size or the degree of uncertainty (i.e., randomness, in our case) was quantified using probability measure, defined as real-valued σ-additive normalized set function ascribing real numbers from the unit interval to (some) subsets of the space Ω of all elementary events under consideration. Keeping the idea that degrees of randomness are defined by sizes of (some) subsets of Ω, let us abandon the assumption that the values ascribed to these subsets are real numbers and let us take into consideration also non-numerical values from a set equipped by a structure weaker than the structures definable over the unit interval of real numbers. E.g., degrees of randomness from a partially ordered set or lattice-valued degrees may be taken into consideration. The operation of addition of real numbers will be replaced by that of supremum, definable in partially ordered sets and in lattices. Pursuing this way of reasoning in more detail, we will analyze, in the next chapters, whether, and in which sense and degree, the model of statistical decision functions, briefly sketched in this chapter, can be modified to the case when probability measure is replaced by a lattice-valued possibilistic measure.

25 Possibilistic Decision Functions

In this chapter our aim will be to reconsider, again, the problem of decision making under uncertainty as introduced in Chapter 24, but this time with the phenomenon of uncertainty (randomness) formally described and processed, and with the degrees of uncertainty quantified, by the tools offered by lattice-valued possibilistic measures and variables. Hence, let the symbols S (the set of possible interval states of a system, hypotheses, solutions, ...), D (the set of possible decisions being at the subject's disposal), and E (the set of possible values of empirical data or observations) keep their meanings and the intuition behind as introduced above in Chapter 24. Also decision functions keep their former notation and meaning, so that decision function δ is a mapping which takes E into D. On the other side, loss function will be re-defined in a way enabling to quantify the loss suffered, when the actual internal state is s and the decision d is taken, by a non-numerical value, as a rule, by the value from a complete lattices.

Given a complete lattice $\mathcal{T} = \langle T, \leq \rangle$, a $\mathcal{T}$-valued loss function is a mapping ρ which takes the Cartesian product $S \times D$ into T, i.e., $\rho(s,d) \in T$ for every $s \in S$ and $d \in D$. Aiming to use the symbol λ for t-norms on T, we introduce here the symbol ρ for lattice-valued loss functions. Combining these notations together, we obtain that the loss suffered when the actual internal state is s, the subject observes the empirical value e and applies the decision function δ, can be denoted by $\rho(s, \delta(e))$, this time the value being an element of the complete lattice $\mathcal{T}$.

Recalling the example introduced in Chapter 24, when decision making consists in the identification of the actual internal state, and applying, again, the most simple two-valued loss function just with real numbers 0 and 1 replaced by $\mathbf{0}_{\mathcal{T}}$ and $\mathbf{1}_{\mathcal{T}}$ (the zero and the unit of the complete lattice $\mathcal{T}$), i.e., setting $S = D$ and $\rho(s,d) = \mathbf{0}_{\mathcal{T}}$, if $s = d$, $\rho(s,d) = \mathbf{1}_{\mathcal{T}}$ otherwise we can easily see that also in the case of lattice-valued loss functions a decision function δ minimizing the loss value $\rho(s, \delta(e))$ uniformly for each $s \in S$ does not exist (up to the trivial cases mentioned in Chapter 24) so that a weaker optimality criterion must be taken into consideration.

As in the case of statistical decision functions, let us implement the phenomenon of uncertainty (in the sense of randomness) into our model of decision making, but this time described, quantified, and processed by lattice-valued possibilistic measures, possibilistic variables and other tools related to them. To achieve this goal, let us fix a $\mathcal{T}$-possibilistic space $\langle \Omega, \mathcal{A}, \Pi \rangle$, where $\mathcal{A}$ is a nonempty ample field of subsets of a nonempty space Ω and Π is a complete $\mathcal{T}$-valued possibilistic measure on $\mathcal{A}$, i. e., $\mathcal{T}$ is a complete lattice and $\Pi(\bigcup \mathcal{R}) = \bigvee \{\Pi(A) : A \in \mathcal{R}\}$ for any $\emptyset \neq \mathcal{R} \subset \mathcal{A}$. Moreover, let us suppose that

(i) the actual internal state s is the value taken by an S-valued possibilistic variable σ defined on $\langle \Omega, \mathcal{A}, \Pi \rangle$, and

(ii) the empirical value e, being at the subject's disposal when choosing a decision, is the value taken by an E-valued possibilistic variable η defined on $\langle \Omega, \mathcal{A}, \Pi \rangle$.

In both the cases we suppose, for the sakes of simplicity, that the ample fields over S and E are the power-sets $\mathcal{P}(S)$ and $\mathcal{P}(E)$. Hence, we suppose that $\sigma : \Omega \to S$ and $\eta : \Omega \to E$ are mappings such that, for every $S_0 \subset S$ and every $E_0 \subset E$, the relations

$$\{\omega \in \Omega : \sigma(\omega) \in S_0\} \in \mathcal{A}, \ \{\omega \in \Omega : \eta(\omega) \in E_0\} \in \mathcal{A} \tag{25.1}$$

are valid.

Under these notations and assumptions, the loss suffered when ρ is the lattice-valued loss function, δ is the decision function and σ, η are the possibilistic variables just defined, becomes a T-valued function taking Ω into T, its value being, for $\omega \in \Omega$,

$$\rho(\sigma(\omega), \ \delta(\eta(\omega))). \tag{25.2}$$

Let us prove that it is a T-valued possibilistic variable supposing that the ample field $\mathcal{Z}_T$ over T is identical with the power-set $\mathcal{P}(T)$ (consequently, the mapping defined in (25.2) is then a T-valued possibilistic variable also for every ample field $\mathcal{Z}_T \subset \mathcal{P}(T)$).

Indeed, given $d \in D$ and $t \in T$, set

$$\delta^{-1}(d) = \{e \in E : \delta(e) = d\}. \tag{25.3}$$

$$R(t) = \{\langle s, d\rangle \in S \times D : \rho(s, d) = t\}. \tag{25.4}$$

Consequently

$$\begin{aligned}
& \{\omega \in \Omega : \rho(\sigma(\omega),\ \delta(\eta(\omega))) = t\} = \\
= \ & \{\omega \in \Omega : \langle\sigma(\omega),\ \delta(\eta(\omega))\rangle \in R(t)\} = \\
= \ & \bigcup_{\langle s,d\rangle \in R(t)} \{\omega \in \Omega : \sigma(\omega) = s,\ \delta(\eta(\omega)) = d\} = \\
= \ & \bigcup_{\langle s,d\rangle \in R(t)} (\{\omega \in \Omega : \sigma(\omega) = s\} \cap \{\omega \in \Omega : \delta(\eta(\omega)) = d\}) = \\
= \ & \bigcup_{\langle s,d\rangle \in R(t)} \left(\{\omega \in \Omega : \sigma(\omega) = s\} \cap \bigcup_{e \in \delta^{-1}(d)} \{\omega \in \Omega : \eta(\omega) = e\} \right).
\end{aligned} \tag{25.5}$$

As the sets $\{\omega \in \Omega : \sigma(\omega) = s\}$ and $\{\omega \in \Omega : \eta(\omega) = e\}$ are in $\mathcal{A}$ for every $s \in S$ and $e \in E$, due to (25.1), and $\mathcal{A}$ is closed with respect to arbitrary intersections and unions, also the set

$$\{\omega \in \Omega : \rho(\sigma(\omega),\ \delta(\eta(\omega))) = t\} \tag{25.6}$$

is in $\mathcal{A}$ for every $t \in T$. Hence, for each $B \subset T$ we obtain that the set

$$\{\omega \in \Omega : \rho(\sigma(\omega),\ \delta(\eta(\omega))) \in B\} = \bigcup_{t \in B} \{\omega \in \Omega : \rho(\sigma(\omega),\ \delta(\eta(\omega))) = t\} \tag{25.7}$$

is also in $\mathcal{A}$, so that the mapping $\rho^\star$ (or $\rho^\star(\sigma, \eta, \delta)$, to explicitate all its components), defined by

$$\rho^\star(\omega) = \rho(\sigma(\omega),\ \delta(\eta(\omega))) \tag{25.8}$$

for every $\omega \in \Omega$, is a $\mathcal{T}$-possibilistic variable defined on the T-possibilistic space $\langle \Omega, \mathcal{A}, \Pi\rangle$.

Hence, like as in the case of statistical decision functions, we can define the expected value of the T-valued possibilistic loss function $\rho^\star$ with respect to a t-norm λ on T, setting

$$E_\lambda \rho^\star = \bigvee_{t \in T} \lambda(t, \Pi(\{\omega \in \Omega : \rho^\star(\omega) \geq t\})). \tag{25.9}$$

and using this value as a global characteristic of the quality of the decision function δ (we will write $E_\lambda \rho^\star_\delta$ in what follows, to make the role of δ explicit). Due to the conditions imposed on $\mathcal{T} = \langle T, \leq \rangle$ this value is always defined and the decision function δ_1 is taken as at least as good as (better than, resp.) a decision function δ_2 w. r. to σ, η, ρ, and λ, if the inequality $E_\lambda \rho^\star_{\delta_1} \leq E_\lambda \rho^\star_{\delta_2} (E_\lambda \rho^\star_{\delta_1} < E_\lambda \rho^\star_{\delta_2}$, resp.) holds. Obviously, contrary to the case of real-valued loss functions, some pairs δ_1, δ_2 of decision functions may be incomparable w. r. to the expected values. As $\mathcal{T}$ is complete lattice, the value

$$E^{inf}_\lambda \rho^\star = \bigwedge_{\delta: E \to D} E_\lambda \rho^\star_\delta = \bigwedge_{\delta: E \to D} \bigvee_{t \in T} \lambda[t, \Pi(\{\omega \in \Omega : \rho(\sigma(\omega),\ \delta(\eta(\omega))) \geq t\})] \tag{25.10}$$

is defined in T, consequently, due to elementary properties of supremum operation in $\mathcal{T} = \langle T, \leq \rangle$, for every $t_0 > E^{inf}_\lambda \rho^\star$ there exists a decision function $\delta_0 : E \to D$ such that the inequality $E_\lambda \rho^\star_{\delta_0} < t_0$ is valid. However, in general, a decision function $\delta_1 : E \to D$ such that $E_\lambda \rho^\star_{\delta_1} = E^{inf}_\lambda \rho^\star$ need not exist, as the following simple example proves.

Let $\langle \Omega, \mathcal{A}, \Pi\rangle$ be a $\mathcal{T}$-possibilistic space, let $S = D = \{s_1, s_2\}$, let $E = \{e\}$ be the degenerated observational space so that no empirical information is at the subject's disposal, as $\eta(\omega) = e$ for every $\omega \in \Omega$.

Let $\rho(s,d) = \mathbf{0}_{\mathcal{T}}$, if $s = d$, $\rho(s,d) = \mathbf{1}_{\mathcal{T}}$ otherwise. Let

$$\Pi(\{\omega \in \Omega : \sigma(\omega) = s_i\}) = t_i > \mathbf{0}_{\mathcal{T}} \tag{25.11}$$

for both $i = 1,2$, let $t_1 \wedge t_2 = \mathbf{0}_{\mathcal{T}}$. Such a possibilistic measure can be easily defined, take, e. g., $\mathcal{T} = \langle \mathcal{P}(/\Omega), \subset \rangle$, i. e., $\mathcal{T}$ is the complete lattice (complete Boolean algebra, as a matter of fact) of all subsets of Ω partially ordered by the set inclusion, and define Π as the identity of $\mathcal{A}$. Then

$$\begin{aligned} & \Pi(\{\omega \in \Omega : \sigma(\omega) = s_1\}) \wedge \Pi(\{\omega \in \Omega : \sigma(\omega) = s_2\}) = \\ = \ & \{\omega \in \Omega : \sigma(\omega) = s_1\} \cap \{\omega \in \Omega : \sigma(\omega) = s_2\} = \emptyset = \mathbf{0}_{\mathcal{T}}, \end{aligned} \tag{25.12}$$

even if the sets $\{\omega \in \Omega : \sigma(\omega) = s_1\}$ are nonempty for both $i = 1,2$.

As E is a singleton, only the two decision functions are possible: $\delta_{s_1}(e) = s_1$ and $\delta_{s_2}(e) = s_2$, so that $\delta_{s_i}(\eta(\omega)) = s_i$ for both $i = 1,2$ and every $\omega \in \Omega$. Consequently,

$$\begin{aligned} \rho(\sigma(\omega),\ \delta_{s_1}(\eta(\omega))) = \rho(\sigma(\omega), s_1) = \mathbf{0}_{\mathcal{T}},\ \text{if } \rho(\omega) = s_1, \\ \rho(\sigma(\omega),\ \delta_{s_1}(\eta(\omega))) = \rho(\sigma(\omega), s_1) = \mathbf{1}_{\mathcal{T}},\ \text{if } \rho(\omega) = s_2, \end{aligned} \tag{25.13}$$

and dually for δ_{s_2}. So, given a t-norm λ on T,

$$\begin{aligned} E_\lambda \rho^\star_{\delta_1} &= \bigvee_{t \in T} \lambda[t, \Pi(\{\omega \in \Omega : \rho(\sigma(\omega),\ \delta_{s_1}(\eta(\omega))) \geq t\})] \leq \\ &\leq \bigvee_{t \in T} [t \wedge \Pi(\{\omega \in \Omega : \rho(\sigma(\omega),\ \delta_{s_1}(\eta(\omega))) \geq t\})] = \\ &= \bigvee_{t \in T} [t \wedge \Pi(\{\omega \in \Omega : \rho(\sigma(\omega), s_1) \geq t\})] = \\ &= \mathbf{1}_{\mathcal{T}} \wedge \Pi(\{\omega \in \Omega : \sigma(\omega) = s_2\}) = \mathbf{1}_{\mathcal{T}} \wedge t_2 = t_2 \end{aligned} \tag{25.14}$$

and, analogously, $E_\lambda \rho^\star_{\delta_2} \leq t_1$ holds, hence, the relation

$$E^{inf}_\lambda \rho^\star = (E_\lambda \rho^\star \delta_1) \wedge (E_\lambda \rho^\star \delta_2) = t_1 \wedge t_2 = \mathbf{0}_{\mathcal{T}} \tag{25.15}$$

follows, but the value $E^{inf}_\lambda \rho^\star$ is reachable neither by δ_1 nor by δ_2.

26 Classifications of Possibilistic Decision Function Based on the Minimax Principle

The formalization of the idea of possibilistic decision functions, as introduced in the last Chapter, has been inspired by the bayesian approach to statistical decision functions. In this case, the actual state of the system under consideration is supposed to be the value taken by a random (or possibilistic) variable and the greatest portion of the critical argumentation related to bayesian statistical decision functions can be applied also to the possibilistic case. So, it may be of interest to see, whether also some alternative models of statistical decision functions, based on the minimax or the worst-case "pessimistic" principle, could be translated into the language of possibilistic measures and possibilistic decision functions.

Let us recall that, under the notations and conditions introduced in Chapter 25, the loss suffered when $\sigma(\omega)$ is the actual state of the system, $\eta(\omega)$ is the empirical value being at the subject's disposal, and δ is the decision function which he/she applies, reads as $\rho(\sigma(\omega),\ \delta(\eta(\omega)))$, and it is a $\mathcal{T}$-valued possibilistic variable defined on the possibilistic space $\langle \Omega, \mathcal{A}, \Pi \rangle$. The expected value of this possibilistic variable, taken with respect to the fixed t-norm λ on T, is then considered as a T-valued criterion of quality of the decision function δ, our aim being to choose δ in such a way that this expected value would be as small as possible (in the sense of the partial ordering on T defined by the complete lattice $\mathcal{T} = \langle T, \leq \rangle$ under consideration). This criterion will be denoted by χ^B_σ (B for bayesian with respect to the apriori possibilistic variable σ), so that

$$\chi^B_\sigma(\delta) = E_\lambda(\rho(\sigma(\cdot),\ \delta(\eta(\cdot)))) = \bigvee_{t\in T} \lambda[t, \Pi(\{\omega \in \Omega : \rho(\sigma(\omega),\ \delta(\eta(\omega))) \geq t\})]. \tag{26.1}$$

One way how to introduce the minimax principle into our reasoning reads as follows. Instead of the loss $\rho(s,d)$ suffered when s is the actual state and d is the decision we consider its "pessimistic" approximation from above, supposing that the loss $\bigvee_{s\in S}\rho(s,d)$ is suffered. This value depends only on d, hence, let us define the function $\hat{\rho} : D \to T$ by $\hat{\rho}(d) = \bigvee_{s\in S}\rho(s,d)$. If the set S of states is finite and if $\leq$ defines a linear ordering on T, e. g., if T is the unit interval of real numbers with their standard ordering, then there exists, given $d \in D$, a state $s_d \in S$ such that $\rho(s_d,d) \geq \rho(s,d)$ holds for each $s \in S$. Hence, s_d is the worst case with respect to d and the loss function ρ and we suppose, with respect to the "pessimistic" minimax principle, that just this s_d is the actual state of the system under consideration. So, with respect to the loss function $\hat{\rho}$, the loss suffered when $\eta(\omega)$ is the empirical value under consideration is $\hat{\rho}(\delta(\eta(\omega)))$ no matter which the actual state $s_0 \in S$ may be. The expected value of this T-valued possibilistic variable, again with respect to the t-norm λ on T, denoted by $\chi^{MM}(\delta)$ (MM for minimax), can serve as a T-valued degree of quality of the decision function δ. In symbols,

$$\begin{aligned}\chi^{MM}(\delta) &= E_\lambda\hat{\rho}(\delta(\eta(\omega))) = E_\lambda\left[\bigvee_{s\in S}\rho(s,\delta(\eta(\cdot)))\right] = \\ &= \bigvee_{t\in T}\lambda\left[t, \Pi\left(\left\{\omega\in\Omega : \left(\bigvee_{s\in S}\rho(s,\delta(\eta(\omega)))\right) \geq t\right\}\right)\right].\end{aligned} \tag{26.2}$$

Another criterion of quality of the decision function δ, obeying the minimax principle, reads as follows. Given $s \in S$, take the expected value of the loss function $\rho(s,\delta(\eta(\cdot)))$ and denote by $\chi^{MM}(\delta)$ the supremum of these expected values for s ranging over S, so that

$$\begin{aligned}\chi^{mm}(\delta) &= \bigvee_{s\in S} E_\lambda\rho(s,\delta(\eta(\cdot))) = \\ &= \bigvee_{s\in S}\left(\bigvee_{t\in T}\lambda[t,\Pi(\{\omega\in\Omega : \rho(s,\delta(\eta(\omega))) \geq t\})]\right).\end{aligned} \tag{26.3}$$

The relations among the three criteria $\chi^B_\sigma(\delta), \chi^{MM}(\delta)$ and $\chi^{mm}(\delta)$ are as follows.

Theorem 26.1 *Let S, D and E be as in Chapter 24, let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let λ be a t-norm on T, let $\rho : S \times D \to T$ be a T-valued loss function, let $\langle \Omega, \mathcal{A}, \Pi \rangle$ be a $\mathcal{T}$-possibilistic space. Then, for each possibilistic variables $\sigma : \Omega \to S,\ \eta : \Omega \to E$, and each decision function $\delta : E \to D$ the relation*

$$\chi^B_\sigma(\delta) = \chi^{mm}(\delta) \leq \chi^{MM}(\delta) \tag{26.4}$$

holds.

When proving this assertion, the following lemma will be of use.

Lemma 26.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\langle \Omega, \mathcal{A}, \Pi \rangle$ be a $\mathcal{T}$-possibilistic space, let $\mathcal{F}$ be a nonempty set of T-valued possibilistic variables on $\langle \Omega, \mathcal{A}, \Pi \rangle$, let λ be a t-norm on T. Then the relation*

$$\bigvee_{f\in\mathcal{F}}(E_\lambda f) \leq E_\lambda\left(\bigvee\mathcal{F}\right) \tag{26.5}$$

is valid, where $(\bigvee \mathcal{F})(\omega) = \bigvee_{f\in\mathcal{F}}(f(\omega))$ *for every* $\omega \in \Omega$*. If there exists, for every* $\omega \in \Omega$*, at most one* $f \in \mathcal{F}$ *such that* $f(\omega) > \mathbf{0}_{\mathcal{T}}$ *holds, then the equality is valid in (26.5).*

Proof. For each $f \in \mathcal{F}$ and each $\omega \in \Omega$ the inequality $f(\omega) \leq (\bigvee \mathcal{F})(\omega)$ is obvious, so that, for every $t \in T$, the inequality

$$\Pi(\{\omega \in \Omega : f(\omega) \geq t\}) \leq \Pi(\{\omega \in \Omega : \left(\bigvee \mathcal{F}\right)(\omega) \geq t\}) \tag{26.6}$$

holds,. Hence, the inequality

$$\begin{aligned} E_\lambda f &= \bigvee_{t\in T} \lambda[t, \Pi(\{\omega \in \Omega : f(\omega) \geq t\})] \leq \\ &\leq \bigvee_{t\in T} \lambda\left[t, \Pi\left(\left\{\omega \in \Omega : \left(\bigvee \mathcal{F}\right)(\omega) \geq t\right\}\right)\right] = E_\lambda\left(\bigvee \mathcal{F}\right) \end{aligned} \tag{26.7}$$

and, consequently, (26.5) immediately follow.

Let there exist, for every $\omega \in \Omega$, at most one $f \in \mathcal{F}$ such that $f(\omega) > \mathbf{0}_{\mathcal{T}}$ holds (in other terms, the supports of all variables in $\mathcal{F}$ are mutually disjoint). Then, for each $t \in T,\ t > \mathbf{0}_{\mathcal{T}}$,

$$\left\{\omega \in \Omega : \left(\bigvee \mathcal{F}\right)(\omega) \geq t\right\} = \bigcup_{f\in\mathcal{F}} \{\omega \in \Omega : f(\omega) \geq t\}, \tag{26.8}$$

so that

$$\Pi\left(\left\{\omega \in \Omega : \left(\bigvee \mathcal{F}\right)(\omega) \geq t\right\}\right) = \bigvee_{f\in\mathcal{F}} \Pi(\{\omega \in \Omega : f(\omega) \geq t\}), \tag{26.9}$$

follows. Moreover, for each $t > \mathbf{0}_{\mathcal{T}}$, if $\{\omega \in \Omega : (\bigvee \mathcal{F})(\omega) \geq t\} \neq \emptyset$, then there exists just one $f_t \in \mathcal{F}$ such that

$$\left\{\omega \in \Omega : \left(\bigvee \mathcal{F}\right)(\omega) \geq t\right\} = \{\omega \in \Omega : f_t(\omega) \geq t\}. \tag{26.10}$$

Hence, the inequalities

$$\begin{aligned} &\lambda\left[t, \Pi\left(\left\{\omega \in \Omega : \left(\bigvee \mathcal{F}\right)(\omega) \geq t\right\}\right)\right] = \lambda[t, \Pi(\{\omega \in \Omega : f_t(\omega) \geq t\})] \leq \\ \leq &\bigvee_{f\in\mathcal{F}} \lambda[t, \Pi(\{\omega \in \Omega : f(\omega) \geq t\})] \leq \\ \leq &\bigvee_{f\in\mathcal{F}} \bigvee_{t\in T} \lambda[t, \Pi(\{\omega \in \Omega : f(\omega) \geq t\})] = \bigvee_{f\in\mathcal{F}} (E_\lambda f) \end{aligned} \tag{26.11}$$

and

$$E_\lambda\left(\bigvee \mathcal{F}\right) = \bigvee_{t\in T} \lambda\left[t, \Pi\left(\left\{\omega \in \Omega : \left(\bigvee \mathcal{F}\right)(\omega) \geq t\right\}\right)\right] \leq \bigvee_{f\in\mathcal{F}} (E_\lambda f) \tag{26.12}$$

follow. So, the equality in (26.5) and the Lemma as a whole are proved. □

Proof of Theorem 26.1. Set, for each $s \in S$ and each $\omega \in \Omega$,

$$\hat{\rho}_s(\omega) = \rho(s, \delta(\eta(\omega))),\ \text{if } \sigma(\omega) = s,\ \hat{\rho}_s(\omega) = \mathbf{0}_{\mathcal{T}},\ \text{if } \sigma(\omega) \neq s. \tag{26.13}$$

Hence, for each $\omega \in \Omega$,

$$\hat{\rho}_s(\omega) \leq \rho(s, \delta(\eta(\omega))), \tag{26.14}$$

$$\rho(\sigma(\omega),\ \delta(\eta(\omega))) = \bigvee_{s\in S} \hat{\rho}_s(\omega), \tag{26.15}$$

moreover, for every $\omega \in \Omega$ there exists at most one $s \in S$ such that $\hat{\rho}_s(\omega) > \mathbf{0}_{\mathcal{T}}$ holds. Applying Lemma 26.1 to $\mathcal{F} = \{\hat{\rho}_s : s \in S\}$ we obtain that □

$$\bigvee_{s \in S} (E_\lambda \hat{\rho}_s(\cdot)) = E_\lambda \left(\bigvee_{s \in S} \hat{\rho}_s(\cdot) \right), \tag{26.16}$$

so that the relation

$$\begin{aligned} \chi_\sigma^B(\delta) &= E_\lambda \rho(\sigma(\cdot),\ \delta(\eta(\cdot))) = E_\lambda \left(\bigvee_{s \in S} \hat{\rho}_s(\cdot) \right) = \\ &= \bigvee_{s \in S} (E_\lambda \hat{\rho}_s(\cdot)) \leq \bigvee_{s \in S} (E_\lambda \rho(s, \delta(\eta(\cdot)))) = \\ &= \chi^{mm}(\delta) \end{aligned} \tag{26.17}$$

easily follows. Applying (26.5) again, now to $\mathcal{F} = \{\rho(s, \delta(\eta(\cdot))) : s \in S\}$, we obtain that

$$\chi^{mm}(\delta) \leq E_\lambda \left(\bigvee_{s \in S} \rho(s, \delta(\eta(\cdot))) \right) = \chi^{MM}(\delta) \tag{26.18}$$

holds. The assertion of Theorem 26.1 is proved.

The equality $\chi^{mm}(\delta) = \chi^{MM}(\delta)$ does not hold in general, as the following example demonstrates. Let all the sets $S = \{s_1, s_2\},\ D = \{d_1, d_2\}$, and $E = \{e_1, e_2\}$ consist of two elements, let the decision function δ be such that $\delta(e_i) = d_i$, for both $i = 1, 2$.

Denote, again for both $i = 1, 2$

$$\Pi(\{\omega \in \Omega : \eta(\omega) = e_i\}) = t_i \tag{26.19}$$

and suppose that $\mathbf{0}_{\mathcal{T}} < t_1, t_2 < \mathbf{1}_{\mathcal{T}}$ and $t_1 \wedge t_2 = \mathbf{0}_{\mathcal{T}}$ hold (the relation $t_1 \vee t_2 = \mathbf{1}_{\mathcal{T}}$ easily follows, as Π is a $\mathcal{T}$-possibilistic measure). Let the loss function ρ be such that

$$\rho(s_1, d_1) = \rho(s_2, d_2) = t_2,\ \rho(s_1, d_2) = \rho(s_2, d_1) = t_1. \tag{26.20}$$

Then, for each $\omega \in \Omega$,

$$\rho(s_1, \delta(\eta(\omega))) \vee \rho(s_2, \delta(\eta(\omega))) = t_1 \vee t_2 = \mathbf{1}_{\mathcal{T}}, \tag{26.21}$$

so that $\chi^{MM}(\delta) = \mathbf{1}_{\mathcal{T}}$. However,

$$\begin{aligned} \{\omega \in \Omega : \rho(s_i, \delta(\eta(\omega))) = t_2\} &= \{\omega \in \Omega : \eta(\omega) = e_i\}, \\ \{\omega \in \Omega : \rho(s_i, \delta(\eta(\omega))) = t_1\} &= \{\omega \in \Omega : \eta(\omega) \neq e_i\}, \end{aligned} \tag{26.22}$$

consequently,

$$\begin{aligned} & E_\lambda \rho(s_1, \delta(\eta(\cdot))) = \bigvee_{t \in T} \lambda[t, \Pi(\{\omega \in \Omega : \rho(s_1, \delta(\eta(\omega))) \geq t\})] = \\ &= \lambda[t_1, \Pi(\{\omega \in \Omega : \eta(\omega) = e_2\})] \vee \lambda[t_2, \Pi(\{\omega \in \Omega : \eta(\omega) = e_1\})] = \\ &= \lambda[t_1, t_2] \vee \lambda[t_2, t_1] \leq (t_1 \wedge t_2) \vee (t_2 \wedge t_1) = \mathbf{0}_{\mathcal{T}}. \end{aligned} \tag{26.23}$$

The proof that $E_\lambda \rho(s_2, \delta(\eta(\cdot))) = \mathbf{0}_{\mathcal{T}}$ is quite analogous, so that

$$\begin{aligned} \chi^{mm}(\delta) &= (E_\lambda \rho(s_1, \delta(\eta(\cdot)))) \vee (E_\lambda \rho(s_2, \delta(\eta(\cdot)))) = \\ &= \mathbf{0}_{\mathcal{T}} < \mathbf{1}_{\mathcal{T}} = \chi^{MM}(\delta). \end{aligned} \tag{26.24}$$

Theorem 26.2 *Let the notations and conditions of Theorem 26.1 hold, let the loss function* ρ *take only the values* $\mathbf{0}_{\mathcal{T}}$ *or* $\mathbf{1}_{\mathcal{T}}$*. Then, for each decision function* $\delta : E \to D$*,*

$$\chi^{mm}(\delta) = \chi^{MM}(\delta). \tag{26.25}$$

Proof. For every $s \in S$,

$$\begin{aligned} E_\lambda \rho(s, \delta(\eta(\cdot))) &= \bigvee_{t \in T} \lambda[t, \Pi(\{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) \geq t\})] = \\ &= \lambda[\mathbf{1}_{\mathcal{T}}, \Pi(\{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\})] = \\ &= \Pi(\{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\}), \end{aligned} \tag{26.26}$$

as $\rho(s, \delta(\eta(\omega)))$ takes only the values $\mathbf{0}_{\mathcal{T}}$ or $\mathbf{1}_{\mathcal{T}}$ on Ω. Hence,

$$\begin{aligned} \chi^{mm}(\delta) &= \bigvee_{s \in S} E_\lambda \rho(s, \delta(\eta(\cdot))) = \\ &= \bigvee_{s \in S} \Pi(\{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\}) = \\ &= \Pi\left(\bigcup_{s \in S} \{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\}\right) = \\ &= \Pi\left(\left\{\omega \in \Omega : \bigvee_{s \in S} \rho(s, \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\right\}\right) = \\ &= \lambda\left[\mathbf{1}_{\mathcal{T}}, \Pi\left(\left\{\omega \in \Omega : \bigvee_{s \in S} \rho(s, \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\right\}\right)\right] = \\ &= \bigvee_{t \in T} \lambda\left[t, \Pi\left(\left\{\omega \in \Omega : \bigvee_{s \in S} \rho(s, \delta(\eta(\omega))) \geq t\right\}\right)\right] = \\ &= E_\lambda\left(\bigvee_{s \in \mathcal{S}} \rho(s, \delta(\eta(\cdot)))\right) = \chi^{MM}(\delta), \end{aligned} \tag{26.27}$$

as $\bigvee_{s \in S} \rho(s, \cdot)$ is also a mapping which takes $S \times D$ into $\{\mathbf{0}_{\mathcal{T}}, \mathbf{1}_{\mathcal{T}}\}$. The assertion is proved. □

Corollary 26.1 *Let the notations and conditions of Theorem 25.1 hold, let* S *contain at least two elements, let* $S = D$ *and let* $\rho : S \times D \to T$ *be such that* $\rho(s, d) = \mathbf{0}_{\mathcal{T}}$*, if* $s = d$*, and* $\rho(s, d) = \mathbf{1}_{\mathcal{T}}$ *otherwise, i.e., if* $s \neq d$*. Then, for every decision function* $\delta : E \to D$*,*

$$\chi^{mm}(\delta) = \chi^{MM}(\delta) = \mathbf{1}_{\mathcal{T}}. \tag{26.28}$$

Proof. The conditions of Theorem 26.2 are obviously satisfied, so that only the relation $\chi mm(\delta) = \mathbf{1}_{\mathcal{T}}$ remains to be proved. Applying (26.27) we obtain that

$$\begin{aligned} \chi^{mm}(\delta) &= \Pi\left(\bigcup_{s \in S} \{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\}\right) = \\ &= \Pi\left(\bigcup_{s \in S} \{\omega \in \Omega : \delta(\eta(\omega)) \neq s\}\right) = \\ &= \Pi\left(\bigcup_{s \in S} (\Omega - \{\omega \in \Omega : \delta(\eta(\omega)) = s\})\right) = \\ &= \Pi\left(\Omega - \bigcap_{s \in S} \{\omega \in \Omega : \delta(\eta(\omega)) = s\}\right) = \Pi(\Omega) = \mathbf{1}_{\mathcal{T}}, \end{aligned} \tag{26.29}$$

as S contains at least two elements s_1, s_2 and for no $\omega \in \Omega$, $\delta(\eta(\omega))$ can take both these values simultaneously. The assertion is proved. □

Let us analyze, in more detail, the case when $\chi^{mm}(\delta) < \mathbf{1}_{\mathcal{T}}$ holds for some decision function δ. Under the conditions of Theorem 26.2 we obtain, applying (26.27), that

$$\begin{aligned} \chi^{mm}(\delta) &= \bigvee_{s \in S} \Pi(\{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\}) = \\ &= \Pi\left(\bigcup_{s \in S} \{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\}\right) = \\ &= \Pi\left(\Omega - \bigcap_{s \in S} \{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{0}_{\mathcal{T}}\}\right) < \mathbf{1}_{\mathcal{T}} \end{aligned} \tag{26.30}$$

holds. Consequently, denoting by t_0 the value $\chi^{mm}(\delta)$, the inequality

$$\Pi(\{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\}) \leq t_0 < \mathbf{1}_{\mathcal{T}} \tag{26.31}$$

is valid for every $s \in S$, moreover, setting

$$t_1 = \Pi\left(\bigcap_{s \in S} \{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{0}_{\mathcal{T}}\}\right), \tag{26.32}$$

we obtain that the inequality

$$\mathbf{0}_{\mathcal{T}} < t_1 \leq \Pi(\{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{0}_{\mathcal{T}}\}) \tag{26.33}$$

holds for each $s \in S$. If the complete lattice $\mathcal{T} = \langle T, \leq \rangle$ defines a linear ordering on T, i.e., if $t_1 \leq t_2$ or $t_2 \leq t_1$ holds for each $t_1, t_2 \in T$ (as it is the case of the unit interval of real numbers equipped by their standard linear ordering), then the identity $\Pi(\Omega - A) = \mathbf{1}_{\mathcal{T}}$ easily follows for every $A \in \mathcal{A}$ such that $\Pi(A) < \mathbf{1}_{\mathcal{T}}$ holds. Indeed, it is the only way to satisfy the relation

$$\Pi(\Omega) = \mathbf{1}_{\mathcal{T}} = \Pi(A) \vee \Pi(\Omega - A), \tag{26.34}$$

obviously valid for each possibilistic measure Π taking the ample field $\mathcal{A} \subset \mathcal{P}(\Omega)$ into T. In this particular case, (26.32) yields that

$$\begin{aligned} t_1 &= \Pi\left(\bigcap_{s \in S} \{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{0}_{\mathcal{T}}\}\right) = \\ &= \Pi(\{\omega \in \Omega : \rho(s, \delta(\eta(\omega))) = \mathbf{0}_{\mathcal{T}}\}) = \mathbf{1}_{\mathcal{T}} \end{aligned} \tag{26.35}$$

holds for each $s \in S$.

27 Possibilistic Decision Functions for State Identification Under Bayesian Classification

In this Chapter, we will go on with our effort to analyze, in more detail, the most simple lattice-valued possibilistic decision functions related to the problem of identification of the actual state of the system under investigation. Hence, as above, we suppose that the space D of decisions is identical with the space S of possible states and that the most simple two-valued loss function is taken into consideration, i.e., that $\rho(s, d) = \mathbf{0}_{\mathcal{T}}$, if $s = d$, and $\rho(s, d) = \mathbf{1}_{\mathcal{T}}$, if $s \neq d$. In the last chapter we analyzed such possibilistic decision functions according to the "pessimistic" minimax principle based on the worst-case analysis, and we obtained some more or less elementary results according to which only under rather strong optimistic conditions the loss $\chi^{mm}(\delta)(= \chi^{MM}(\delta)$ in this case) can be kept below the maximal possible value, i.e., below $\mathbf{1}_{\mathcal{T}}$. Let us analyze, now, these possibilistic decision functions with respect to the Bayesian principle, i.e., supposing that the actual state of the system is defined by the value taken by an S-valued possibilistic variable defined on the fixed $\mathcal{T}$-possibilistic space $\langle \Omega, \mathcal{A}, \Pi \rangle$ (the apriori possibilistic distribution) and that this fact can be, more or less sophistically, taken into consideration when choosing and optimizing a possibilistic decision function δ in order to solve the identification problem sketched above.

As introduced in Chapter 25, the quality of a decision function $\delta : E \rightarrow D$ is defined by the expected value of the loss function $\rho : S \times D \rightarrow T$ with respect to the possibilistic variables σ and η, and with respect to the fixed t-norm λ on T. So, applying (25.9) and restricting ourselves to the case when $S = D$ and ρ takes only the values $\mathbf{0}_{\mathcal{T}}$ or $\mathbf{1}_{\mathcal{T}}$ in the way defined above, we obtain that

$$\begin{aligned}
& \chi^B_\sigma(\delta) = E_\lambda \rho^\star = \bigvee_{t \in T} \lambda[t, \Pi(\{\omega \in \Omega : \rho^\star(\omega) \geq t\})] = \\
= \;& \bigvee_{t \in T} \lambda[t, \Pi(\{\omega \in \Omega : \rho(\sigma(\omega),\ \delta(\eta(\omega))) \geq t\})] = \\
= \;& \lambda[\mathbf{0}_{\mathcal{T}}, \Pi(\{\omega \in \Omega : \rho(\sigma(\omega),\ \delta(\eta(\omega))) \geq \mathbf{0}_{\mathcal{T}}\})] \vee \\
\vee \;& \lambda[\mathbf{1}_{\mathcal{T}}, \Pi(\{\omega \in \Omega : \rho(\sigma(\omega),\ \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\})] = \\
= \;& \lambda[\mathbf{0}_{\mathcal{T}}, \mathbf{1}_{\mathcal{T}}] \vee \Pi(\{\omega \in \Omega : \rho(\sigma(\omega),\ \delta(\eta(\omega))) = \mathbf{1}_{\mathcal{T}}\})] = \\
= \;& \Pi(\{\omega \in \Omega : \sigma(\omega) \neq \delta(\eta(\omega))\}),
\end{aligned} \tag{27.1}$$

applying the most elementary properties of t-norms. Hence, quite according to the intuition behind and to the case of statistical decision functions applied to the same decision problem, the expected loss is defined by the size of the set of those elementary events (those $\omega \in \Omega$), for which the decision function δ fails, i.e., wrongly claims that the actual state of the system under investigation is some $s_1 \in S$ different from the true actual state $s_0 \in S$. The only difference between the statistical and the possibilistic cases consists in different conditions imposed on the set functions quantifying the sizes of (some) subsets of Ω.

The following attributes will be related only to the specific two-valued loss functions and to decision functions occuring in our specific case of state-identification decision problems.

Decision function $\delta : E \rightarrow D$ is called *optimal* in $e \in E$, if the relation

$$\Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = \delta(e)\}) = \bigvee_{s \in S} \Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = s\}) \tag{27.2}$$

holds. Decision function δ is *(uniformly) optimal on* E, if (27.2) is valid for every $e \in E$. Decision function δ is *weakly optimal in* E, if there is no $s \in S$ such that the inequality

$$\Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = s\}) > \Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = \delta(e)\}) \tag{27.3}$$

would hold. Decision function δ is *(uniformly) weakly optimal on* E, if it is weakly optimal in every $e \in E$.

If the state space S is finite, then there always exists a decision function $\delta : E \rightarrow D (= S$, in our particular case) which is uniformly weakly optimal on E. Indeed, denote by A^e the set of all greatest elements among the values ascribed by the possibilistic measure Π to the subsets of Ω on which $\eta(\omega) = e$ is observed and $\sigma(\omega) = s$ is the actual state of the system under consideration. In symbols,

$$A^e = \{\Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = s\}) : s \in S\}^{\leq} \subset T. \tag{27.4}$$

Let S^e be defined by

$$S^e = \{s \in S : \Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = s\}) \in A^e\}. \tag{27.5}$$

As S is finite, the sets $A^e \subset T$ and, consequently, also $S^e \subset S$ are nonempty for each $e \in E$, so that, for each $e \in E$, a value $\delta(e) \in S^e$ can be chosen. The resulting mapping $\delta : E \rightarrow S$ is obviously a uniformly weakly optimal decision function on E.

Contrary to the case of weakly optimal decision function, if $e \in E$ is such that the set A^e, defined by (27.4), contains at least two elements, then obviously there is no decision function δ optimal in e. Indeed, let t_1, t_2 be two (different, hence, incomparable by the partial ordering relation $\leq$ on T) elements of A^e and, for both $i = 1, 2$, let $s_i \in S^e$ be such that

$$\Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = s_i\}) = t_i. \tag{27.6}$$

Suppose, in order to arrive at contradiction, that $\delta : E \to S$ is optimal in e. Then the inequality

$$t_i < \Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = \delta(e)\}) \tag{27.7}$$

must be valid for both $i = 1, 2$, but this contradicts the assumption that both t_1, t_2 are greatest (i.e., non-dominated) elements in the set of values taken by Π for the given $e \in E$ and for s ranging over S.

In the particular case when the complete lattice $\mathcal{T} = \langle T, \leq \rangle$ defines a linear ordering on T and when the set S of possible internal states of the system under consideration is finite, a uniformly optimal on E decision function δ_{opt} exists and minimizes the expected loss $\aleph_\sigma^B(\delta)$ over the space of all decision functions $\delta : E \to D(= S)$, as the following assertion claims.

Theorem 27.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice such that $\leq$ defines a linear ordering on T, let λ be a t-norm on T, let $\langle \Omega, \mathcal{A}, \Pi \rangle$ be a possibility space with $\mathcal{T}$-valued possibilistic measure Π on $\mathcal{A}$, let σ be an S-valued possibilistic variable on $\langle \Omega, \mathcal{A}, \Pi \rangle$, where S is a finite set of possible internal states of the system under consideration, let η be an E-valued possibilistic variable on $\langle \Omega, \mathcal{A}, \Pi \rangle$, where E is the set of all possible empirical values (observations), let the set D of decisions be identical with S, let the loss function $\rho : S \times D \to T$ be such that $\rho(s, d) = \mathbf{0}_{\mathcal{T}}$, if $s = d$, $\rho(s, d) = \mathbf{1}_{\mathcal{T}}$ if $s \neq d$. Let $\delta_{opt} : E \to D(= S)$ be defined in such a way that $\delta_{opt}(e) \in S^e$ holds for each $e \in E$, where S^e is defined by (27.5). Then δ_{opt} is a uniformly on E optimal decision function in the sense that (27.2) is valid for every $e \in E$, and for every decision function $\delta_0 : E \to D(= S)$ the inequality*

$$\chi_\sigma^B(\delta_0) = E_\lambda \rho(\sigma(\cdot),\ \delta_0(\eta(\cdot))) \geq E_\lambda \rho(\sigma(\cdot),\ \delta_{opt}(\eta(\cdot))) = \chi_\sigma^B(\delta_{opt}) \tag{27.8}$$

holds.

Proof. The set S being finite, also the set

$$\{\Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = s\}) : s \in S\} \tag{27.9}$$

defines, for each $e \in E$, a finite subset of T. As $\leq$ defines a linear ordering on T there exists, for every $e \in E$, $s_e \in S$ such that the relation

$$\Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = s_e\}) = \bigvee_{s \in S} \Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = s\}) \tag{27.10}$$

holds, consequently, the set S^e is nonempty for each $e \in E$ and a function δ_{opt} such that $\delta_{opt}(e) \in S^e$ holds for each $e \in E$ can be defined. It follows immediately that each such decision function δ_{opt} is uniformly optimal on E. Let us recall that under our conditions the set A^e defined by (27.4) is a singleton for each $e \in E$, but the set S^e may contain, in general, more elements mutually equivalent in the sense that the value $\Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = s\})$ is the same for all $s \in S$.

Due to (27.1), for any $\delta : E \to D(= S)$ the relation

$$chi_\sigma^B = \Pi(\{\omega \in \Omega : \sigma(\omega) \neq \delta(\eta(\omega))\}) \tag{27.11}$$

is valid. For every $e \in E$ we obtain that

$$\begin{aligned} &\Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = \delta_{opt}(e)\}) = \\ = &\bigvee_{s \in S} \Pi(\{\omega \in \Omega : \eta(\omega) = e,\ \sigma(\omega) = s\}) = \end{aligned}$$

$$
\begin{aligned}
&= \Pi\left(\bigcup_{s\in S}\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)=s\}\right)= \\
&= \Pi(\{\omega\in\Omega:\eta(\omega)=e\}), \qquad (27.12)
\end{aligned}
$$

as $\sigma(\omega)\in S$ holds for each $\omega\in\Omega$. Let $\delta_0 : E\rightarrow D(=S)$ and $e\in E$ be such that $\delta_0(e)$ is not in S^e. Consequently, the inequality

$$
\begin{aligned}
&\Pi(\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)=\delta_0(e)\})< \\
&< \bigvee_{s\in S}\Pi(\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)=s\})= \\
&= \Pi(\{\omega\in\Omega:\eta(\omega)=e\}) \qquad (27.13)
\end{aligned}
$$

follows from (27.12). However, the relation

$$
\begin{aligned}
\Pi(\{\omega\in\Omega:\eta(\omega)=e\}) &= \Pi(\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)=\delta_0(e)\})\cup \\
&\cup\ \{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)\neq\delta_0(e)\}= \\
&= \Pi(\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)=\delta_0(e)\})\vee \\
&\vee\ \Pi(\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)\neq\delta_0(e)\}) \qquad (27.14)
\end{aligned}
$$

follows from the fact that Π is a $\mathcal{T}$-possibilistic measure on $\mathcal{A}$. As $\leq$ is a linear ordering on T and (27.13) holds, (27.14) can be satisfied only when

$$
\Pi(\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)\neq\delta_0(e)\})=\Pi(\{\omega\in\Omega:\eta(\omega)=e\}) \qquad (27.15)
$$

holds for each $\delta_0 : E\rightarrow D(=S)$ and each $e\in E$ such that $\delta_0(e)$ is not in S^e. The inequality

$$
\Pi(\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)\neq\delta_{opt}(e)\})\leq\Pi(\{\omega\in\Omega:\eta(\omega)=e\}) \qquad (27.16)
$$

follows trivially from the set inclusion between the subsets of Ω in question. Hence, for each $\delta_0 : E\rightarrow D(=S)$ and each $e\in E$ the inequality

$$
\Pi(\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)\neq\delta_{opt}(e)\})\leq\Pi(\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)\neq\delta_0(e)\}) \qquad (27.17)
$$

obviously holds.

Combining the relation (27.17) together for different values $e\in E$, we obtain that for each $\delta_0 : E\rightarrow D(=S)$

$$
\begin{aligned}
\chi_\sigma^B(\delta_0) &= \Pi(\{\omega\in\Omega:\sigma(\omega)\neq\delta_0(\eta(\omega))\})= \\
&= \Pi\left(\bigcup_{e\in E}\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)\neq\delta_0(\eta(\omega))\}\right)= \\
&= \bigvee_{e\in E}\Pi(\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)\neq\delta_0(e)\})\geq \\
&\geq \bigvee_{e\in E}\Pi(\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)\neq\delta_{opt}(e)\})= \\
&= \Pi\left(\bigcup_{e\in E}\{\omega\in\Omega:\eta(\omega)=e,\ \sigma(\omega)\neq\delta_{opt}\eta(\omega)\}\right)= \\
&= \Pi(\{\omega\in\Omega:\sigma(\omega)\neq\delta_{opt}(\eta(\omega))\})=\chi_\sigma^B(\delta_{opt}). \qquad (27.18)
\end{aligned}
$$

The assertion is proved. □

28 Robustness of Possibilistic Decision Functions over Lattice-Valued Possibilistic Measures and Loss Functions

There are numerous decision problems under uncertainty when the demand of a robustness, imposed on decision functions applied when solving these problems, is quite intuitive and legitimate. It is to say that a "small enough" change of the values taken by the loss function in question, or these changes being restricted to the cases which occur "rather rarely" (the probability or possibility degree related to the occurence of such cases is "small enough", say) results in a "rather small" change of the expected value of the loss function under consideration; this expected value quantifies the quality of the applied decision function. This demand of robustness avoids from consideration loss functions and, consequently, decision functions with singularities of the kind that a very small change of conditions can cause great changes in the loss suffered when applying such decision functions (a very small mechanical tremor or an almost negligible increase of the temperature can involve an explosion with terrible consequences). Let us investigate, in more detail, the most simple and to the idea of robustness related properties of possibilistic decision functions over lattice-valued possibilistic measures. The following assertion, simplifying the operations with the expected values of $\mathcal{T}$-valued functions over $\mathcal{T}$-valued possibilistic spaces, may be of use in our further considerations (cf. [10]).

Lemma 28.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\langle \Omega, \mathcal{A}, \Pi \rangle$ be a $\mathcal{T}$-possibilistic space with complete $\mathcal{T}$-possibilistic measure Π, let λ be a convex t-norm on T, i.e., $\lambda(t, \bigvee_{s \in A} s) = \bigvee_{s \in A} \lambda(t, s)$ holds for each $t \in T$ and each $\emptyset \neq A \subset T$, let $f : \Omega \rightarrow T$ be a mapping such that the inclusion*

$$\{\{\omega \in \Omega : f(\omega) = t\} : t \in T\} \subset \mathcal{A} \tag{28.1}$$

is valid. Then the expected value $E_\lambda f(\cdot)$ satisfies the relation

$$E_\lambda f(\cdot) = \bigvee_{t \in T} \lambda[t, \Pi(\{\omega \in \Omega : f(\omega) = t\})]. \tag{28.2}$$

Proof. By definition and due to the properties of the measure Π we obtain that

$$\begin{aligned} E_\lambda f(\cdot) &= \bigvee_{t \in T} \lambda[t, \Pi(\{\omega \in \Omega : f(\omega) \geq t\})] = \\ &= \bigvee_{t \in T} \lambda \left[t, \bigvee_{t_1 \geq t} \Pi(\{\omega \in \Omega : f(\omega) = t_1\}) \right]. \end{aligned} \tag{28.3}$$

Each possibilistic measure is monotone with respect to set inclusion, so that the relations

$$\Pi(\{\omega \in \Omega : f(\omega) \geq t\}) \geq \Pi(\{\omega \in \Omega : f(\omega) = t\}) \tag{28.4}$$

and

$$\lambda[t, \Pi(\{\omega \in \Omega : f(\omega) \geq t\})] \geq \lambda[t, \Pi(\{\omega \in \Omega : f(\omega) = t\})] \tag{28.5}$$

are valid for each $t \in T$. Hence, also the inequality

$$E_\lambda f(\cdot) = \bigvee_{t \in T} \lambda[t, \Pi(\{\omega \in \Omega : f(\omega) = t\})] \tag{28.6}$$

immediately follows.

Let $t, t_1 \in T$ be such that $t_1 \geq t$ holds. Then we obtain that for each such t the inequality

$$\begin{aligned} &\lambda[t, \Pi(\{\omega \in \Omega : f(\omega) = t_1\})] \leq \lambda[t_1, \Pi(\{\omega \in \Omega : f(\omega) = t_1\})] \leq \\ \leq &\bigvee_{t_1 \in T} \lambda[t_1, \Pi(\{\omega \in \Omega : f(\omega) = t_1\})] \end{aligned} \tag{28.7}$$

is valid. Hence, also the inequality

$$\lambda \left[t, \bigvee_{t_1 \geq t} \Pi(\{\omega \in \Omega : f(\omega) = t_1\}) \right] = \lambda[t, \Pi(\{\omega \in \Omega : f(\omega) \geq t\})] \leq$$
$$\leq \bigvee_{t_1 \in T} \lambda[t, \Pi(\{\omega \in \Omega : f(\omega) = t_1\})] \tag{28.8}$$

holds for each $t \in T$. Consequently, we obtain that

$$\bigvee_{t \in T} \lambda[t, \Pi(\{\omega \in \Omega : f(\omega) \geq t\})] = E_\lambda f(\cdot) \leq$$
$$\leq \bigvee_{t_1 \in T} \lambda[t_1, \Pi(\{\omega \in \Omega : f(\omega) = t_1\})] \tag{28.9}$$

easily follows; combining (28.6) and (28.9) we complete the proof. □

Within the framework of statistical decision functions the expected values of loss functions are real numbers so that the changes resulting when replacing the loss function in question by another one can be quantified by the absolute value of the difference of the expected values of these loss functions. In the case of lattice-valued loss functions and possibilistic measures the expected values of loss functions are elements of the complete lattice $\mathcal{T} = \langle T, \leq \rangle$, so that the difference or distance between two elements of T must be appropriately defined. Let us take an inspiration from the idea of symmetric difference of two sets, defined in the particular case when $\mathcal{T} = \langle \mathcal{P}(X), \subset \rangle$ over a nonempty set X by

$$A \dot{-} B = (A - B) \cup (B - A) = (A \cap (X - B)) \cup (B \cap (X - A)) \tag{28.10}$$

for each $A, B \subset X$. The generalization to the case of a Boolean algebra is straighforward.

As there is no primary operation of complement in complete lattices, let us introduce the notion of (pseudo-) complement as follows. Let λ be a t-norm on T, where $\mathcal{T} = \langle T, \leq \rangle$ is a complete lattice, let $t \in T$. The λ-pseudo-complement $t^{\lambda,c}$ of t is defined by

$$t^{\lambda,c} = \bigvee \{s \in T : \lambda(s,t) = \mathbf{0}_\mathcal{T}\}. \tag{28.11}$$

Evidently, $t^{\lambda,c}$ is defined for each $t \in T$, as the set $\{s \in T : \lambda(s,t) = \mathbf{0}_\mathcal{T}\}$ contains at least the element $\mathbf{0}_\mathcal{T}$. If λ is a convex t-norm on T, then $\lambda(t, t^{\lambda,c}) = \mathbf{0}_\mathcal{T}$ holds for each $t \in T$. Indeed, given $t \in T$,

$$\lambda(t, t^{\lambda,c}) = \lambda \left(t, \bigvee \{s \in T : \lambda(s,t) = \mathbf{0}_\mathcal{T}\} \right) =$$
$$= \bigvee \{\lambda(t,s) : s \in T,\ \lambda(s,t) = \mathbf{0}_\mathcal{T}\} = \bigvee \{\mathbf{0}_\mathcal{T}\} = \mathbf{0}_\mathcal{T}. \tag{28.12}$$

In general, for each t-norm λ on T we obtain that

$$\mathbf{0}_\mathcal{T}{}^{\lambda,c} = \bigvee \{s \in T : \lambda(s, \mathbf{0}_\mathcal{T}) = \mathbf{0}_\mathcal{T}\} = \mathbf{1}_\mathcal{T}, \tag{28.13}$$

as $\lambda(s, \mathbf{0}_\mathcal{T}) \leq s \wedge \mathbf{0}_\mathcal{T} = \mathbf{0}_\mathcal{T}$ holds for each $s \in T$. Dually

$$\mathbf{1}_\mathcal{T}{}^{\lambda,c} = \bigvee \{s \in T : \lambda(s, \mathbf{1}_\mathcal{T}) = \mathbf{0}_\mathcal{T}\} = \mathbf{0}_\mathcal{T}, \tag{28.14}$$

as $\lambda(s, \mathbf{1}_\mathcal{T}) = s \neq \mathbf{0}_\mathcal{T}$, if $s \neq \mathbf{0}_\mathcal{T}$.

In order to simplify our further reasoning and not to dissolve the idea of decision functions based on lattice-valued possibilistic measures and loss functions into numerous technicalities involved when considering a t-norm λ in general, i.e., as a free parameter of all our considerations, let us limit ourselves to the particular case of the "greatest" t-norm on T defined by the infimum operation $\wedge$

in the complete lattice $\mathcal{T} = \langle T, \leq \rangle$. So, the index λ will be omitted if $\wedge$ is the t-norm in question. Moreover, let us suppose that $\wedge$ is a convex t-norm on T, so that

$$\bigvee_{s \in A} (t \wedge s) = t \wedge \bigvee_{s \in A} s \left(= t \wedge \bigvee A \right) \tag{28.15}$$

holds for each $t \in T$ and each $\mathbf{0}_{\mathcal{T}} \neq A \subset T$. In this case, the complete lattice $\mathcal{T} = \langle T, \leq \rangle$ is called *semi-Boolean*. So, the definition of (pseudo-)complement reads that

$$t^c = \bigvee \{s \in T : s \wedge t = \mathbf{0}_{\mathcal{T}}\}, \tag{28.16}$$

so that t^c is defined for each $t \in T$ and (28.12) yields that $t \wedge t^c = \mathbf{0}_{\mathcal{T}}$ is valid for each $t \in T$. The complete lattice $\mathcal{T} = \langle T, \leq \rangle$ is called *Boolean-like*, if it is semi-Boolean and, moreover, $t \vee t^c = \mathbf{1}_{\mathcal{T}}$ holds for each $t \in T$. In general, a semi-Boolean complete lattice need not be Boolean-like. Indeed, take $\mathcal{T} = \langle [0,1], \leq \rangle$, i.e., the unit interval of real numbers with their standard linear ordering. Then $x^c = \bigvee \{y \in [0,1] : y \wedge x = 0\} = 0$, if $x > 0$, and $0^c = 1$. Hence, $x \wedge x^c = 0$ for every $x \in [0,1]$, but $x^c \vee x = 0 \vee x = x < 1$ for every $0 < x < 1$.

So, keeping in mind the idea of symmetric difference of two sets, we introduce the binary operation $\triangle$ on T, i.e., $\triangle : T \times T \rightarrow T$, in this way: for every $s, t \in T$,

$$\triangle(s,t) = (s \wedge t^c) \vee (t \wedge s^c). \tag{28.17}$$

As can be easily proved (cf.[34] for more detail), the mapping $\triangle$ can be taken as $\mathcal{T}$-valued metric (function) on T in the sense that the three following conditions are fulfilled: for each $s, t, u \in T$

$$\begin{aligned} \triangle(t,t) &= \mathbf{0}_{\mathcal{T}} \text{ (reflexivity)}, \\ \triangle(s,t) &= \triangle(t,s) \text{ (symmetry)}, \\ \triangle(s,t) &\leq \triangle(s,u) \vee \triangle(u,t) \end{aligned} \tag{28.18}$$

(triangular inequality in the lattice sense). Let us note that the condition according to which $\mathcal{T} = \langle T, \leq \rangle$ is semi-Boolean is substantial when proving these relations.

Let $\mathcal{T} = \langle T, \leq \rangle$ be a semi-Boolean complete lattice, let $\langle \Omega, \mathcal{A}, \Pi \rangle$ be a possibility space with a complete $\mathcal{T}$-valued possibilistic measure Π on the ample field $\mathcal{A}$ of subsets of Ω, let $f_1, f_2 : \Omega \rightarrow T$ be mappings such that, for both $i = 1, 2$, the inclusion

$$\{\{\omega \in \Omega : f_i(\omega) = t\} : t \in T\} \subset \mathcal{A} \tag{28.19}$$

holds. Consequently, also the inclusion

$$\{\{\omega \in \Omega : f_i(\omega) \in A\} : A \subset T\} \subset \mathcal{A} \tag{28.20}$$

is valid for both $i = 1, 2$. Define the values $D_1(f_1, f_2)$ and $D_2(f_1, f_2)$ as follows:

$$\begin{aligned} D_1(f_1, f_2) &= \int \triangle(f_1(\cdot)) d\Pi = \\ &= \bigvee_{t \in T} [t \wedge \Pi(\{\omega \in \Omega : \triangle(f_1(\omega),\ f_2(\omega)) \geq t\})], \end{aligned} \tag{28.21}$$

and

$$D_2(f_1, f_2) = \triangle \left(\int f_1(\cdot)) d\Pi, \int f_2(\cdot) d\Pi \right), \tag{28.22}$$

where, for both $i = 1, 2$,

$$\int f_i(\cdot) d\Pi = \bigvee_{t \in T} [t \wedge \Pi(\{\omega \in \Omega : f_i(\omega) \geq t\})]. \tag{28.23}$$

Lemma 28.2 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a Boolean-like complete lattice. Then the identity*

$$(s \wedge t)^c = s^c \vee t^c \tag{28.24}$$

holds for each $s, t \in T$.

Proof. Let $t_1, t_2 \in T$ be such that $t_1 \leq t_2$ holds. Then, for every $s \in T$ such that $s \wedge t_1 = \mathbf{0}_{\mathcal{T}}$ holds, the identity $s \wedge t_2 = \mathbf{0}_{\mathcal{T}}$ is valid as well, so that the inequality $t_1^c \geq t_2^c$ for the corresponding (pseudo-)complements immediately follows.

Given $s, t \in T$, the inequalities $s \wedge t \leq t,\ s \wedge t \leq s$ yield that also the inequalities $(s \wedge t)^c \geq t^c$, $(s \wedge t)^c \geq s^c$ and, consequently, $(s \wedge t)^c \geq s^c \vee t^c$ are valid in every complete lattice $\mathcal{T} = \langle T, \leq \rangle$. If $\mathcal{T}$ is Boolean-like, it is semi-Boolean and $s^c \vee s = \mathbf{1}_{\mathcal{T}}$ holds for each $s \in T$. Hence,

$$\begin{aligned}(s \wedge t)^c &= (s \wedge t)^c \wedge \mathbf{1}_{\mathcal{T}} = (s \wedge t)^c \wedge (s^c \vee s) = \\ &= ((s \wedge t)^c \wedge s^c) \vee (s \wedge t)^c \wedge s).\end{aligned} \tag{28.25}$$

As $(s \wedge t)^c \geq s^c$ holds, $(s \wedge t)^c \wedge s^c$ follows. Moreover,

$$((s \wedge t)^c \wedge s) \wedge t = (s \wedge t)^c \wedge (s \wedge t) = \mathbf{0}_{\mathcal{T}}, \tag{28.26}$$

as $\mathcal{T}$ is semi-Boolean, so that the inequality

$$(s \wedge t)^c \wedge s \leq t^c \tag{28.27}$$

follows. So, (28.25) yields that $(s \wedge t)^c \leq s^c \vee t^c$ holds and the assertion is proved. □

Theorem 28.1 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a Boolean-like complete lattice, let Π be a complete $\mathcal{T}$-valued possibilistic measure on the power-set $\mathcal{P}(\Omega)$ of all subsets of a nonempty set Ω, let f_1, f_2 be mappings which take Ω into T, let $D_1(f_1, f_2)$ and $D_2(f_1, f_2)$ be defined by (28.21) and (28.22). Then the inequality $D_2(f_1, f_2) \leq D_1(f_1, f_2)$ holds.*

Proof. Being complete, the $\mathcal{T}$-possibilistic measure Π is obviously defined by the $\mathcal{T}$-possibilistic distribution $\pi : \Omega \rightarrow T$ such that $\pi(\omega) = \Pi(\{\omega\})$ for every $\omega \in \Omega$, consequently, $\Pi(A) = \bigvee_{\omega \in A} \pi(\omega)$. Applying (28.2) to the case when $\wedge$ is taken as the t-norm λ, we obtain that, for every $f : \Omega \rightarrow T$,

$$\begin{aligned}\int f d\Pi &= E \wedge f(\cdot) = \bigvee_{t \in T} [t \wedge \Pi(\{\omega \in \Omega : f(\omega) = t\})] = \\ &= \bigvee_{t \in T} \left[t \wedge \bigvee \{\pi(\omega) : f(\omega) = t\}\right] = \\ &= \bigvee_{t \in T} \left[f(\omega) \wedge \bigvee \{\pi(\omega) : f(\omega) = t\}\right] = \\ &= \bigvee_{\omega \in \Omega} (f(\omega) \wedge \pi(\omega)).\end{aligned} \tag{28.28}$$

Hence, applying appropriately (28.24), we obtain that

$$\begin{aligned}&D_2(f_1, f_2) = \triangle\left(\int f_1 d\Pi, \int f_2 d\Pi\right) = \\ &= \left(\left(\int f_1 d\Pi\right) \wedge \left(\int f_2 d\Pi\right)^c\right) \vee \left(\left(\int f_2 d\Pi\right) \wedge \left(\int f_1 d\Pi\right)^c\right) = \\ &= \left[\left(\bigvee_{\omega \in \Omega} (f_1(\omega) \wedge \pi(\omega))\right) \wedge \left(\bigvee_{\omega \in \Omega} (f_2(\omega) \wedge \pi(\omega))\right)^c\right] \vee\end{aligned}$$

$$\begin{aligned}
\vee & \quad \left[\left(\bigvee_{\omega\in\Omega}(f_2(\omega)\wedge\pi(\omega))\right)\wedge\left(\bigvee_{\omega\in\Omega}(f_1(\omega)\wedge\pi(\omega))\right)^c\right] \leq \\
\leq & \quad \left[\left(\bigvee_{\omega\in\Omega}(f_1(\omega)\wedge\pi(\omega)))\wedge(f_2(\omega)\wedge\pi(\omega)\right)^c\right] \vee \\
\vee & \quad \left[\left(\bigvee_{\omega\in\Omega}(f_2(\omega)\wedge\pi(\omega)))\wedge(f_1(\omega)\wedge\pi(\omega)\right)^c\right] = \\
= & \quad \left[\left(\bigvee_{\omega\in\Omega}(f_1(\omega)\wedge\pi(\omega))\right)\wedge((f_2(\omega))^c\vee(\pi(\omega))^c)\right] \vee \\
\vee & \quad \left[\left(\bigvee_{\omega\in\Omega}(f_2(\omega)\wedge\pi(\omega))\right)\wedge((f_1(\omega))^c\vee(\pi(\omega))^c)\right] = \\
= & \quad \left[\bigvee_{\omega\in\Omega}(f_1(\omega)\wedge\pi(\omega)\wedge(f_2(\omega))^c)\right]\vee\left[\bigvee_{\omega\in\Omega}(f_1(\omega)\wedge\pi(\omega)\wedge\pi(\omega))^c)\right] \vee \\
\vee & \quad \left[\bigvee_{\omega\in\Omega}(f_2(\omega)\wedge\pi(\omega)\wedge(f_1(\omega))^c)\right]\vee\left[\bigvee_{\omega\in\Omega}(f_2(\omega)\wedge\pi(\omega)\wedge\pi(\omega))^c)\right] = \\
= & \quad \left[\bigvee_{\omega\in\Omega}(f_1(\omega)\wedge(f_2(\omega))^c)\wedge\pi(\omega)\right]\vee\mathbf{0}_{\mathcal{T}}\vee \\
\vee & \quad \left[\bigvee_{\omega\in\Omega}(f_2(\omega)\wedge(f_2(\omega))^c)\wedge\pi(\omega)\right]\vee\mathbf{0}_{\mathcal{T}} \leq \\
\leq & \quad \bigvee_{\omega\in\Omega}[((f_1(\omega))\wedge(f_2(\omega))^c)\vee((f_2(\omega)\wedge(f_1(\omega))^c))\wedge\pi(\omega)] = \\
= & \quad \bigvee_{\omega\in\Omega}[(\triangle(f_1(\omega),\ f_2(\omega)))\wedge\pi(\omega)] = \\
= & \quad \int\triangle(f_1,f_2)d\Pi = D_1(f_1,f_2).
\end{aligned} \tag{28.29}$$

The assertion is proved. □

The equality $D_1(f_1, f_2) = D_2(f_1, f_2)$ does not hold in general, as the following very simple example demonstrates. Let $\Omega = \{\omega_1, \omega_2\}$, let $f_1(\omega_1) = f_2(\omega_2) = \mathbf{0}_{\mathcal{T}}$, $f_1(\omega_2) = f_2(\omega_1) = \mathbf{1}_{\mathcal{T}}$, let $\Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}$, $\Pi(A) = \mathbf{1}_{\mathcal{T}}$ for every $\emptyset \neq A \subset \Omega$, so that $\pi(\omega_1) = \pi(\omega_2) = \mathbf{1}_{\mathcal{T}}$. Then

$$\begin{aligned}
\int f_1 d\Pi &= \bigvee_{\omega\in\Omega}(f_1(\omega)\wedge\pi(\omega)) = (f_1(\omega_1)\wedge\pi(\omega_1))\vee(f_1(\omega_2)\wedge\pi(\omega_2)) = \\
&= f_1(\omega_1)\vee f_1(\omega_2) = \mathbf{1}_{\mathcal{T}} = f_2(\omega_1)\vee f_2(\omega_2) = \int f_2 d\Pi.
\end{aligned} \tag{28.30}$$

Consequently,

$$D_2(f_1, f_2) = \triangle\left(\int f_1 d\Pi,\ \int f_2 d\Pi\right) = \triangle(\mathbf{1}_{\mathcal{T}}, \mathbf{1}_{\mathcal{T}}) = \mathbf{0}_{\mathcal{T}}. \tag{28.31}$$

However,

$$\begin{aligned}
\triangle(f_1(\omega),\ f_2(\omega)) &= \triangle(\mathbf{0}_{\mathcal{T}}, \mathbf{1}_{\mathcal{T}}) = (\mathbf{0}_{\mathcal{T}}\wedge\mathbf{1}_{\mathcal{T}}{}^c)\vee(\mathbf{1}_{\mathcal{T}}\wedge\mathbf{0}_{\mathcal{T}}{}^c) = \\
&= \mathbf{0}_{\mathcal{T}}\vee\mathbf{1}_{\mathcal{T}} = \mathbf{1}_{\mathcal{T}} = \triangle(f_1(\omega_2),\ f_2(\omega_2)),
\end{aligned} \tag{28.32}$$

so that

$$\begin{aligned} D_1(f_1, f_2) &= \int \triangle(f_1(\cdot),\ f_2(\cdot))d\Pi = \mathbf{1}_{\mathcal{T}} \wedge \pi(\omega_1) \vee (\mathbf{1}_{\mathcal{T}} \wedge \pi(\omega_2) = \mathbf{1}_{\mathcal{T}} > \\ &> \mathbf{0}_{\mathcal{T}} = D_2(f_1, f_2). \end{aligned} \tag{28.33}$$

Theorem 28.2 *Let $\mathcal{T} = \langle T, \leq \rangle$ be a semi-Boolean complete lattice, let $\langle \Omega, \mathcal{A}, \Pi \rangle$ be a $\mathcal{T}$-possibility space with a complete $\mathcal{T}$-possibilistic measure Π on the ample field $\mathcal{A}$, let $f_1, f_2 : \Omega \rightarrow T$ be mappings such that the inclusion*

$$\{\{\omega \in \Omega : f_i(\omega) = t\} : t \in T,\ i = 1, 2\} \subset \mathcal{A} \tag{28.34}$$

holds, let $D_1(f_1, f_2)$ be defined by (28.21). Then the relation

$$\begin{aligned} D_1(f_1, f_2) &= \int (I_A(\cdot) \wedge \triangle(f_1(\cdot),\ f_2(\cdot)))d\Pi \leq \\ &\leq \left[\bigvee_{\omega \in A} \triangle(f_1(\omega),\ f_2(\omega)) \right] \wedge \Pi(A) \end{aligned} \tag{28.35}$$

is valid, where $A = \{\omega \in \Omega : f_1(\omega) \neq f_2(\omega)\}$ is supposed to be in $\mathcal{A}$ and I_A is its $\mathcal{T}$-valued characteristic function (identifier), i.e., $I_A(\omega) = \mathbf{1}_{\mathcal{T}}$, if $\omega \in A$, $I_A(\omega) = \mathbf{0}_{\mathcal{T}}$ otherwise.

According to the common conventions we can write $\int_A \triangle(f_1(\cdot),\ f_2(\cdot))d\Pi$ for the left-hand side of (28.35).

Proof. If $\omega \in \Omega - A$, then $f_1(\omega) = f_2(\omega)$, so that the relation

$$\triangle(f_1(\omega),\ f_2(\omega)) = \mathbf{0}_{\mathcal{T}} = I_A(\omega) \wedge \triangle(f_1(\omega),\ f_2(\omega)) \tag{28.36}$$

follows. If $\omega \in A$, then $I_A(\omega) = \mathbf{1}_{\mathcal{T}}$, so that the equality

$$I_A(\omega) \wedge \triangle(f_1(\omega),\ f_2(\omega)) = \triangle(f_1(\omega),\ f_2(\omega)) \tag{28.37}$$

holds again. Hence, we obtain that the relation

$$\begin{aligned} D_1(f_1, f_2) &= \int \triangle(f_1(\cdot),\ f_2(\cdot))d\Pi = \int (I_A(\cdot) \wedge \triangle(f_1(\cdot),\ f_2(\cdot)))d\Pi \leq \\ &\leq \int I_A(\cdot)d\Pi = \bigvee_{t \in T} [t \wedge \Pi(\{\omega \in \Omega : I_A(\omega) \geq t\})] = \Pi(A) \end{aligned} \tag{28.38}$$

obviously holds. Also the inequality

$$\begin{aligned} D_1(f_1, f_2) &= \bigvee_{t \in T} [t \wedge \Pi(\{\omega \in \Omega : s(f_1(\omega),\ f_2(\omega)) \geq t\})] \leq \\ &\leq \bigvee_{\omega \in A} \triangle(f_1(\omega),\ f_2(\omega)) \end{aligned} \tag{28.39}$$

immediately follows, so that the assertion is proved. □

Let us return ourselves to the case of possibilistic decision functions as introduced and analyzed in the foregoing chapters with respect to the expected value of the loss function taken as the main criterion of their quality. Restricting ourselves to the case when the infimum operation on $\mathcal{T} = \langle T, \leq \rangle$ is taken as the t-norm on T and considering two decision functions δ_1, δ_2, both of them taking the space E of empirical observations (data) into the space D of decisions, we obtain, for both $i = 1, 2$, that

$$\chi_\sigma^B(\delta_i) = \int \rho(\sigma(\cdot),\ \delta_i(\eta(\cdot)))d\Pi = = \bigvee_{t\in T}[t \wedge \Pi(\{\omega\in\Omega : \rho(\sigma(\omega),\ \delta_i(\eta(\omega))) \geq t\})] \quad (28.40)$$

(cf. (26.1)). The difference between the qualities of these two decision functions can be defined using either D_1 or D_2 defined by (28.21) and (28.23) and applied to the functions $\rho(\sigma(\cdot),\ \delta_1(\eta(\cdot)))$ and $\rho(\sigma(\cdot),\ \delta_2(\eta(\cdot)))$ instead of f_1 and f_2 in (28.21) and (28.23). Hence, setting

$$D_i^\star(\delta_1,\delta_2) = D_i(\rho(\sigma(\cdot),\ \delta_1(\eta(\cdot))),\ \rho(\sigma(\cdot),\ \delta_2(\eta(\cdot)))) \quad (28.41)$$

for both $i = 1, 2$, we obtain that

$$D_1^\star(\delta_1,\delta_2) = \int(\triangle(\rho(\sigma(\cdot),\ \delta_1(\eta(\cdot))),\ \rho(\sigma(\cdot),\ \delta_2(\eta(\cdot))))d\Pi = = \bigvee_{t\in T}[t\wedge\Pi(\{\omega\in\Omega : \triangle(\rho(\sigma(\cdot),\ \delta_1(\eta(\cdot))),\ \rho(\sigma(\cdot),\ \delta_2(\eta(\cdot)))) \geq t\})], \quad (28.42)$$

and

$$D_2^\star(\delta_1,\delta_2) = \triangle(\int \rho(\sigma(\cdot),\ \delta_1(\eta(\cdot)))d\Pi,\ \int\rho(\sigma(\cdot),\ \delta_2(\eta(\cdot)))d\Pi) = = \triangle(\chi_\sigma^B(\delta_1),\ \chi_\sigma^B(\delta_2)), \quad (28.43)$$

let us recall that $\triangle(t_1,t_2) = (t_1\wedge t_2^c)\vee(t_2\wedge t_1^c)$ for every $t_1, t_2 \in T$.

Applying Theorem 28.1 to the particular case when $f_i(\omega) = \rho(\sigma(\cdot),\ \delta_i(\eta(\omega)))$ for any $\omega\in\Omega$ and both $i = 1, 2$, and keeping in mind that under the conditions of Theorem 28.1 also Theorem 28.2 holds, we arrive at the following corollary.

Corollary 28.1 *Let $\mathcal{T} = \langle T, \leq\ \rangle$ be a Boolean-like complete lattice, let Π be a $\mathcal{T}$-valued possibilistic measure on the power-set $\mathcal{P}(\Omega)$ of all subsets of a nonempty set Ω, let S be the set of possible internal states of the system under consideration, let D be the set of possible decisions, let E be the set of possible empirical values (data), all the sets, S, D and E are supposed to be nonempty. Let $\sigma : \Omega\to S$, $\eta : \Omega\to E$ and $\delta_i : E\to D$ be mappings with the intuition behind as above, let $\rho : S\times D\to T$ be a $\mathcal{T}$-valued loss function, let $D_1^\star(\delta_1,\delta_2)$ and $D_2^\star(\delta_1,\delta_2)$ be defined by (28.42)and (28.43), and let*

$$A = \{\omega\in\Omega : \rho(\sigma(\omega),\ \delta_1(\eta(\omega))) \neq \rho(\sigma(\omega),\ \delta_2(\eta(\omega)))\}. \quad (28.44)$$

Then the relation

$$D_2^\star(\delta_1,\delta_2) \leq D_1^\star(\delta_1,\delta_2) \leq \leq \left[\bigvee_{\omega\in A}\triangle(\rho(\sigma(\omega),\ \delta_1(\eta(\omega))),\ \rho(\sigma(\omega),\ \delta_2(\eta(\omega))))\right]\wedge\Pi(A) \quad (28.45)$$

holds. Obviously, the right-hand side inequality in (28.45) is valid also under weaker conditions of Theorem 28.2.

Informally told, when considering decision problems under uncertainty and when quantifying and processing this uncertainty using lattice-valued possibilistic measures, the resulting possibilistic decision functions are robust in a sense close to that in the case of statistical decision functions. Namely, if the losses suffered when applying different decision functions differ only rarely, i.e., if the possibility degree of such cases is small enough, or when the differences between the suffered losses are small, then the qualities of the possibilistic decision functions in question also do not differ too much from each other. To be more precise, for statistical decision functions such robustness follows only when the loss function is uniformly bound from above, e.g., if it takes values in a finite interval $[0, a]$ (in particular, $[0, 1]$) of real numbers. Indeed, for loss functions taking values in $[0, \infty]$ a rarely occurring but very large difference in the losses suffered can make the global quantities of the statistical decision functions under consideration rather significantly differing.

As a matter of fact, in the case of possibilistic decision functions the robustness with respect to differences of suffered losses in rarely occurring cases is still more strong than claimed by Corollary 7.1. Namely, if the losses suffered when applying various decision function are perhaps different but in all cases rather large (in the sense of the partial ordering in the complete lattice $\mathcal{T} = \langle T, \leq \rangle$ under consideration), then the global quality of the resulting possibilistic decision functions will be the same no matter which among the alternative decision functions (i.e., mappings δ taking E into D) is chosen. Let us demonstrate this fact by a simple example.

Let the notations and conditions of Corollary 28.1 hold, let δ_1 and δ_2 be such hat the losses $\rho(\sigma(\omega),\ \delta_i(\eta(\omega))),\ i = 1, 2$ differ from each other only when $s_1 \in S$ is the actual state of the system under investigation. In symbols, the inclusion

$$\begin{aligned} A &= \{\omega \in \Omega : \rho(\sigma(\omega),\ \delta_1(\eta(\omega))) \neq \rho(\sigma(\omega),\ \delta_2(\eta(\omega)))\} \subset \\ &\subset \{\omega \in \Omega : \sigma(\omega) = s_1\} \end{aligned} \tag{28.46}$$

is valid. Consider the case when the ineqaulity

$$\Pi(A) \leq \rho(\sigma(\omega),\ \delta_1(\eta(\omega))) \wedge \rho(\sigma(\omega),\ \delta_2(\eta(\omega))) \tag{28.47}$$

holds for each $\omega \in A$. Then $\chi^B_\sigma(\delta_1) = \chi^B_\sigma(\delta_2)$.
Indeed, for both $i = 1, 2$ we obtain that

$$\begin{aligned} \chi^B_\sigma(\delta_i) &= \bigvee_{\omega \in \Omega} [\rho(\sigma(\omega),\ \delta_i(\eta(\omega))) \wedge \pi(\omega)] = \\ &= \bigvee_{\omega \in \Omega - A} [\rho(\sigma(\omega),\ \delta_i(\eta(\omega))) \wedge \pi(\omega)] \vee \\ &\vee \bigvee_{\omega \in A} [\rho(\sigma(\omega),\ \delta_i(\eta(\omega))) \wedge \pi(\omega)]. \end{aligned} \tag{28.48}$$

For each $\omega \in A$ the relation

$$\pi(\omega) = \Pi(\{\omega\}) \leq \Pi(A) \leq \rho(\sigma(\omega),\ \delta_i(\eta(\omega))) \tag{28.49}$$

holds for both $i = 1, 2$, so that

$$\rho(\sigma(\omega),\ \delta_i(\eta(\omega))) \wedge \pi(\omega) = \pi(\omega) \tag{28.50}$$

holds for both $i = 1, 2$ and each $\omega \in A$. Then (28.48) yields that, for $i = 1, 2$,

$$\chi^B_\sigma(\delta_i) = \left[\bigvee_{\omega \in \Omega - A} (\rho(\sigma(\omega),\ \delta_i(\eta(\omega))) \wedge \pi(\omega)) \right] \vee \Pi(A). \tag{28.51}$$

As the values $\rho(\sigma(\omega),\ \delta_i(\eta(\omega)))$ are the same for both $i = 1, 2$, if $\omega \in \Omega - A$, the identity $\chi^B_\sigma(\delta_1) = \chi^B_\sigma(\delta_2)$ follows.

Having sketched, very briefly, the classical model of decision making under uncertainty with uncertainty quantified and processed by the tools of the axiomatic (Kolmogorov) probability theory leading to the notion and theory of statistical decision functions, we have submitted an attempt to rewrite this model for the case when the underlying uncertainty is quantified and processed using a lattice-valued possibilistic measure. The reasons for this approach read that, as a matter of fact, both the non-numerical nature of the lattice-valued degrees of uncertainty as well as the specific features in which the axioms imposed on possibilistic measures differ from the probabilistic (and, in its nature and origins, measure-theoretic) ones, emphasize rather the qualitative than the quantitative aspects of the degrees of uncertainty under consideration. E.g., applying lattice-valued possibilistic measures we are not forced to make any two degrees of uncertainty mutually comparable as far as their sizes are concerned, what may be of use in numerous cases when there are not sufficient (or even any) reasons at our disposal to make these uncertainty degrees comparable. Moreover, complete lattice is perhaps the most specific mathematical structure still covering the two most often used structures for quantification and processing of sizes: the unit interval $[0, 1]$ of real numbers with their standard linear ordering $\leq$, and the complete Boolean algebra with the corresponding partial ordering induced by the supremum and infimum operations, as a matter of fact, this structure can be identified with that of all subsets of a fixed nonempty set partially ordered by the relation of set inclusion. Let us recall that it was just the qualitatively different ways in which the operations of complement (abstraction $1 - .$ in $[0, 1]$, set complement $X - A$ in $\mathcal{P}(X)$) are defined in both these structures what has brought us to the idea of pseudo-complement definable as a secondary notion within the framework of each complete lattice.

Still going on with the inspiration borrowed from the theory of statistical decision functions, we have introduced the possibilistic modifications of the two classical criteria used in order to define and measure the qualities of procedures for decision making under uncertainty, hence, in our case, the qualities of possibilistic decision functions. Using the minimax (the worst-case) principle we have proved that as least in the simple case when we have to identify the actual internal state of the system under consideration, the maximum likelihood (in the possibilistic sense) principle optimizes the resulting possibilistic decision function in the sense that the loss suffered in the worst (the least favorable) case is minimized. For the possibilistic variant of the Bayes principle we have proved that the expected (in the sense of Sugeno pssibilistic integral) loss suffered is robust with respect to the apriori possibilistic distribution and to the loss function applied. Informally told, if two decision functions are such that the differences in the resulting losses are "small" and/or occuring only "rarely", then also the global quality values of the two possibilistic decision function do not differ "too much".

Among the possible directions for further investigation let us mention explicitly just the following ones.

(1) To apply the general model from above to a particular decision problem under uncertainty so that the specific features of the domain under consideration would allow to choose an appropriate particular loss function and/or apriori possibilistic distribution or measure.

(2) To consider richer and more powerful structures for the possibility degrees, in particular, to choose either the unit interval of reals (i.e., the usual real-valued possibilistic measure), or Boolean (in particular, set-valued) possibilistic measures, giving up the idea that our results should be general enough to cover both these cases.

(3) The opposite, in a sense, way of reasoning could be to investigate, which of the constructions realized, and results obtained, above would be realizable, and remained to be valid, also in structures weaker than complete lattices (in lattices, lower or upper semilattices, partially ordered sets, . . .).

(4) Among the more specific, but important and interesting problems closely related to possibilistic decision functions let us mention this one: which would be the possibilistic modification (if it is possible at all) of the well-known Laplace principle? Let us recall that this principle, if applied to the case of statistical decision functions, suggests to take the uniform probability distribution on a finite set as the "default" probability distribution if no arguments in favor of another apriori probability distribution are known. Obviously, the possibilistic distribution ascribing the unit (i.e., maximum) value to every element of the universe of discourse does not solve the problem, as no non-trivial results can be achieved in this case.

For the reader's convenience, the list of references from Part I is completely copied under Nos. [1]–[46]. Thematically relevant author's paper having appeared during the last year (2006) are listed below as Nos. [47]–[51].

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