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Abstract:

Fuzzy Class Theory was proposed as a formal background of an axiomatic approach to fuzzy mathematics built inside a sufficiently strong formal fuzzy logic. The goal of this text is to provide a thorough introduction to this theory for non-specialists in formal fuzzy logic. First, we give an informal exposition of the apparatus of the theory and explain how its results can be interpreted from the point of view of traditional fuzzy mathematics. Secondly, we address notable features of this theory, starting with the importance of graded theories, continuing through natural embedding and then fuzzifying of classical mathematical theories, ending with basic methodological guidelines regarding the latter. Finally, we show how to prove the results in Fuzzy Class Theory, by employing not only proof methods for particular first-order logic but also new strong methods which heavily utilize the advantages of formal build-up of the Fuzzy Class Theory.

This is version 1.0 of the Primer. A newer version may be downloadable from `www.cs.cas.cz/hp`.

Keywords:
Fuzzy Class Theory, graded properties, proof methods

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1 The apparatus of Fuzzy Class Theory

Fuzzy Class Theory (FCT for short), introduced in [3], axiomatizes the notion of fuzzy set. Here we give an informal exposition of the apparatus and an explanation as to how the results in FCT can be interpreted from the point of view of traditional fuzzy mathematics.[1] Before we proceed towards this goal we need to recall what are the *models* of FCT.

1.1 Models of FCT

As said above, Fuzzy Class Theory is an axiomatic theory aimed to capture the notion of fuzzy set. Therefore its models will be systems of traditional fuzzy sets.[2] Let us be more specific: given some crisp universe U, fuzzy sets of the *first order* are all functions from U to real unit interval [0,1]. Taking the set of all fuzzy sets as a new universe we obtain fuzzy sets of the *second order* (fuzzy sets of fuzzy sets); iterating this process we obtain fuzzy sets of any order. The collections of fuzzy sets of all orders over a given universe U are the intended models of FCT; we call them *Zadeh models,* as they correspond exactly to Zadeh's [30] original notion of fuzzy set.

Notice two special features of Zadeh models: (i) they take the unit interval $[0,1]$ as the set of truth values, and (ii) they contain *all* possible fuzzy sets and relations. General models of FCT can relax both of these features: (i) the set of truth values can be any $\mathrm{MTL}_\triangle$-chain (consult [12] or Appendix 4 for the definition of $\mathrm{MTL}_\triangle$-chain), and (ii) they need not contain all fuzzy sets (but only those which are definable in FCT, more details later). The models over [0,1] are called *standard* and those which contain all fuzzy sets (of all orders) are called *full.* Thus, Zadeh models are full standard models.

FCT is *sound* w.r.t. all of its models, including Zadeh ones.[3] Thus whatever we prove in FCT is true about $\mathbf{L}$-valued fuzzy sets, for any $\mathrm{MTL}_\triangle$-chain $\mathbf{L}$; in particular, it is true about the usual $[0,1]$-valued fuzzy sets. Since in general models of FCT some (undefinable) fuzzy subsets of the universe may be missing, we call the objects of FCT fuzzy *classes* rather than fuzzy sets.[4] Nevertheless, in virtue of the soundness of FCT w.r.t. Zadeh models, the theorems of FCT are always valid for fuzzy *sets.* Thus whenever we speak of classes, the reader can always safely substitute usual fuzzy sets for our "classes".

1.2 Variables and atomic expressions

The language of FCT contains:

- Variables for atomic objects from some crisp universe of discourse U (denoted by lowercase letters $x, y, \dots$)
- Variables for fuzzy classes of atomic objects (denoted by uppercase letters $A, B, \dots$)
- Variables for fuzzy classes of fuzzy classes of objects, which are also called fuzzy classes of the second order (denoted by calligraphic letters $\mathcal{A}, \mathcal{B}, \dots$)
- Variables for fuzzy classes of the third order, etc.

If necessary, the order can be explicitly superscribed to the variable: e.g., $X^{(4)}$ is a variable of the 4th order, i.e., for a fuzzy class of fuzzy classes of fuzzy classes of objects.

[1] We adapt some parts of [4], [1], and [6] here.

[2] We regard n-ary fuzzy relations as a special kind of fuzzy sets, namely fuzzy sets of n-tuples of individuals.

[3] It is not a *complete* theory of Zadeh models, as it is not axiomatizable due to Gödel's Incompleteness Theorem (natural numbers are definable in Zadeh models over $\mathrm{MTL}_\triangle$). Nevertheless, it seems sufficiently strong for all practical purposes.

[4] By this we also follow the terminological tradition of set theory, in which collections of objects of some fixed universe are called classes, while the word "set" has a more specific meaning (of a member of the cumulative hierarchy of sets).

An important feature by which FCT differs from traditional fuzzy mathematics is screening off direct references to truth values; this ensures that FCT renders fuzzy sets (classes) as a primitive notion, rather than modelling them by membership functions. Thus there are *no variables for truth degrees* in the language of FCT. The degree to which x belongs to A is expressed simply by the atomic formula $x \in A$ (which can alternatively be written in a more traditional way as Ax). The theory is typed, so only such atomic formulae are well-formed which express the membership of an object of a lesser order to an object of a higher order.[5]

The crisp identity of individuals is expressed by the predicate =. For atomic objects, $x = y$ holds in a model iff x, y are the same object in the universe U of the model. For fuzzy classes of arbitrary orders, $A = B$ iff the membership functions of A and B are identical.[6] The identity of membership degrees Ax and By is also expressible by a formula of FCT, namely the formula $\triangle(Ax \leftrightarrow By)$.[7] Identical individuals, classes, or membership degrees are freely intersubstitutable in formulae.[8] Many non-crisp equality notions are definable in FCT as well, as shown later in this Primer, but these do not ensure free intersubstitutivity in formulae.

In order to express statements about fuzzy relations, FCT contains the usual apparatus for tuples of atomic individuals or fuzzy classes (of any order). The n-tuple of elements $x_1, \dots, x_k$ is denoted by $\langle x_1, \dots, x_k \rangle$, or shortly $x_1 \dots x_k$. As usual, the identity of tuples is component-wise (see Section 1.8).

1.3 Logical connectives

Membership degrees, expressed by atomic formulae, can be combined by means of logical connectives. The logical connectives can come from any suitable fuzzy logic. Originally [3], FCT was formulated over the fuzzy logic ŁΠ, which contains as definable connectives all usual arithmetical operations as well as a wide class of t-norms. In this Primer we use the logic $\mathrm{MTL}_\triangle$, which is (roughly speaking) the weakest fuzzy logic with good deductive properties for Fuzzy Class Theory. If the expressive means of $\mathrm{MTL}_\triangle$ are not sufficient for a particular purpose, a stronger logic (e.g., $\mathrm{MTL}_\sim$ [16] or ŁΠ [14]) can be used instead of $\mathrm{MTL}_\triangle$. Since $\mathrm{MTL}_\triangle$ is weaker than these logics, all results obtained in $\mathrm{MTL}_\triangle$ can be reinterpreted in the stronger fuzzy logic and used without changes in the stronger apparatus. (Recall that the results in $\mathrm{MTL}_\triangle$ are valid in the standard models based on *any* left-continuous t-norm. Consult Appendix 4 for more details on $\mathrm{MTL}_\triangle$.)

Thus in the present Primer we shall assume that the formulae of FCT take truth values in an arbitrary $\mathrm{MTL}_\triangle$-chain $\mathbf{L} = (L, *, \Rightarrow, \wedge, \vee, 0, 1, \triangle)$, and logical connectives appearing in the formulae are interpreted by operations of the $\mathrm{MTL}_\triangle$-chain $\mathbf{L}$:

strong conjunction &	by the monoidal operation $*$ of $\mathbf{L}$
implication $\rightarrow$	by its residuum $\Rightarrow$
weak connectives $\wedge$ and $\vee$	by the lattice operations of minimum resp. maximum, etc.

In particular, the connectives can be interpreted as operations on $[0, 1]$ based on a left-continuous t-norm T, where

strong conjunction &	is interpreted as T
implication $\rightarrow$	as its residuum $\overrightarrow{T}$
weak conjunction $\wedge$ and disjunction $\vee$	as the minimum and the maximum, respectively
the Baaz $\triangle$ connective	as the $\{0, 1\}$-projector $\triangle x = 1 - \mathrm{sgn}(1 - x)$

Furthermore, $\neg\varphi$ and $\varphi \leftrightarrow \psi$ are defined as $\varphi \rightarrow 0$ and $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, respectively.

Example 1.3.1 The minimum of the membership degrees expressed by the formulae Ax and Bx is expressed as usual, by the formula $Ax \wedge Bx$.

[5] I.e., $X^{(n)} \in A^{(m)}$ is a well-formed formula iff $n < m$.
[6] The latter is ensured by the axiom of extensionality, see Section 1.8.
[7] See Section 1.3 for the meaning of propositional connectives in complex formulae.
[8] This is ensured by the rules of the logic $\mathrm{MTL}_\triangle$, see Definition 4.2.4 in Appendix 4.

Example 1.3.2 The truth degree

$$Ax * Ax * Bx \ \Rightarrow Ax * Bx$$

in an MTL$_\triangle$-chain **L**, as well as the truth degree

$$\overrightarrow{T}(T(Ax, T(Ax, Bx)), T(Ax, Bx))$$

on $[0, 1]$, are both expressed by the formula

$$Ax \,\&\, Ax \,\&\, Bx \ \rightarrow \ Ax \,\&\, Bx$$

(the associativity of & in MTL$_\triangle$ and the precedence rules of Convention 4.1.1 are used to avoid unnecessary brackets).

Example 1.3.3 The comparison of the truth values of formulae φ, ψ is expressible in MTL$_\triangle$ by the formula $\triangle(\varphi \rightarrow \psi)$, since in MTL$_\triangle$-chain, the formula $\varphi \rightarrow \psi$ has the truth value 1 iff the truth value of φ is less than or equal to the truth value of ψ.

Thus we can define logical connectives that compare truth degrees:

$$\begin{aligned} \varphi \leq \psi \quad &\equiv_{\mathrm{df}} \quad \triangle(\varphi \rightarrow \psi) \\ \varphi = \psi \quad &\equiv_{\mathrm{df}} \quad \triangle(\varphi \leftrightarrow \psi) \end{aligned}$$

and their combinations like $\varphi < \psi \equiv_{\mathrm{df}} (\varphi \leq \psi) \,\&\, \neg(\varphi = \psi)$. The truth constants 0 and 1 are defined in MTL$_\triangle$, thus we can write the formulae like $Ax = 1$ or $Bx > 0$ with the obvious meaning.

1.4 Logical quantifiers

Infima and suprema of truth degrees are symbolized by the logical symbols $\forall$ and $\exists$, respectively. Thus, for example, instead of $\inf_x Ax$ we write $(\forall x)Ax$, and instead of $\sup_y(Ay * By)$ we write $(\exists y)(Ay \,\&\, By)$ or, by Convention 1.6.1, $(\exists y \in A)(y \in B)$. It should be noticed that *unless φ is crisp,* the expressions of the form $(\forall x)\varphi$ should not be read "for all x it holds that φ", since the meaning of the formula is a (possibly intermediate) truth degree, rather than a statement which either holds or not. Similarly, $(\exists x)\varphi$ must be understood as the *supremum of degrees* to which there is an x such that φ (unless φ is crisp, i.e., unless $\triangle(\varphi \vee \neg\varphi)$ is proved or assumed).

1.5 Comprehension terms

In virtue of the comprehension axioms of FCT (see Section 1.8), fuzzy classes (of any order $n+1$) can be denoted by the comprehension terms $\{x \mid \varphi(x)\}$, where φ is any formula of FCT and x is a variable of order n. The notation $A = \{x \mid \varphi(x)\}$ means that $Ax = \varphi(x)$ for all x. The usual abbreviations can be used, e.g.:

$$\begin{array}{rcl} \{x \in B \mid \varphi(x)\} & \text{abbreviates} & \{x \mid x \in B \,\&\, \varphi(x)\} \\ \{X \subseteq Y \mid \varphi(X)\} & \text{''} & \{X \mid X \subseteq Y \,\&\, \varphi(X)\} \\ \{xy \mid \varphi(x, y)\} & \text{''} & \{z \mid (\exists x, y)(z = \langle x, y \rangle \,\&\, \varphi(x, y))\}, \text{ etc.} \end{array}$$

In a more traditional fuzzy notation, the fuzzy class $A = \{x \in B \mid \varphi(x)\}$ would be denoted by $A = \sum_{x_i \in B} \varphi(x_i)/x_i$ if A is finite, or $A = \int_B \varphi(x)/x$ if A is infinite.

Again, unless the formula φ expresses a crisp condition, the term $\{x \mid \varphi(x)\}$ should not be read "the set of all those x for which φ holds", but rather "the (fuzzy) class to which any object x belongs to the same degree to which φ is true about the object x".

1.6 Abbreviations

Various abbreviations which are common in formal fuzzy logic, classical mathematics, or traditional fuzzy set theory can be used in formulae of FCT. This makes many of them look quite similar to the usual statements about fuzzy sets. Some of such abbreviations are listed in Convention 1.6.1:

Convention 1.6.1 Besides the abbreviations introduced earlier, we shall use the following ones:

- The formulae $(\forall x)(x \in A \to \varphi)$ and $(\exists x)(x \in A \mathbin{\&} \varphi)$ are abbreviated by $(\forall x \in A)\varphi$ and $(\exists x \in A)\varphi$, respectively. Similar notation can be used for relativization by other binary predicates as well, e.g., $(\forall X \subseteq A)\varphi$.
- The formulae $\varphi \mathbin{\&} \dots \mathbin{\&} \varphi$ (n times) are abbreviated φ^n; instead of $(x \in A)^n$ we can write $x \in^n A$ (and similarly for other predicates).
- $x \notin A$ is shorthand for $\neg(x \in A)$, and similarly for other binary predicates.
- A chain of implications $\varphi_1 \to \varphi_2, \varphi_2 \to \varphi_3, \dots, \varphi_{n-1} \to \varphi_n$ can be written as $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \dots \longrightarrow \varphi_n$, and similarly for the equivalence connective.

1.7 Defined notions

It could be seen in the previous subsections that the primitive notions of FCT can express only the most basic concepts of fuzzy set theory. Further notions can be introduced by means of defined constants, predicates, and functors.[9]

Important class constants are the *empty class* $\emptyset$ and the *universal class* V, defined as follows:

Definition 1.7.1 In FCT, we define the following class constants:

$$\begin{array}{rcll} \emptyset & =_{\mathrm{df}} & \{x \mid 0\} & \text{empty class} \\ \mathrm{V} & =_{\mathrm{df}} & \{x \mid 1\} & \text{universal class} \end{array}$$

Thus $x \in \emptyset$ has the truth degree 0 and $\mathrm{V}x = 1$, for any x (both classes are crisp).

Usual fuzzy class operations like unions and intersections can be defined in FCT by means of simple comprehension terms, like in classical mathematics. For instance, we define the intersection of two fuzzy sets as $A \cap B =_{\mathrm{df}} \{x \mid x \in A \mathbin{\&} x \in B\}$. It can be seen that in a model, $C = A \cap B$ iff $Cx = Ax * Bx$ for all x as usual. For a list of defined class operations see Definition 1.7.2.

Definition 1.7.2 In FCT, we define the following elementary fuzzy class operations:

$$\begin{array}{rcll} \mathrm{Ker}(A) & =_{\mathrm{df}} & \{x \mid \triangle(x \in A)\} & \text{kernel} \\ \mathrm{Supp}(A) & =_{\mathrm{df}} & \{x \mid \neg\triangle\neg(x \in A)\} & \text{support} \\ A_\alpha & =_{\mathrm{df}} & \{x \mid \triangle(\alpha \to x \in A)\} & \alpha\text{-cut} \\ A_{=\alpha} & =_{\mathrm{df}} & \{x \mid \triangle(\alpha \leftrightarrow x \in A)\} & \alpha\text{-level} \\ \backslash A & =_{\mathrm{df}} & \{x \mid \neg(x \in A)\} & \text{complement} \\ A \cap B & =_{\mathrm{df}} & \{x \mid x \in A \mathbin{\&} x \in B\} & \text{intersection} \\ A \sqcap B & =_{\mathrm{df}} & \{x \mid x \in A \wedge x \in B\} & \text{min-intersection} \\ A \sqcup B & =_{\mathrm{df}} & \{x \mid x \in A \vee x \in B\} & \text{max-union} \\ A \setminus B & =_{\mathrm{df}} & \{x \mid x \in A \mathbin{\&} x \notin B\} & \text{difference} \end{array}$$

In stronger logics where strong disjunction is available, we also define strong union as $A \cup B =_{\mathrm{df}} \{x \mid x \in A \veebar x \in B\}$. Notice that unless we work in a logic with truth constants (like ŁΠ$\frac{1}{2}$), the α in the definitions of α-cuts and α-levels is a truth degree expressed by a *formula* (e.g., $A_{(Bx \& Cy)}$) rather then a number (the 0.4-cut $A_{0.4}$ is only meaningful in a given model, not in the syntax of FCT).

[9] For the theoretical foundation for introducing defined notions in formal theories over fuzzy logic see [21].

Many usual properties of fuzzy classes are expressible by suitable formulae of FCT. For example, the *normality* of A is expressed by the formula $(\exists x)\triangle(x \in A)$. We can define that A is *crisp* iff $(\forall x)(\triangle Ax \vee \neg\triangle Ax)$, and *fuzzy* iff it is not crisp. It can be noticed that these properties are themselves crisp: any set either is, or is not normal (crisp, fuzzy). Besides crisp properties of fuzzy sets, traditional fuzzy set theory also defines some functions that assign a truth degree to a fuzzy set. For example the *height* of a fuzzy set is the supremum of membership degrees it hits. In FCT, these notions are expressed by *fuzzy properties* of fuzzy classes: the height of a fuzzy class A is defined by the formula $(\exists x)(x \in A)$. Again this is not to be read as "there is an x in A", but interpreted as the supremum of the truth degrees of Ax. We list some definable properties of fuzzy classes (both crisp and fuzzy ones) in the following definition:

Definition 1.7.3 In FCT we define the following elementary properties of fuzzy sets:

$\mathrm{Hgt}(A)$	$\equiv_{\mathrm{df}}$	$(\exists x)(x \in A)$	height
$\mathrm{Norm}(A)$	$\equiv_{\mathrm{df}}$	$(\exists x)\triangle(x \in A)$	normality
$\mathrm{Crisp}(A)$	$\equiv_{\mathrm{df}}$	$(\forall x)\triangle(x \in A \vee x \notin A)$	crispness
$\mathrm{Fuzzy}(A)$	$\equiv_{\mathrm{df}}$	$\neg\,\mathrm{Crisp}(A)$	fuzziness
$\mathrm{Ext}_E(A)$	$\equiv_{\mathrm{df}}$	$(\forall x, y)(Exy \,\&\, x \in A \to y \in A)$	E-extensionality

The usual relations between fuzzy classes (e.g., inclusion, disjointness, etc.) can be defined by formulae of FCT as well:

Definition 1.7.4 We define in FCT the following elementary relations between fuzzy sets:

$A \subseteq B$	$\equiv_{\mathrm{df}}$	$(\forall x)(x \in A \to x \in B)$	inclusion
$A \approxeq B$	$\equiv_{\mathrm{df}}$	$(A \subseteq B) \,\&\, (B \subseteq A)$	(strong) bi-inclusion
$A \approx B$	$\equiv_{\mathrm{df}}$	$(\forall x)(x \in A \leftrightarrow x \in B)$	weak bi-inclusion
$A \parallel B$	$\equiv_{\mathrm{df}}$	$(\exists x)(x \in A \,\&\, x \in B)$	compatibility

Usual relational notions are definable in FCT, too:

Definition 1.7.5 In FCT, we define the following relational operations:

$A \times B$	$=_{\mathrm{df}}$	$\{\langle x, y\rangle \mid x \in A \,\&\, y \in B\}$	Cartesian product
$\mathrm{Dom}(R)$	$=_{\mathrm{df}}$	$\{x \mid Rxy\}$	domain
$\mathrm{Rng}(R)$	$=_{\mathrm{df}}$	$\{y \mid Rxy\}$	range
$R^{\to} A$	$=_{\mathrm{df}}$	$\{y \mid (\exists x \in A)Rxy\}$	image
$R^{\leftarrow} B$	$=_{\mathrm{df}}$	$\{x \mid (\exists y \in B)Rxy\}$	pre-image
$R \circ S$	$=_{\mathrm{df}}$	$\{\langle x, y\rangle \mid (\exists z)(Rxz \,\&\, Szy)\}$	composition
R^{-1}	$=_{\mathrm{df}}$	$\{\langle x, y\rangle \mid Ryx\}$	converse relation
Id	$=_{\mathrm{df}}$	$\{\langle x, y\rangle \mid x = y\}$	identity relation

Definition 1.7.6 In FCT, we define the basic properties of relations as follows:

$\mathrm{Refl}(R)$	$\equiv_{\mathrm{df}}$	$(\forall x)Rxx$	reflexivity
$\mathrm{Irrefl}(R)$	$\equiv_{\mathrm{df}}$	$(\forall x)(\neg Rxx)$	irreflexivity
$\mathrm{Sym}(R)$	$\equiv_{\mathrm{df}}$	$(\forall x, y)(Rxy \to Ryx)$	symmetry
$\mathrm{Trans}(R)$	$\equiv_{\mathrm{df}}$	$(\forall x, y, z)(Rxy \,\&\, Ryz \to Rxz)$	transitivity
$\mathrm{AntiSym}_E(R)$	$\equiv_{\mathrm{df}}$	$(\forall x, y)(Rxy \,\&\, Ryx \to Exy)$	E-antisymmetry
$\mathrm{ASym}(R)$	$\equiv_{\mathrm{df}}$	$(\forall x, y)\neg(Rxy \,\&\, Ryx)$	asymmetry

Besides the min-intersection and max-union of a pair of fuzzy classes, the inf-intersection and sup-union of a *fuzzy class* of fuzzy classes is definable in FCT:

Definition 1.7.7 The union and intersection of a class of classes are defined in FCT as follows:

$$\bigcup \mathcal{A} =_{\mathrm{df}} \{x \mid (\exists A \in \mathcal{A})(x \in A)\}$$

$$\bigcap \mathcal{A} =_{\mathrm{df}} \{x \mid (\forall A \in \mathcal{A})(x \in A)\}$$

(Observe that the functions $\bigcup, \bigcap$ assign a fuzzy class to a fuzzy class of fuzzy classes.)

In virtue of comprehension axioms (see Section 1.8), all defined notions of FCT are themselves certain fuzzy classes (and thus objects of the theory). E.g., the fuzzy property Hgt defines the fuzzy class $\mathcal{H}gt =_{\mathrm{df}} \{A \mid \mathrm{Hgt}(A)\}$ of fuzzy classes, to which a fuzzy class A belongs to the degree of its height.

1.8 Axioms of FCT

The axiomatic system of FCT is very simple: it contains the *axioms of comprehension* for all formulae of any order, which express the fact that any fuzzy property defines a fuzzy class; and the *axioms of extensionality* for fuzzy classes of all orders, which express the fact that fuzzy classes are determined by their membership functions.[10] We state the definition of the axioms formally:

Definition 1.8.1 Besides the logical axioms of multi-sorted first-order logic $\mathrm{MTL}_\triangle$ with identity (for which see Appendix 4), FCT contains the following axioms:

- The *comprehension axioms:* $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$, for any formula φ and any variables x, y of the same order. (Here the formula $\varphi(x)$ can contain further variables of any orders besides x.)
- The *extensionality axioms:* $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \rightarrow A = B$, for variables A, B of each order.

For handling tuples of individuals, the usual axioms for the identity of tuples (component-wise) are needed (for all orders):

- The *tuple identity axioms:* $\langle x_1, \ldots, x_k \rangle = \langle y_1, \ldots, y_k \rangle \rightarrow x_1 = y_1 \;\&\; \ldots \;\&\; x_k = y_k$, for all $k \geq 0$ and all orders of the variables.

Remark 1.8.2 The account of the axioms here is somewhat simplified for didactic purposes. When defining the apparatus for tuples, one actually needs to distinguish sorts of variables for each arity of tuples. However, these subtleties are only necessary in the formal introduction of the apparatus, as in the actual use these sorts are always uniquely determined by the context. Thus the reader can rely on the account provided here; full details can be found in [3].

Remark 1.8.3 The axioms listed in Definition 1.8.1 do not form a *complete* axiomatization of the notion of fuzzy set in full models (cf. footnote 3). Nevertheless, it seems sufficient for most practical purposes. Even though further axioms can be added to those of Definition 1.8.1, e.g., the axiom of infinity or various versions of the axiom of choice, no need for the additional axioms has arisen in the development of fuzzy mathematics within FCT so far.

Recall that the deduction of the theorems of FCT from the axioms must follow the rules of the logic $\mathrm{MTL}_\triangle$ (rather than classical Boolean logic), since we are dealing with fuzzy connectives and quantifiers. Some methods for actual proofs are provided in Section 3.

The following observation justifies why we defined the notions in Section 1.7 only for the first-order classes:

Observation 1.8.4 Since the language of FCT is the same at each order, defined symbols of any order can be shifted to all higher orders as well. Since furthermore the axioms of FCT are the same at each order, all theorems on FCT-definable notions are preserved by uniform upward order-shifts.

2 Features of FCT

2.1 Graded notions

A crucial feature of FCT is that not only the membership predicate $\in$, but generally all defined predicates are *graded* (unless they are defined as provably crisp).

[10] These axioms are analogous to the axioms for crisp classes in classical mathematics. From the point of view of formal logic, FCT can also be characterized as *(Henkin-style) higher-order fuzzy logic* or *simple type theory over fuzzy logic* (cf. [28]).

Example 2.1.1 Define the inclusion of fuzzy classes by the same formula as in classical mathematics, i.e.,

$$A \subseteq B \ \equiv_{\mathrm{df}} \ (\forall x)(x \in A \to x \in B).$$

Then $A \subseteq B$ is true to degree 1 in models iff the membership function of A is majorized by that of B (which is the non-graded notion of subsethood). The graded formula $A \subseteq B$, however, gets a large degree of truth also whenever A does not exceed B too much (where the exact meaning of "too much" depends on the $\mathrm{MTL}_\triangle$-chain used, or in standard models on the t-norm used). The graded notion of subsethood thus conveys more information than non-graded crisp subsethood, which can be defined as $\triangle(A \subseteq B)$.

Example 2.1.2 Traditionally, the reflexivity of a fuzzy relation R is defined by the condition that Rxx has the truth value 1 for all x. In FCT, the reflexivity of R can be defined by the formula $(\forall x)Rxx$. If the condition of the traditional definition is not satisfied, then the reflexivity of R simply does not hold (its truth value is 0). The defining formula in FCT, however, can even in such a case yield a meaningful non-zero truth value. For instance, if the truth value of Rxx is 0.999 for all x in a standard model, then the truth value of $(\forall x)Rxx$ is 0.999. It is clear that such a relation is "almost reflexive" (each pair $\langle x, x\rangle$ is almost fully in the relation R), even though it does not satisfy the traditional definition. Since furthermore the formula $(\forall x)Rxx$ has the same form as the formula which defines reflexivity in classical mathematics, it is quite natural to take its truth value for the degree of *graded* reflexivity of R, and say that R is 0.999-reflexive. The graded reflexivity is 1-true in a model iff the traditional condition of reflexivity holds; thus if we denote the graded reflexivity of R by $\mathrm{Refl}(R)$, then the corresponding non-graded notion is expressed in FCT by $\triangle\,\mathrm{Refl}(R)$.

Some graded notions have already been studied in fuzzy mathematics. Graded properties of fuzzy relations have been introduced in Gottwald's paper [17] and systematically investigated in his monograph [18]; more recently they have been elaborated in Gottwald's [19, §18.6] and Bělohlávek's [7, §4.1]. Many graded notions are also investigated in fuzzy topology, see e.g. [24, 29] and others.

Graded notions are important for several reasons:

- Graded properties generalize the traditional (non-graded) ones. A traditional non-graded property holds in a model iff the truth value of the corresponding graded property is exactly 1. In all other cases, graded properties provide a fine-grained scale of the degrees of their validity, while non-graded properties are then simply false.
- The graded approach allows to infer relevant information when the traditional conditions are almost, but still not completely, fulfilled. E.g., in Example 2.1.2, R is 0.999-reflexive: if we prove in FCT that the graded reflexivity of R implies some property φ, we shall know that φ holds at least to the degree 0.999 (as follows from the semantics of provable implication in $\mathrm{MTL}_\triangle$). On the contrary, from the non-graded reflexivity of R we cannot infer anything as it is simply false.
- Graded properties can easily be handled in FCT: valid inferences about them can be proved by the formal rules of first-order $\mathrm{MTL}_\triangle$ (see Section 3). The semantics of $\mathrm{MTL}_\triangle$ then translates the formal theorems into the laws valid for real-valued fuzzy relations.
- Graded properties are "fuzzier" than their non-graded counterparts: if we take seriously the idea of general fuzziness of concepts, there is no reason to presuppose that properties of fuzzy relations should only be crisp (i.e., either true or false as in the non-graded traditional definitions).

The gradedness of all notions in FCT allows proving more general theorems which are not available for non-graded notions in traditional fuzzy mathematics. A typical non-graded theorem of traditional fuzzy mathematics has the following form:

> *If* a (non-graded) assumption is true (i.e., fully true, since non-graded),
> *then* a (non-graded) conclusion is (fully) true.

With graded notions we can formulate (and prove in FCT) a much stronger theorem of the following form:

The more a (graded) assumption is true (even if partially),
the more a (graded) conclusion is true (i.e., at least as true as the assumption).

The latter can be expressed in FCT by means of implication $\varphi \to \psi$, where φ is the formula which expresses the assumption and ψ is the formula which expresses the conclusion. By the semantics of implication, if $\varphi \to \psi$ is provable in FCT, than the truth value of ψ is at least as large as the truth value of φ in any model of FCT. Provable implications thus express exactly the graded theorems of the above form. Since the full truth of φ is expressed by $\triangle\varphi$, the former non-graded theorem of traditional fuzzy mathematics is expressed in FCT by the formula $\triangle\varphi \to \triangle\psi$. The graded theorem $\varphi \to \psi$ is generally stronger than the non-graded theorem $\triangle\varphi \to \triangle\psi$, since the latter is an immediate consequence of the former in $\mathrm{MTL}_\triangle$ (by the rule of $\triangle$-necessitation and the axiom $\triangle 5$ of $\mathrm{MTL}_\triangle$, see Appendix 4), but not vice versa.

By the latter considerations, proving graded theorems amounts to proving formulae in the form of implication in FCT. Some methods for making such proofs are described in Section 3.

Example 2.1.3 Recall from Definition 1.7.2 that $\mathrm{Id} = \{\langle x, y\rangle \mid x = y\}$. Then:

- Traditional fuzzy mathematics proves that if a fuzzy relation R is reflexive (in the traditional sense), then Id is a fuzzy subset of R; i.e., if $Rxx = 1$ for each x, then $\mathrm{Id}\, xy \le Rxy$ for each x, y.
- In FCT we can easily prove that the more a fuzzy relation R is reflexive (in the graded sense), the more Id is a fuzzy subclass of R; in symbols, FCT proves $\mathrm{Refl}(R) \to \mathrm{Id} \subseteq R$. Thus for any left-continuous t-norm T we get $\inf\limits_x Rxx \le \inf\limits_{x,y} \overrightarrow{T}(\mathrm{Id}\, xy, Rxy)$.

Notice that the second result is indeed more general than the first one, as the first one follows from the second one (but obviously not vice-versa): assume that $Rxx = 1$, then $\inf_x Rxx = 1$ and so $\overrightarrow{T}(\mathrm{Id}\, xy, Rxy) = 1$ for each x, y, which entails that $\mathrm{Id}\, xy \le Rxy$. However, for R from Example 2.1.2, the traditional theorem asserts nothing (as R is *not* reflexive in the traditional sense), while the graded theorem of FCT ensures that Id is a fuzzy subclass of R at least to degree 0.999. (Much more complex examples of this kind can be found in the paper [2]).

2.2 Embedding of crisp structures

Since FCT contains the classical theory of classes (for classes which are crisp), we can introduce all concepts which are definable in classical class theory (i.e., in classical simple type theory, or Boolean higher-order logic—cf. footnote 10). The only thing we need to do is adding new predicate and functional symbols of the appropriate sorts add axioms saying that all predicates and functions appearing in the theory are crisp. The following definition is the formalization of this approach for the first-order theories (see Example 2.2.5 for an example of a higher order). For further generalizations of this approach see [10].

Definition 2.2.1 *Let* $\mathbf{\Gamma}$ *be a classical one-sorted predicate language and* T *a* $\mathbf{\Gamma}$*-theory. We define the language* $\mathrm{FCT}(\mathbf{\Gamma})$ *as the language of* FCT *restricted to symbols of order less than* 2 *and extended by* $\mathbf{\Gamma}$. *We define the theory* $\mathrm{FCT}(T)$ *in the language* $\mathrm{FCT}(\mathbf{\Gamma})$ *as the theory with the following axioms:*

- *The axioms of* FCT
- *The axioms of* T
- $\mathrm{Crisp}(\bar{Q})$ *for each predicate symbol* $Q \in \mathbf{\Gamma}$

Lemma 2.2.2 *Let* $\mathbf{\Gamma}$ *be a classical predicate language,* T *a* $\mathbf{\Gamma}$*-theory,* $\mathbf{L}$ *an* $\mathrm{MTL}_\triangle$*-algebra. If* $\mathbf{M}$ *is an* $\mathbf{L}$*-model of* $\mathrm{FCT}(T)$*, then the classical model* $\mathbf{M}^c$ *in the language* $\mathbf{\Gamma}$ *with the domain* M *and* $S_{\mathbf{M}^c} = S_{\mathbf{M}}$ *for each* $S \in \mathbf{\Gamma}$*, is a model (in the sense of classical logic) of the theory* T*. Vice versa, for each model* $\mathbf{M}$ *of* T *there is an* $\mathbf{L}$*-model* $\mathbf{N}$ *of* $\mathrm{FCT}(T)$ *such that* $\mathbf{N}^c$ *is isomorphic to* $\mathbf{M}$.

Therefore, $T \vdash \varphi$ *iff* $\mathrm{FCT}(T) \vdash \varphi$*, for any* $\mathbf{\Gamma}$*-formula* φ.

Example 2.2.3 *Let R be a binary first-order predicate symbol. Then in each* **L***-model of the theory* $\mathrm{Crisp}(R),\ \mathrm{Refl}(R), \mathrm{Trans}(R), \mathrm{AntiSym}(R)$*, the symbol R is represented by a crisp ordering on the universe of objects.*

Example 2.2.4 *If T is a classical theory of the real closed field, then in each* **L***-model* **M** *of the theory* $\mathrm{FCT}(T)$*, the universe of objects with $\leq_{\mathbf{M}}, +_{\mathbf{M}}, -_{\mathbf{M}}, \cdot_{\mathbf{M}}, 0_{\mathbf{M}}, 1_{\mathbf{M}}$ is a real closed field.*

Example 2.2.5 *Let τ be a constant[11]for a class of classes and T the theory with the axioms:*

- $\mathrm{Crisp}(\tau)$
- $X \in \tau \to \mathrm{Crisp}(X)$
- $\mathrm{Crisp}(\mathcal{X})\ \&\ \mathcal{X} \subseteq \tau \to \bigcup \mathcal{X} \in \tau$
- $X_1 \in \tau\ \&\ X_2 \in \tau \to X_1 \cap X_2 \in \tau$

Then in each **L***-model of the theory T, the constant τ is represented by a classical topology on the universe of objects.*

2.3 Natural fuzzification

It can be observed that the defining formulae of most notions in FCT are exactly the same as the definitions of analogous properties of crisp relations in classical mathematics. This correlates with the motivation of fuzzy logic as a generalization of classical logic to non-crisp predicates: classical mathematical notions are fuzzified in a natural way just by reinterpreting the classical definitions in fuzzy logic. This methodology has been foreshadowed in [22, §5] by Höhle, much later formalized in [3, §7], and suggested as an important guideline for formal fuzzy mathematics in [5].

If we examine the definition in the above section, we see the crucial rôle of the predicate Crisp. If we remove this predicate from the above definitions we get the "natural" fuzzification of the concepts described by the theory. In order to illustrate the methodology of fuzzification, let us concentrate on the concept of ordering. If we remove the predicate Crisp from the definition, then we get the concept of fuzzy ordering, as it was introduced by Zadeh. However, some carefulness is due here not to overlook some "hidden" crispness. There is crisp identity used in the antisymmetry axiom, and also in the reflexivity axiom which can be written as $(\forall x, y)(x = y \to Rxy)$. This concept of fuzzy ordering was studied mainly by Bodenhoffer, see e.g., [8]. For more details about removing hidden crispness see [1].

2.4 Split notions

Even though an important guideline, the method of natural fuzzification described in Section 2.3 cannot be applied mechanically, as some classically equivalent definitions may no longer be equivalent in fuzzy logic. Then the classically equivalent definitions are concurrent candidates for the definition of the fuzzy notion. Of these, in some cases one is behaving best and can be chosen as the fuzzy counterpart of the crisp notion. In other cases, two or more variants of the definition are meaningful and well-behaved in fuzzy logic: then the notion of classical mathematics splits into several notions in FCT. This can be exemplified by the notion of equality of fuzzy classes. Besides the primitive crisp identity = of fuzzy classes, at least two graded notions of natural fuzzy equality can be defined (see Definition 1.7.4):

$$\begin{array}{llll} A \approx B & \equiv_{\mathrm{df}} & (\forall x)[(Ax \to Bx)\ \&\ (Bx \to Ax)], & \text{i.e., } (\forall x)(Ax \leftrightarrow Bx) \\ A \approxeq B & \equiv_{\mathrm{df}} & (\forall x)(Ax \to Bx)\ \&\ (\forall x)(Bx \to Ax), & \text{i.e., } (A \subseteq B)\ \&\ (B \subseteq A) \end{array}$$

These notions are not equivalent in FCT, as shown by the following counter-example:

[11] Recall that any n-ary predicate R of order m can be identified with a constant symbol for a class of order $m+1$ of n-tuples.

Example 2.4.1 Let A, B be interpreted in a model over the standard $\mathrm{MV}_\triangle$-algebra (see Example 4.1.5) by the following assignment of truth values: $Ap = Bq = 1$, $Aq = Bp = 0.5$, and 0 otherwise. Then the truth value of $A \approx B$ is 0.5, while the truth value of $A \approxeq B$ is 0.

In traditional non-graded fuzzy mathematics both notions coincide, since they are fully true under the same conditions; however, under the graded approach they differ, since in graded fuzzy mathematics we do not require them to be true to degree 1. It can be noticed that Gottwald [19] uses $\approxeq$ while Bělohlávek [7] uses $\approx$ as a graded equality of fuzzy sets. For the interrelation of both notions see [6].

3 How to prove results in Fuzzy Class Theory

In this section we give some hints as to how to make valid proofs of graded theorems in FCT.

3.1 Derivations in fuzzy logic

As argued above, many formulae of FCT look very similar to those of classical set theory. This is made possible by the design of the theory which hid the references to truth degrees into the atomic formulae and their combinations. It is the meaning of connectives and quantifiers in which the formulae of FCT differ from those of classical set theory: the connectives are interpreted by the rules of fuzzy logic rather than classical Boolean logic. Consequently, formulae that express laws valid for fuzzy sets can formally be deduced from the axioms of FCT, but the derivation must use the logic $\mathrm{MTL}_\triangle$ instead of classical Boolean logic.[12]

$\mathrm{MTL}_\triangle$ is not much different from classical logic; many proof methods of classical logic therefore work in $\mathrm{MTL}_\triangle$ as well. E.g., by the transitivity of implication, one can prove in steps (i.e., in order to establish $\varphi \to \psi$ one can prove the chain of implications $\varphi \longrightarrow \chi_1 \longrightarrow \cdots \longrightarrow \chi_n \longrightarrow \psi$). Provable implications are therefore useful for transitions between these successive steps; some of them are listed for reference at the end of this section. Since furthermore many theorems of FCT have exactly the same form as in classical mathematics, classical proofs (or their slightly adapted variants that avoid rules that are invalid in $\mathrm{MTL}_\triangle$) often work for them.

The main difference between $\mathrm{MTL}_\triangle$ and classical logic as regards proof methods is that some of classically valid implications and equivalences are invalid in $\mathrm{MTL}_\triangle$. These must be avoided in proofs in FCT; only the laws that are provable in $\mathrm{MTL}_\triangle$ can be used in FCT proofs. A list of rules that are provable in $\mathrm{MTL}_\triangle$ can be found in Section 3.2. There are two main groups of classically valid, but $\mathrm{MTL}_\triangle$-invalid logical laws which must be avoided:

1. Most intuitionistically invalid laws are also invalid in $\mathrm{MTL}_\triangle$, since $\mathrm{MTL}_\triangle$ lacks the law of double negation ($\mathrm{MTL}_\triangle$ can actually be characterized as intuitionistic logic with globalization, minus the rule of contraction, plus the rule of prelinearity). These rules are recovered in extensions of $\mathrm{MTL}_\triangle$ that enjoy the double negation law, e.g., $\mathrm{IMTL}_\triangle$ or Łukasiewicz logic.

2. The rule of contraction $\varphi \leftrightarrow \varphi \,\&\, \varphi$ is in general invalid in $\mathrm{MTL}_\triangle$. Consequently, multiple occurrences of the same formula among the premises cannot be cancelled and must all appear in the theorem. By the rules of $\mathrm{MTL}_\triangle$ for implication (see Lemma 3.2.2(3) and comments below it), if a premise is used n times in an implicational proof (i.e., a proof that does not use $\triangle$-necessitation), it has to appear n times among the premises of the theorem. Only crisp formulae can be contracted and handled by classically valid inference rules in $\mathrm{MTL}_\triangle$.

[12] In using fuzzy logic for proving theorems Fuzzy Class Theory differs radically from mainstream (traditional) fuzzy mathematics. Traditional fuzzy mathematics models fuzzy sets by membership functions which themselves are crisp, therefore it can use classical logic for proving its theorems. FCT, on the other hand, takes fuzzy sets as a primitive fuzzy notion; since fuzzy logic is designed as the logic for reasoning about fuzzy notions, it is the latter which must be used for proofs in FCT rather than classical logic.

3.2 Proof methods in $\mathrm{MTL}_\triangle$ and FCT

Like in classical mathematics, proofs in FCT can either be formal or informal. Formal proofs are always primary, in that the validity of a theorem of FCT is founded on a formal proof in first-order $\mathrm{MTL}_\triangle$ from the axioms of FCT. Informal proofs are descriptions (in symbols or even natural language) of a formal proof or hints that can lead the reader to the reconstruction of a formal proof, if such a formal proof is requested. Since formal proofs (i.e., sequences of formulae) are hard to read, informal proofs are often much more preferable for human readers.

Each step in an informal proof is based on some formal law derivable in first-order $\mathrm{MTL}_\triangle$. Reasoning by rules based on the laws of $\mathrm{MTL}_\triangle$ is sound w.r.t. derivability in $\mathrm{MTL}_\triangle$, and the conclusions of such reasoning are therefore valid theorems of FCT. Some laws of $\mathrm{MTL}_\triangle$ which are useful for making informal (or even formal) proofs in FCT are listed below (the list is by no means exhaustive). Most of them are restatements or easy corollaries of theorems proved in [20] and [12].

Lemma 3.2.1 *The following are theorems of* $\mathrm{MTL}_\triangle$*:*

(T1) $\varphi \to (\psi \to \varphi)$
(T2) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(T3) $\varphi \to \varphi$

(T4a) $(\varphi \to \psi) \to (\varphi \mathbin{\&} \chi \to \psi \mathbin{\&} \chi))$
(T4b) $(\varphi \to \psi) \to (\varphi \wedge \chi \to \psi \wedge \chi))$
(T4c) $(\varphi \to \psi) \to (\varphi \vee \chi \to \psi \vee \chi))$
(T5) $(\varphi \to (\psi \to \varphi \mathbin{\&} \psi)$

(T6) $\varphi \mathbin{\&} \psi \to \varphi \wedge \psi$
(T7) $\varphi \wedge \psi \to \psi$
(T8) $((\chi \to \varphi) \wedge (\chi \to \psi)) \to (\chi \to \varphi \wedge \psi)$
(T8′) $((\chi \to \varphi) \mathbin{\&} (\chi \to \psi)) \to (\chi \to \varphi \wedge \psi)$
(T9a) $(\varphi \to \psi) \to (\varphi \leftrightarrow \varphi \wedge \psi)$
(T9b) $(\varphi \to \psi) \leftrightarrow (\varphi \to \varphi \wedge \psi)$
(T9c) $\varphi \to (\varphi \wedge \psi \leftrightarrow \psi)$

(T10) $\varphi \to \varphi \vee \psi$
(T11) $((\varphi \to \chi) \wedge (\psi \to \chi)) \to (\varphi \vee \psi \to \chi)$
(T11′) $((\varphi \to \chi) \mathbin{\&} (\psi \to \chi)) \to (\varphi \vee \psi \to \chi)$
(T12) $(\varphi \to \psi) \vee (\psi \to \varphi)$
(T13) $(\varphi \to \psi)^n \vee (\psi \to \varphi)^n$
(T14a) $(\varphi \to \psi) \to (\psi \leftrightarrow \varphi \vee \psi)$
(T14b) $(\varphi \to \psi) \leftrightarrow (\varphi \vee \psi \to \psi)$

(T15) $\varphi \to \neg\neg\varphi$
(T16) $\neg\varphi \to (\varphi \to \psi)$
(T17) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
(T17′) $(\varphi \leftrightarrow \psi) \to (\neg\psi \leftrightarrow \neg\varphi)$
(T18a) $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$
(T18b) $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$

(T19) $\varphi \mathbin{\&} 1 \leftrightarrow \varphi$
(T20) $(1 \to \varphi) \leftrightarrow \varphi$

(T21) $(\varphi \leftrightarrow \psi) \to ((\varphi \mathbin{\&} \chi) \leftrightarrow (\psi \mathbin{\&} \chi))$
(T21′) $(\varphi \leftrightarrow \psi) \mathbin{\&} (\chi \leftrightarrow \delta) \to ((\varphi \mathbin{\&} \chi) \leftrightarrow (\psi \mathbin{\&} \delta))$
(T22) $(\varphi \leftrightarrow \psi) \to ((\varphi \to \chi) \leftrightarrow (\psi \to \chi))$
(T23) $(\varphi \leftrightarrow \psi) \to ((\chi \to \varphi) \leftrightarrow (\chi \to \psi))$
(T23′) $(\varphi \leftrightarrow \psi) \mathbin{\&} (\chi \leftrightarrow \delta) \to ((\varphi \to \chi) \leftrightarrow (\psi \to \delta))$

(T24a) $\varphi \,\&\, (\psi \wedge \chi) \leftrightarrow (\varphi \,\&\, \psi) \wedge (\varphi \,\&\, \chi)$
(T24b) $\varphi \,\&\, (\psi \vee \chi) \leftrightarrow (\varphi \,\&\, \psi) \vee (\varphi \,\&\, \chi)$
(T25a) $\varphi \vee (\psi \wedge \chi) \leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \chi)$
(T25b) $\varphi \wedge (\psi \vee \chi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
(T26a) $(\varphi \wedge \psi) \,\&\, (\varphi \wedge \psi) \leftrightarrow (\varphi \,\&\, \varphi) \wedge (\psi \,\&\, \psi)$
(T26b) $(\varphi \vee \psi) \,\&\, (\varphi \vee \psi) \leftrightarrow (\varphi \,\&\, \varphi) \vee (\psi \,\&\, \psi)$
(T27) $(\varphi \,\&\, \psi) \,\&\, \chi \leftrightarrow \varphi \,\&\, (\psi \,\&\, \chi)$
(T28) $(\varphi \to \psi) \,\&\, (\chi \to \delta) \to ((\psi \to \chi) \to (\varphi \to \delta))$
(T29) $(\varphi \to \psi) \,\&\, (\chi \to \delta) \to ((\varphi \,\&\, \chi) \to (\psi \,\&\, \delta))$

(T△1) $\triangle\varphi \leftrightarrow \triangle(\varphi \,\&\, \varphi)$
(T△2) $\triangle\varphi \leftrightarrow \triangle\varphi \,\&\, \triangle\varphi$
(T△3) $\triangle(\varphi \,\&\, \psi) \leftrightarrow (\triangle\varphi \,\&\, \triangle\psi)$
(T△4) $(\triangle\varphi \wedge \triangle\psi) \leftrightarrow (\triangle\varphi \,\&\, \triangle\psi)$
(T△5) $\triangle(\varphi \wedge \psi) \leftrightarrow (\triangle\varphi \wedge \triangle\psi)$

Let I and J be finite sets. Then the following are theorems of $\mathrm{MTL}_\triangle$*:*

(S1) $\bigwedge_{i\in I} \varphi_i \,\&\, \bigwedge_{j\in J} \psi_j \leftrightarrow \bigwedge_{i\in I, j\in J} (\varphi_i \,\&\, \psi_j)$
(S2) $\bigvee_{i\in I} \varphi_i \,\&\, \bigvee_{j\in J} \psi_j \leftrightarrow \bigvee_{i\in I, j\in J} (\varphi_i \,\&\, \psi_j)$
(S3) $(\bigvee_{i\in I} \varphi_i \to \bigwedge_{j\in J} \psi_j) \leftrightarrow \bigwedge_{i\in I, j\in J} (\varphi_i \to \psi_j)$
(S4) $(\bigwedge_{i\in I} \varphi_i \to \bigvee_{j\in J} \psi_j) \leftrightarrow \bigvee_{i\in I, j\in J} (\varphi_i \to \psi_j)$
(S5) $\bigwedge_{i\in I} \varphi_i \vee \bigwedge_{j\in J} \psi_j \leftrightarrow \bigwedge_{i\in I, j\in J} (\varphi_i \vee \psi_j)$
(S6) $\bigvee_{i\in I} \varphi_i \wedge \bigvee_{j\in J} \psi_j \leftrightarrow \bigvee_{i\in I, j\in J} (\varphi_i \wedge \psi_j)$

The following lemmata justify certain proof methods in $\mathrm{MTL}_\triangle$; they are often used without explicit mention in informal proofs. Besides giving the lemmata we shall also comment on how they can be employed in both formal and informal proofs in FCT.

Lemma 3.2.2 *The theorems of the following forms are provable in the propositional logic* $\mathrm{MTL}_\triangle$*:*

1. $[\varphi \to (\psi \to \chi)] \longleftrightarrow [\varphi \,\&\, \psi \to \chi] \longleftrightarrow [\psi \,\&\, \varphi \to \chi] \longleftrightarrow [\psi \to (\varphi \to \chi)]$
2. $\triangle(\varphi \vee \psi) \,\&\, (\varphi \to \chi) \,\&\, (\psi \to \chi) \to \chi$
3. $[(\varphi_1 \to \psi_1) \,\&\, (\varphi_2 \to \psi_2)] \to (\varphi_1 \,\&\, \varphi_2 \to \psi_1 \,\&\, \psi_2)$
4. $[(\varphi_1 \to \psi_1) \wedge (\varphi_2 \to \psi_2)] \to (\varphi_1 \wedge \varphi_2 \to \psi_1 \wedge \psi_2)$
5. $[(\varphi_1 \to (\psi_1 \to \psi_2)) \wedge (\varphi_2 \to (\psi_2 \to \psi_1))] \to (\varphi_1 \wedge \varphi_2 \to (\psi_1 \leftrightarrow \psi_2))$
6. $\triangle(\varphi \vee \neg\varphi) \to [(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))]$
7. $\triangle(\varphi \vee \neg\varphi) \to [(\varphi \,\&\, (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \varphi \,\&\, \chi)]$

By Lemma 3.2.2(1), one can arbitrarily reorder the premises in proofs. Lemma 3.2.2(2) justifies proofs by cases (one must be, however, careful not to take cases on φ and $\neg\varphi$, as $\varphi \vee \neg\varphi$ is generally *not* provable in $\mathrm{MTL}_\triangle$ unless φ is crisp). Lemma 3.2.2(3) shows that a strong conjunction can be proved by proving each conjunct separately. Similarly Lemma 3.2.2(4) shows that to prove a min-conjunction it is enough to take min-conjunction of the premisses. Lemma 3.2.2(5) demonstrates that equivalence is implied by a min-conjunction of premises needed to prove both implications. Lemma 3.2.2(6) and (7) allow certain distributions for *crisp* formulae.

Lemma 3.2.2(3) and Lemma 3.2.2(4) demonstrate an important feature of formal fuzzy mathematics: counting the premisses. They justify two important proof methods: if we can prove

$$\varphi_1^{i_1} \,\&\, \varphi_2^{i_2} \,\&\, \ldots \,\&\, \varphi_n^{i_n} \to \psi \text{ and } \varphi_1^{j_1} \,\&\, \varphi_2^{j_2} \,\&\, \ldots \,\&\, \varphi_n^{j_n} \to \chi$$

then we can also prove

$$\varphi_1^{i_1+j_1} \,\&\, \varphi_2^{i_2+j_2} \,\&\, \ldots \,\&\, \varphi_n^{i_n+j_n} \to \psi \,\&\, \chi$$

as well as

$$\varphi_1^{\max(i_1,j_1)} \,\&\, \varphi_2^{\max(i_2,j_2)} \,\&\, \ldots \,\&\, \varphi_n^{\max(i_n,j_n)} \to \psi \wedge \chi$$

(Notice two different ways of "counting" based on whether we prove conjunction or min-conjunction of conclusions.)

Lemma 3.2.3 together with the transitivity of implication makes it possible to prove theorems by successive transformations of the formula using provably equivalent substitutes of subformulae:

Lemma 3.2.3 *If $\varphi \leftrightarrow \psi$ is provable in the first-order logic* $\mathrm{MTL}_\triangle$, *then so is $\chi[\varphi/\psi] \leftrightarrow \chi$, where $\chi[\varphi/\psi]$ is the result of replacing an occurrence of the subformula φ by ψ in χ.*

Proof: By induction on the complexity of χ, using the fact that all connectives and quantifiers of first-order $\mathrm{MTL}_\triangle$ are congruent w.r.t. provable equivalence. QED

Now we list several important theorem of first-order $\mathrm{MTL}_\triangle$. These theorems are either proven in [12] or are simple consequences of theorems listed there.

Lemma 3.2.4 *For arbitrary formulae φ, ψ, ν where ν does not contain x freely, the following are theorems of first-order* $\mathrm{MTL}_\triangle$*:*

($T\forall 1$)	$(\forall x)(\nu \to \varphi) \leftrightarrow (\nu \to (\forall x)\varphi)$
($T\forall 2$)	$(\forall x)(\varphi \to \nu) \leftrightarrow ((\exists x)\varphi \to \nu)$
($T\forall 3$)	$(\exists x)(\nu \to \varphi) \to (\nu \to (\exists x)\varphi)$
($T\forall 4$)	$(\exists x)(\varphi \to \nu) \to ((\forall x)\varphi \to \nu)$
($T\forall 5$)	$(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$
($T\forall 6$)	$(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$
($T\forall 7$)	$(\forall x)\varphi \,\&\, (\exists x)\psi \to (\exists x)(\varphi \,\&\, \psi)$
($T\forall 8$)	$(\forall x)\varphi(x) \leftrightarrow (\forall y)\varphi(y)$
($T\forall 8'$)	$(\exists x)\varphi(x) \leftrightarrow (\exists y)\varphi(y)$
($T\forall 9$)	$(\exists x)(\varphi \,\&\, \nu) \leftrightarrow ((\exists x)\varphi \,\&\, \nu)$
($T\forall 10$)	$(\exists x)\varphi^n \leftrightarrow ((\exists x)\varphi)^n$
($T\forall 11$)	$(\exists x)\varphi \to \neg(\forall y)\neg\varphi$
($T\forall 12$)	$\neg(\exists x)\varphi \leftrightarrow (\forall y)\neg\varphi$
($T\forall 13$)	$(\exists x)(\nu \wedge \varphi) \leftrightarrow (\nu \wedge (\exists x)\varphi)$
($T\forall 14$)	$(\exists x)(\nu \vee \varphi) \leftrightarrow (\nu \vee (\exists x)\varphi)$
($T\forall 15$)	$(\forall x)(\nu \wedge \varphi) \leftrightarrow (\nu \wedge (\forall x)\varphi)$
($T\forall 16$)	$(\exists x)(\varphi \vee \psi) \leftrightarrow ((\exists x)\varphi \vee (\exists x)\varphi)$
($T\forall 17$)	$(\forall x)(\varphi \wedge \psi) \leftrightarrow ((\forall x)\varphi \wedge (\forall x)\psi)$
($T\forall 18$)	$(\exists x)(\varphi \wedge \psi) \to ((\exists x)\varphi \wedge (\exists x)\varphi)$
($T\forall 19$)	$((\forall x)\varphi \vee (\forall x)\psi) \to (\forall x)(\varphi \vee \psi)$
($T\forall 20$)	$(\forall x)(\varphi \to (\forall y)(\psi \to \chi)) \leftrightarrow (\forall y)(\psi \to (\forall x)(\varphi \to \chi))$
($T\forall 21$)	$(\exists x)(\varphi \,\&\, (\forall y)(\psi \,\&\, \chi)) \leftrightarrow (\forall y)(\psi \,\&\, (\forall x)(\varphi \,\&\, \chi))$

3.3 Proof methods in FCT

Besides proof methods described above, which are suitable for any theory in first-order logic $\mathrm{MTL}_\triangle$, there are methods specially tailored for FCT. First notice that theorems ($T\forall 20$) and ($T\forall 21$) justify exchanging relativized quantifiers of the same kind, e.g., $(\forall x \in A)(\forall y \in B)\varphi \leftrightarrow (\forall y \in B)(\forall x \in A)\varphi$ or $(\exists A \subseteq B)(\exists C \in \mathcal{D})\varphi \leftrightarrow (\exists C \in \mathcal{D})(\exists A \subseteq B)\varphi$, etc. Let us continue by few simple examples:

Lemma 3.3.1 *Formulae of the following forms are provable in* FCT*:*

1. $\bigcup\{B \mid \varphi(B)\} \subseteq A \leftrightarrow (\forall B)(\varphi(B) \to B \subseteq A)$
2. $A \subseteq \bigcap\{B \mid \varphi(B)\} \leftrightarrow (\forall B)(\varphi(B) \to A \subseteq B)$
3. $\varphi(C) \to \bigcap\{B \mid \varphi(B)\} \subseteq C$
4. $\varphi(C) \to C \subseteq \bigcup\{B \mid \varphi(B)\}$
5. $\mathrm{Crisp}\, A \to [(\forall x \in A)(\varphi \to \psi) \to [(\forall x \in A)\varphi \to (\forall x \in A)\psi]]$

Now we present a method a reducing some proof to the propositional case as shown in [3]. Before we start we give a few general definitions from [3].

Convention 3.3.2 *Let $\varphi(p_1, \dots, p_n)$ be a propositional formula with $p_1, \dots, p_n$ its only propositional variables, and let $\psi_1, \dots, \psi_n$ be any formulae. By $\varphi(\psi_1, \dots, \psi_n)$ we denote the formula φ in which all occurrences of p_i are replaced by ψ_i (for all $i \le n$).*

Furthermore, if n is known from the context, we shall sometimes write just $\vec{X}$ for $X_1, \dots, X_n$.

Definition 3.3.3 *Let $\varphi(p_1, \dots, p_n)$ be a propositional formula. We define the n-ary class operation induced by φ as*

$$\mathrm{Op}_\varphi(X_1, \dots, X_n) =_{\mathrm{df}} \{x \mid \varphi(x \in X_1, \dots, x \in X_n)\}.$$

Furthermore we define two relations between $X_1, \dots, X_n$ induced by φ:

$$\begin{aligned} \mathrm{Rel}^{\forall}_\varphi(X_1, \dots, X_n) &\equiv_{\mathrm{df}} (\forall x)\varphi(x \in X_1, \dots, x \in X_n) \\ \mathrm{Rel}^{\exists}_\varphi(X_1, \dots, X_n) &\equiv_{\mathrm{df}} (\exists x)\varphi(x \in X_1, \dots, x \in X_n) \end{aligned}$$

Example 3.3.4

$$\begin{aligned} \mathrm{Rel}^{\forall}_{p \to q}(X, Y) &\equiv_{\mathrm{df}} (\forall x)(x \in X \to x \in Y), \text{ i.e., } X \subseteq Y \\ \mathrm{Op}_{p \& q}(X, Y) &\equiv_{\mathrm{df}} \{x \mid x \in X \,\&\, x \in Y\}, \text{ i.e., } X \cap Y \end{aligned}$$

The following metatheorems show that a large part of elementary fuzzy set theory can be reduced to fuzzy propositional calculus.

Theorem 3.3.5 *Let $\varphi, \psi_1, \dots, \psi_n$ be propositional formulae.*

Then $\mathrm{MTL}_\triangle$ *proves* $\varphi(\psi_1, \dots, \psi_n)$

$$\text{iff FCT proves } \mathrm{Rel}^{\forall}_\varphi(\mathrm{Op}_{\psi_1}(X_{1,1}, \dots, X_{1,k_1}), \dots, \mathrm{Op}_{\psi_n}(X_{n,1}, \dots, X_{n,k_n})) \tag{3.1}$$

$$\text{iff FCT proves } \mathrm{Rel}^{\exists}_\varphi(\mathrm{Op}_{\psi_1}(X_{1,1}, \dots, X_{1,k_1}), \dots, \mathrm{Op}_{\psi_n}(X_{n,1}, \dots, X_{n,k_n})) \tag{3.2}$$

Corollary 3.3.6 *Let φ and ψ be propositional formulae.*

If $\mathrm{MTL}_\triangle$ *proves* $\varphi \to \psi$ *then* FCT *proves* $\mathrm{Op}_\varphi(X_1, \dots, X_n) \subseteq \mathrm{Op}_\psi(X_1, \dots, X_n)$.

If $\mathrm{MTL}_\triangle$ *proves* $\varphi \leftrightarrow \psi$ *then* FCT *proves* $\mathrm{Op}_\varphi(X_1, \dots, X_n) = \mathrm{Op}_\psi(X_1, \dots, X_n)$.

If $\mathrm{MTL}_\triangle$ *proves* $\varphi \vee \neg\varphi$ *then* FCT *proves* $\mathrm{Crisp}(\mathrm{Op}_\varphi(X_1, \dots, X_n))$.

By virtue of Theorem 3.3.5, the properties of propositional connectives directly translate to the properties of class relations and operations. For example:

Provability in $\mathrm{MTL}_\triangle$ of:	Proves in FCT:
$\triangle p \to p$	$\mathrm{Ker}(X) \subseteq X$
$p \to p \veebar q$	$X \subseteq X \cup Y$
$p \to p \veebar q$	$\mathrm{Norm}(X) \to \mathrm{Norm}(X \cup Y)$
$0 \to p$	$\emptyset \subseteq X$
$p \,\&\, q \to p \wedge q$	$X \cap Y \subseteq X \sqcap Y$

In order to translate monotonicity and congruence properties of propositional connectives to the same properties of class operations, we need another theorem:

Theorem 3.3.7 *Let $\varphi_i, \varphi'_i, \psi_{i,j}, \psi'_{i,j}$ be propositional formulae. Then* MTL$_\triangle$ *proves*

$$\mathop{\&}_{i=1}^{k} \varphi_i(\psi_{i,1}, \ldots, \psi_{i,n_i}) \rightarrow \bigwedge_{i=1}^{k'} \varphi'_i(\psi'_{i,1}, \ldots, \psi'_{i,n'_i}) \tag{3.3}$$

iff FCT *proves*

$$\mathop{\&}_{i=1}^{k} \mathrm{Rel}^{\forall}_{\varphi_i}\left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X})\right) \rightarrow \bigwedge_{i=1}^{k'} \mathrm{Rel}^{\forall}_{\varphi'_i}\left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X})\right)$$

Examples of direct corollaries of the theorem:

Provability in MTL$_\triangle$ of	Proves in FCT
$(p \rightarrow q) \rightarrow ((p \,\&\, r) \rightarrow (q \,\&\, r))$	$X \subseteq Y \rightarrow X \cap Z \subseteq Y \cap Z$
$(p \rightarrow q) \rightarrow (p \rightarrow (p \wedge q))$	$X \subseteq Y \rightarrow X \subseteq X \sqcap Y$
$[(p \rightarrow q) \,\&\, (q \rightarrow p)] \rightarrow (p \leftrightarrow q)$	$(X \subseteq Y \,\&\, Y \subseteq X) \rightarrow X \approx Y$
$(p \leftrightarrow q) \rightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$	$X \approx Y \rightarrow (X \subseteq Y \wedge Y \subseteq X)$
$[(p \rightarrow r) \,\&\, (q \rightarrow r)] \rightarrow (p \veebar q \rightarrow r)$	$(X \subseteq Z \,\&\, Y \subseteq Z) \rightarrow X \cup Y \subseteq Z$
$\triangle(p \rightarrow q) \rightarrow [\triangle(\alpha \rightarrow p) \rightarrow \triangle(\alpha \rightarrow q)]$	$\triangle(X \subseteq Y) \rightarrow X_\alpha \subseteq Y_\alpha$
transitivity of $\rightarrow, \leftrightarrow$	transitivity of $\subseteq, \approx$, etc.

To derive theorems about $\mathrm{Rel}^{\exists}$, we slightly modify Theorem 3.3.7:

Theorem 3.3.8 *Let $\varphi_i, \varphi'_i, \psi_{i,j}, \psi'_{i,j}$ be propositional formulae. Then* MTL$_\triangle$ *proves*

$$\mathop{\&}_{i=1}^{k} \varphi_i(\psi_{i,1}, \ldots, \psi_{i,n_i}) \rightarrow \bigvee_{i=1}^{k'} \varphi'_i(\psi'_{i,1}, \ldots, \psi'_{i,n'_i}) \tag{3.4}$$

iff FCT *proves*

$$\mathop{\&}_{i=1}^{k-1} \mathrm{Rel}^{\forall}_{\varphi_i}\left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X})\right) \,\&\, \mathrm{Rel}^{\exists}_{\varphi_k}\left(\mathrm{Op}_{\psi_{k,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{k,n_k}}(\vec{X}\right) \rightarrow$$

$$\rightarrow \bigvee_{i=1}^{k'} \mathrm{Rel}^{\exists}_{\varphi'_i}\left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X})\right) \tag{3.5}$$

Examples of direct corollaries:

Provability in MTL$_\triangle$ of	Proves in FCT
$((p \rightarrow q) \,\&\, p) \rightarrow q$	$((X \subseteq Y) \,\&\, \mathrm{Hgt}(X)) \rightarrow \mathrm{Hgt}(Y)$
$\triangle(p \vee q) \rightarrow \triangle p \vee \triangle q$	$\mathrm{Norm}(X \sqcup Y) \rightarrow \mathrm{Norm}(X) \vee \mathrm{Norm}(Y)$
$(p \rightarrow r) \,\&\, (p \,\&\, q) \rightarrow (q \,\&\, r)$	$X \subseteq Z \,\&\, X \parallel Y \rightarrow Y \parallel Z$, etc.

4 Appendix: Logic MTL$_\triangle$

Monoidal t-norm based logic (MTL for short) was introduced by Esteva and Godo in [12] as an extension of Höhle's monoidal logic [23] by the axiom of prelinearity (i.e., the axiom (A6) below). In this appendix we recall some of the basic properties of MTL and its expansion by the connective $\triangle$, first propositional and then first-order.

4.1 Propositional logics MTL and $\mathrm{MTL}_\triangle$

The formulae of propositional logic MTL are composed from a countable set of propositional atoms by using three basic binary connectives $\to$, $\wedge$, and $\&$, and a nullary connective 0. Further connectives can be defined as:

$$
\begin{array}{ccl}
\varphi \vee \psi & \text{is} & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), \\
\neg\varphi & \text{is} & \varphi \to 0, \\
\varphi \leftrightarrow \psi & \text{is} & (\varphi \to \psi) \wedge (\psi \to \varphi), \\
1 & \text{is} & \neg 0.
\end{array}
$$

Convention 4.1.1 In order to avoid unnecessary parentheses, we stipulate that unary connectives take precedence over $\wedge$, $\vee$, and $\&$, which in turn bind more closely than $\to$ and $\leftrightarrow$.

The deduction rule of MTL is Modus Ponens (from φ and $\varphi \to \psi$ infer ψ) and the following formulae are axioms of MTL:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $\varphi \,\&\, \psi \to \varphi$
(A3) $\varphi \,\&\, \psi \to \psi \,\&\, \varphi$
(A4a) $\varphi \,\&\, (\varphi \to \psi) \to \varphi \wedge \psi$
(A4b) $\varphi \wedge \psi \to \varphi$
(A4c) $\varphi \wedge \psi \to \psi \wedge \varphi$
(A5a) $(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$
(A5b) $(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $0 \to \varphi$

The logic $\mathrm{MTL}_\triangle$ was introduced in [12] as an expansion of the logic MTL by a new unary connective $\triangle$, the deduction rule of necessitation (from φ infer $\triangle\varphi$), and the following axioms:

(A$\triangle$1) $\triangle\varphi \vee \neg\triangle\varphi$,
(A$\triangle$2) $\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$,
(A$\triangle$3) $\triangle\varphi \to \varphi$,
(A$\triangle$4) $\triangle\varphi \to \triangle\triangle\varphi$,
(A$\triangle$5) $\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$.

Formulae derived from these axioms by means of the mentioned deduction rules are called *theorems* of $\mathrm{MTL}_\triangle$.

Remark 4.1.2 The following logics known from the literature are among the expansions of MTL: BL, SMTL, IMTL, ΠMTL, NM, WNM, SBL, Łukasiewicz logic, product logic, and Gödel logic (for their definitions see [12, 20]); the logic PŁ (extension of Łukasiewicz logic by an additional conjunction—see [25]) and extensions of these logics by truth constants (see [15, 11]).

The following logics known from the literature are among the expansions of $\mathrm{MTL}_\triangle$: the expansions of all fuzzy logics mentioned above by the connective $\triangle$, fuzzy logics with strict negation and an extra involutive negation ($\mathrm{SBL}_\sim$, $\Pi_\sim$, $\mathrm{G}_\sim$, see [13] for more details); and two expressively very rich fuzzy logic ŁΠ and $\text{Ł}\Pi\frac{1}{2}$ (for details see [9, 14]).

Definition 4.1.3 An MTL-*algebra* is a structure $\mathbf{L} = (L, *, \Rightarrow, \wedge, \vee, 0, 1)$, where

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(2) $(L, *, 1)$ is a commutative monoid,
(3) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$ for all x, y, z, (residuation)
(4) $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$. (prelinearity)

Definition 4.1.4 An $\mathrm{MTL}_\triangle$-*algebra* is a structure $\mathbf{L} = (L, *, \Rightarrow, \wedge, \vee, 0, 1, \triangle)$ such that

(0) $(L, *, \Rightarrow, \wedge, \vee, 0, 1)$ is an MTL-algebra,
(1) $\triangle x \vee (\triangle x \Rightarrow 0) = 1$,
(2) $\triangle(x \vee y) \leq (\triangle x \vee \triangle y)$,
(3) $\triangle x \leq x$,
(4) $\triangle x \leq \triangle\triangle x$,
(5) $\triangle(x \Rightarrow y) \leq \triangle x \Rightarrow \triangle y$,
(6) $\triangle 1 = 1$.

If the lattice order of **L** is linear, we say that **L** is an MTL$_\triangle$-*chain.* If the lattice reduct of **L** is the real unit interval with the usual order, we say that **L** is a *standard* MTL$_\triangle$-chain. It can easily be shown that in each MTL$_\triangle$-chain the following holds:

$$\triangle x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The structure $([0,1], *, \Rightarrow, \min, \max, 0, 1, \triangle)$ is a standard MTL$_\triangle$-chain iff $*$ is a left continuous t-norm and $\Rightarrow$ its residuum.

Example 4.1.5 Let $*_{\text{L}}$ be a Łukasiewicz t-norm ($x *_{\text{L}} y = \max(0, x + y - 1)$) and $\Rightarrow_{\text{L}}$ its residuum ($x \Rightarrow_{\text{L}} y = \min(1, 1 - x + y)$). Then the MTL$_\triangle$-chain $([0,1], *_{\text{L}}, \Rightarrow_{\text{L}}, \min, \max, 0, 1, \triangle)$ is called the *standard* MV$_\triangle$-*algebra* (see e.g., [20]).

Given an MTL$_\triangle$-algebra, we can evaluate formulae of MTL$_\triangle$ by assigning elements of **L** to propositional atoms and computing values of compound formulae using the operations of **L**. A formula is a *tautology* of a given MTL$_\triangle$-algebra if always evaluated to 1.

The completeness theorem for MTL and MTL$_\triangle$ w.r.t. standard algebras (a formula is a theorem iff it is a tautology of each standard MTL$_\triangle$-algebra) was proven in [26].

4.2 First-order logics MTL and MTL$_\triangle$

Definition 4.2.1 A *predicate language* $\mathbf{\Gamma}$ is a tuple $(\mathbf{S}, \mathbf{P}, \mathbf{F}, \mathbf{a})$, where **S** is a non-empty set of sorts of variables; **P** is a non-empty set of predicate symbols; **F** is a set of function symbols; and **a** is an *arity function*, which assigns a sequence of sorts $(s_1, \ldots, s_k)$ to each predicate symbol and a sequence of sorts $(s_1, \ldots, s_k, s_{k+1})$ to each function symbol ($k \geq 0$ in both cases). Functions with arity (s_1) are called *object constants* of sort (s_1). The set **P** is supposed to contain a symbol $=$ of arity (s, s) for each sort s. For each sort s there are countably many variables $x_1^s, x_2^s, \ldots$.

Now we define several syntactical notions; notice that they are determined by a predicate language. In order to simplify the definitions let us take a fixed predicate language $\mathbf{\Gamma}$ in this whole section. Analogously, the semantical notions we are going to define are determined by an MTL$_\triangle$-chain, so let us consider a fixed MTL$_\triangle$-chain **L** from now on.

Definition 4.2.2 Any variable x^s of sort s is a *term* of sort s. If $F \in \mathbf{F}$ is a function symbol of arity $(s_1, \ldots, s_k, s_{k+1})$, then for any terms $t_1, \ldots, t_k$ of the respective sorts $s_1, \ldots, s_k$, the expression $F(t_1, \ldots, t_k)$ is a term of sort s_{k+1}.

Atomic formulae have the form $P(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are terms of respective sorts $s_1, \ldots, s_k$ and $P \in \mathbf{P}$ is a predicate symbol of arity $(s_1, \ldots, s_k)$. (We usually use infix notation for binary predicate symbols.)

Formulae are built from atomic formulae by using the connectives of MTL$_\triangle$ and the quantifiers $\forall, \exists$ (for a formula φ and an object variable x, both $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulae).

Definition 4.2.3 An occurrence of a variable x in a formula φ is *bound* if it is in the scope of a quantifier over x; otherwise it is called *free.* A formula φ is called a *sentence* if all occurrences of variables in φ are bound.

A term t is *substitutable* for the object variable x in a formula $\varphi(x)$ iff t is of the same sort as x and no variable occurring in t becomes bound in $\varphi(t)$.

Definition 4.2.4 The first-order logic MTL$_\triangle$ (with crisp identity) has the following axioms:

- (P) The axioms resulting from the axioms of MTL$_\triangle$ by the substitution of $\mathbf{\Gamma}$-formulae for propositional variables,
- ($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable for x in φ,
- ($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ,
- ($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, where x is not free in χ,
- ($\exists$2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, where x is not free in χ,
- ($\forall$3) $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, where x is not free in χ.
- (=1) $x = x$
- (=2) $x = y \to (\varphi(x) \to \varphi(y))$, where y is substitutable for x in φ.

The deduction rules are those of MTL$_\triangle$ and *generalization*: from φ infer $(\forall x)\varphi$.

We define the notion of *theorem* in the same way as in the propositional case. We can also define a more general notion:

Definition 4.2.5 A *theory* is a set of sentences. A formula is *provable in a theory* T if it is derivable by the deduction rules from the axioms of first-order MTL$_\triangle$ and sentences belonging to T. We denote this fact by $T \vdash \varphi$.

Definition 4.2.6 An $\mathbf{L}$-*structure* $\mathbf{M}$ has the form: $\mathbf{M} = ((M_s)_{s\in\mathbf{S}}, (P_\mathbf{M})_{P\in\mathbf{P}}, (F_\mathbf{M})_{f\in\mathbf{F}})$, where each M_s is a non-empty set; each $P_\mathbf{M}$ is a k-ary fuzzy relation $P_\mathbf{M}\colon \prod_{i=1}^k M_{s_i} \to \mathbf{L}$ for each predicate symbol $P \in \mathbf{P}$ of arity $(s_1, \ldots, s_k)$; and $f_\mathbf{M}$ is a k-ary function $F_\mathbf{M}\colon \prod_{i=1}^k M_{s_i} \to M_{s_{k+1}}$ for each function symbol $F \in \mathbf{F}$ of arity $(s_1, \ldots, s_k, s_{k+1})$. Furthermore, $=_\mathbf{M}$ is the crisp identity of the elements of M_s for each $s \in \mathbf{S}$.

In words: an $\mathbf{L}$-structure consists of (i) domains for all sorts of variables, (ii) an interpretation of all predicate symbols by $\mathbf{L}$-fuzzy relations defined on appropriate domains, and (iii) an interpretation of all function symbols by functions between appropriate domains.

Definition 4.2.7 Let $\mathbf{M}$ be an $\mathbf{L}$-structure. An $\mathbf{M}$-*evaluation* is a mapping v which assigns to each object variable x of sort s an element from M_s. For an $\mathbf{M}$-evaluation v, a variable x of sort s, and $a \in M_s$ we define the $\mathbf{M}$-evaluation $v[x \mapsto a]$ as

$$v[x \mapsto a](y) = \begin{cases} a \text{ if } y = x \\ v(y) \text{ otherwise} \end{cases}$$

Definition 4.2.8 Let $\mathbf{M}$ be an $\mathbf{L}$-structure and v an $\mathbf{M}$-evaluation. We define *values* of the terms and *truth values* of the formulae in $\mathbf{M}$ for an $\mathbf{M}$-evaluation v as:

$$\begin{array}{rcll}
\|x\|^\mathbf{L}_{\mathbf{M},v} &=& v(x) & \\
\|F(t_1,\ldots,t_n)\|^\mathbf{L}_{\mathbf{M},v} &=& F_\mathbf{M}(\|t_1\|^\mathbf{L}_{\mathbf{M},v},\ldots,\|t_n\|^\mathbf{L}_{\mathbf{M},v}) & \text{for each } F \in \mathbf{F} \\
\|P(t_1,\ldots,\ t_n)\|^\mathbf{L}_{\mathbf{M},v} &=& P_\mathbf{M}(\|t_1\|^\mathbf{L}_{\mathbf{M},v},\ldots,\|t_n\|^\mathbf{L}_{\mathbf{M},v}) & \text{for each } P \in \mathbf{P} \\
\|c(\varphi_1,\ldots,\varphi_n)\|^\mathbf{L}_{\mathbf{M},v} &=& c_\mathbf{L}(\|\varphi_1\|^\mathbf{L}_{\mathbf{M},v},\ldots,\|\varphi_n\|^\mathbf{L}_{\mathbf{M},v}) & \text{for each connective } c \\
\|(\forall x)\varphi\|^\mathbf{L}_{\mathbf{M},v} &=& \inf_{a\in M} \|\varphi\|^\mathbf{L}_{\mathbf{M},v[x\to a]} & \\
\|(\exists x)\varphi\|^\mathbf{L}_{\mathbf{M},v} &=& \sup_{a\in M} \|\varphi\|^\mathbf{L}_{\mathbf{M},v[x\to a]} &
\end{array}$$

If the infimum or supremum does not exist, we take its value as undefined. We say that a structure $\mathbf{M}$ *safe* iff $\|\varphi\|^\mathbf{L}_{\mathbf{M},v}$ is defined for each formula φ and each $\mathbf{M}$-evaluation v. Notice that in a standard MTL$_\triangle$-algebra (or more generally in any MTL$_\triangle$-algebra whose lattice reduct is a complete lattice) the *safeness* of a structure is a void condition, as the suprema and infima of *all* sets exist.

Definition 4.2.9 Formula φ is *valid* in a structure $\mathbf{M}$ (denoted as $\mathbf{M} \models \varphi$) if $\|\varphi\|^\mathbf{L}_{\mathbf{M},v} = 1$ for each $\mathbf{M}$-evaluation v. A structure $\mathbf{M}$ is a *model* of a theory T if $\mathbf{M} \models \varphi$ for each φ in T.

Finally we present the (strong) completeness theorem which relates syntactical and semantical aspects of the first-order MTL$_\triangle$ logic (see [27, 12] for the proof). Recall that the direction from syntax to semantics is usually called *soundness* whereas the reverse one is called *completeness*.

Theorem 4.2.10 *Let* $\mathbf{\Gamma}$ *be a predicate language,* T *a theory, and* φ *a formula. Then the following are equivalent:*

1. $T \vdash \varphi$.
2. $\mathbf{M} \models \varphi$ *for each* MTL-*chain* $\mathbf{L}$ *and each safe* $\mathbf{L}$-*model* $\mathbf{M}$ *of* T.
3. $\mathbf{M} \models \varphi$ *for each standard* MTL-*chain* $\mathbf{L}$ *and each* $\mathbf{L}$-*model* $\mathbf{M}$ *of* T.

Thus by (1)⇒(2) we get that if a formula is provable in a given theory T, then it is valid in all models of T over *all* MTL$_\triangle$-chains. Conversely, by (3)⇒(1) we get that if a formula is valid in all models of T over all all *standard* MTL$_\triangle$-chains, then it is provable in T.

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