Relative Interpretations over First-Order Fuzzy Logic

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Abstract

The classical notion of relative interpretation (also known as direct syntactic model) is adapted for multi-sorted first-order fuzzy logics. The level of generality is chosen to suit the needs of its applications in Fuzzy Class Theory.

In formal logic, relative interpretations are a powerful tool that can be used not only for the proofs of relative consistency, but also for direct syntactic constructions of notions of one theory in another. Here we adapt the notion for fuzzy logic and show the analogs of key classical metatheorems. These results allow using relative interpretations of fuzzy theories in essentially the same way they are used in classical metamathematics.

Relative interpretations can be defined at varying levels of generality, the price for greater generality being more preconditions in theorems on invariance under an interpretation. The level of generality chosen here follows the needs of the paper [3]. For relative interpretations see [9]; we follow and slightly generalize the exposition given in [8].

Multi-sorted first-order fuzzy logic with subsumption of sorts has been introduced in [1] for the logic ŁΠ [6, 4]. It is nevertheless obvious that the definitions and proofs of [1] work over any fuzzy logic that axiomatically expands MTL or MTLΔ [5]. In what follows, by “fuzzy logic” we shall therefore mean any logic that in this sense contains MTL; we shall only require that all of its propositional connectives be extensional w.r.t. provable equivalence (otherwise some of the metatheorems below could fail). Crisp identity is assumed in the first-order fuzzy logic under consideration; in models it is always realized as the identity of elements and it can be axiomatized e.g. by the axioms of reflexivity $x = x$ and intersubstitutivity *salva veritate* $x = y \to (\varphi(x) \leftrightarrow \varphi(y))$ for any formula φ (for details see [1]).

Besides the theorems of first-order MTL that are listed in [5], we shall need a few more (meta)lemmata. The following lemma shows that it is possible to bind only some occurrences of a term in the existentialization of a formula:

Lemma 1 *Let $\varphi(x, y)$ be a formula and t a term substitutable for both x and y in φ. Then $\varphi(t, t) \to (\exists x)\varphi(x, t)$.*

Proof: Directly by existentialization on x in φ. QED

Lemma 2 *For an arbitrary term t substitutable for x in $\varphi(x)$ it is provable that*

$$\varphi(t) \quad\leftrightarrow\quad (\forall x)(x = t \to \varphi(x)) \tag{1}$$

$$\varphi(t) \quad\leftrightarrow\quad (\exists x)(x = t \mathbin{\&} \varphi(x)) \tag{2}$$

Proof: (1) Left to right: from the identity axiom $\varphi(t) \to (x = t \to \varphi(x))$ by generalization on x and shifting the quantifier. Right to left: by specification of x to t.

(2) Left to right: $\varphi(t)$ implies $t = t \mathbin{\&} \varphi(t)$, which by Lemma 1 implies $(\exists x)(x = t \mathbin{\&} \varphi(x))$. Right to left: from the identity axiom $x = t \mathbin{\&} \varphi(x) \to \varphi(t)$ by generalization on x and shifting the quantifier to the antecedent. QED

Corollary 3 *Any formula is equivalent to a formula in which logical functions are applied only to variables and occur only in atomic subformulae of the form $y = F(x_1, \ldots, x_k)$.*

Proof: Using Lemma 2, inductively decompose nested terms $s(t)$ by $\varphi(s(t)) \leftrightarrow (\exists x)(x = t \mathbin{\&} \varphi(s(x)))$ and finally by $\varphi(F(x_1, \ldots, x_k)) \leftrightarrow (\exists y)(y = F(x_1, \ldots, x_k) \mathbin{\&} \varphi(y))$ for all F. QED

Recall from [1] that the language of a multi-sorted first-order fuzzy logic is a quintuple $(\mathbf{S}, \preceq, \mathbf{P}, \mathbf{F}, \mathbf{A})$, where $\mathbf{S}$ is a non-empty set of sorts, $\preceq$ is a partial ordering of $\mathbf{S}$ indicating the subsumption of sorts, $\mathbf{P}$ and $\mathbf{F}$ are disjoint sets of predicate resp. function symbols, and $\mathbf{A}$ is an arity function that assigns a finite sequence of sorts to each element of $\mathbf{P} \cup \mathbf{F}$ (the sequence must be non-empty for elements of $\mathbf{F}$). If $P \in \mathbf{P}$ and $\mathbf{A}(P) = (s_1, \ldots, s_k)$, then $P(t_1, \ldots, t_k)$ is a well-formed atomic formula iff the term t_i is of sort s_i for all $i = 1, \ldots, k$. If $\mathbf{A}(F) = (s_1, \ldots, s_k, s_{k+1})$, then $F(t_1, \ldots, t_k)$ is a well-formed term, of sort s_{k+1}, iff the term t_i is of sort s_i for all $i = 1, \ldots, k$. For more details on multi-sorted first-order fuzzy logics see [1, §2.2].

Definition 4 (Interpretation of a language) *Let $\mathbf{L} = (\mathbf{S}, \preceq, \mathbf{P}, \mathbf{F}, \mathbf{A})$ and $\mathbf{L}' = (\mathbf{S}', \preceq', \mathbf{P}', \mathbf{F}', \mathbf{A}')$ be two multi-sorted first-order languages. An* interpretation of the language $\mathbf{L}$ in the language $\mathbf{L}'$ *is a (metamathematical) mapping $\star$ which assigns to each sort $s \in \mathbf{S}$ a function symbol $F_s^\star \in \mathbf{F}'$ of arity $\mathbf{A}'(F_s^\star) = (s_\star, s^\star)$ for some $s_\star, s^\star \in \mathbf{S}'$, to each predicate symbol $P \in \mathbf{P}$ a predicate symbol $P^\star \in \mathbf{P}'$, and to each function symbol $F \in \mathbf{F}$ a function symbol $F^\star \in \mathbf{F}'$, and which satisfies the following conditions:*

- *For all $s, r \in \mathbf{S}$, if $s \preceq r$ then $s^\star \preceq r^\star$.*
- *For all $P \in \mathbf{P}$, if $\mathbf{A}(P) = (s_1, \ldots, s_k)$ and $\mathbf{A}'(P^\star) = (r_1, \ldots, r_k)$ then $s_i^\star \preceq r_i$ for all $i = 1, \ldots, k$.*
- *For all $F \in \mathbf{F}$, if $\mathbf{A}(F) = (s_1, \ldots, s_{k+1})$ and $\mathbf{A}'(F^\star) = (r_1, \ldots, r_{k+1})$, then $s_i^\star \preceq r_i$ for all $i = 1, \ldots, k$ and $r_{k+1} \preceq s_{k+1}^\star$.*

An interpretation $\star$ of $\mathbf{L}$ in $\mathbf{L}'$ extends by metamathematical induction on the complexity of terms and formulae of $\mathbf{L}$ to a mapping (also denoted by $\star$) which assigns to each term t of $\mathbf{L}$ a term $t^\star$ of $\mathbf{L}'$ and to each formula φ of $\mathbf{L}$ a formula $\varphi^\star$ of $\mathbf{L}'$ as follows:

- *The i-th variable x_i^s of each sort s in $\mathbf{L}$ is assigned the term $F_s^\star(x_i^{s_\star})$ of sort $s^\star$, where $x_i^{s_\star}$ is the i-th variable of sort $s_\star$.*
- *Each term $F(t_1, \ldots, t_k)$ of $\mathbf{L}$ is assigned the term $F^\star(t_1^\star, \ldots, t_k^\star)$.*
- *Each formula $P(t_1, \ldots, t_k)$ of $\mathbf{L}$ is assigned the formula $P^\star(t_1^\star, \ldots, t_k^\star)$.*
- *Each formula $x = y$ of $\mathbf{L}$ is assigned the formula $x^\star = y^\star$.*

- *For all k-ary propositional connectives $\mathbf{c}$, each formula $\mathbf{c}(\varphi_1, \ldots, \varphi_k)$ of $\mathbf{L}$ is assigned the formula $\mathbf{c}(\varphi_1^\star, \ldots, \varphi_k^\star)$.*
- *Each formula $(\forall x^s)\varphi$ resp. $(\exists x^s)\varphi$ of $\mathbf{L}$ is assigned the formula $(\forall x^{s_\star})\varphi^\star$ resp. $(\exists x^{s_\star})\varphi^\star$.*

Remark 5 Notice that we allow reinterpreting *variables* of sort s by *functions* from $s_\star$ to $s^\star$. This is necessitated by the applications in [3], where we need to interpret variables by functional terms (e.g., when identifying x with the pair $\langle x, 0\rangle$). A straightforward interpretation of a sort s by another sort r is covered by this definition, taking the identity function on sort $s_\star$ for $F_s^\star$ and $s^\star = s_\star = r$.

Remark 6 In Definition 4, the logical symbols (except for variables) are left unaffected by the translation $\star$. The notion of interpretation can be defined more generally to include also the specification of the translations of $[(\forall x)\varphi]^\star$, $[(\exists x)\varphi]^\star$, $(x = y)^\star$, and $[\mathbf{c}(\varphi_1, \ldots, \varphi_k)]^\star$ for each propositional connective $\mathbf{c}$. In Definition 9 we would then require the provability of the interpreted logical axioms and rules.

Notice that in the latter case, the background logic of the interpreted language or theory may be allowed to differ from the background logic of the original language or theory. For empty theories, we then get an interpretation of one logic in another. An example of such kind is the interpretation of the logic $\mathrm{PC}(*)$ of a particular ŁΠ-representable t-norm $*$ in ŁΠ, which takes $\to$ of $\mathrm{PC}(*)$ to $\to_*$ of ŁΠ, $\&$ of $\mathrm{PC}(*)$ to $\&_*$ of ŁΠ, etc. By a recent result (oral presentation by Marchioni and Montagna at IPMU'06), the interpretation is faithful, i.e., $\mathrm{PC}(*) \vdash \varphi$ iff ŁΠ $\vdash \varphi_*$, for any formula φ of $\mathrm{PC}(*)$. Another example of this kind are Gödel-style interpretations, e.g., the $\neg\neg$-interpretation of classical logic in SMTL (or stronger) or the Δ-interpretation of classical logic in MTLΔ (or stronger). (Notice that Gödel-style interpretations require a further generalization of the rule for the interpretation of atomic formulae.) In this paper, however, we shall only use interpretations which leave the logical symbols absolute, and thus do not change the underlying logic.

Definition 7 (Absolute and invariant notions) *Let $\star$ be an interpretation of the language $\mathbf{L}$ in the language $\mathbf{L}'$ and let $\mathbf{T}'$ be a theory in the language $\mathbf{L}'$. Let $\varphi(x_1, \ldots, x_k)$ be a formula of $\mathbf{L}$ and let all non-logical symbols of φ belong to $\mathbf{L}'$ as well. Then the formula φ is called* absolute *(in the theory $\mathbf{T}'$ w.r.t. the interpretation $\star$) iff $\mathbf{T}' \vdash \varphi(x_1^\star, \ldots, x_k^\star) \leftrightarrow \varphi^\star$. Similarly, a predicate P or a functor F is called absolute, if the formula $P(x_1, \ldots, x_k)$ resp. $y = F(x_1, \ldots, x_k)$ is absolute.*

Let furthermore $\mathbf{L}'$ contain the sorts of all variables that occur in φ. Then we will call φ invariant *(in the theory $\mathbf{T}'$ w.r.t. the interpretation $\star$) iff $\mathbf{T}' \vdash \varphi \leftrightarrow \varphi^\star$. A predicate P or a functor F is called invariant, if the formula $P(x_1, \ldots, x_k)$ resp. $y = F(x_1, \ldots, x_k)$ is invariant.*

Observation 8 *If φ is both absolute and invariant w.r.t. $\star$ in $\mathbf{T}'$, then $\mathbf{T}' \vdash \varphi(x_1^\star, \ldots, x_k^\star) \leftrightarrow \varphi(x_1, \ldots, x_k)$.*

Definition 9 (Interpretation of a theory) *Let $\mathbf{T}$ be a theory in the language $\mathbf{L}$ and $\mathbf{T}'$ a theory in the language $\mathbf{L}'$. An interpretation $\star$ of $\mathbf{L}$ in $\mathbf{L}'$ is called an* interpretation of the theory $\mathbf{T}$ in the theory $\mathbf{T}'$ *iff $\mathbf{T}' \vdash \varphi^\star$ for each formula φ which is a logical axiom of identity or an axiom of the theory $\mathbf{T}$.*

The requirement in Definition 9 that the interpreted identity axioms be provable is automatically satisfied if all functions $F_s^\star$ together are injective:

Lemma 10 *Let $\star$ be an interpretation of the language $\mathbf{L}$ in the language $\mathbf{L}'$ and let $\mathbf{T}'$ be a theory in the language $\mathbf{L}'$. If $\mathbf{T}' \vdash F_s^\star(x^{s_\star}) = F_t^\star(y^{t_\star}) \to x^{s_\star} = y^{t_\star}$ for all sorts s, t in $\mathbf{L}$, then the interpreted axioms of identity are provable in $\mathbf{T}'$.*

Proof: The axiom of reflexivity $x = x$ translates into $x^\star = x^\star$, which is an instance of the reflexivity axiom of identity in $\mathbf{T}'$. The intersubstitutivity axiom $x = y \to [\varphi(x) \leftrightarrow \varphi(y)]$ translates into the formula of the form

$$F_s^\star(x^{s_\star}) = F_t^\star(y^{t_\star}) \to [\psi(x^{s_\star}) \leftrightarrow \psi(y^{t_\star})]$$

which is provable in $\mathbf{T}'$ by the assumption of the lemma and the instance for ψ of the intersubstitutivity axiom of $\mathbf{T}'$. QED

The usual theorems on interpretations known from classical logic remain valid for interpretations over fuzzy logics as well. The following theorems give examples of such results.

Observation 11 *A composition of two interpretations (between languages or theories) is an interpretation (between languages or theories, respectively).*

Since the composition is obviously associative and the identical mapping is always an interpretation of a theory in itself, the languages or theories over a fuzzy logic form a category just like in classical logic, allowing categorial constructions on fuzzy theories.

Theorem 12 *Let $\star$ be an interpretation of the theory $\mathbf{T}$ in the theory $\mathbf{T}'$. Then for any formula φ in the language of $\mathbf{T}$, if $\mathbf{T} \vdash \varphi$ then $\mathbf{T}' \vdash \varphi^\star$.*

Proof: By induction on the proof of φ: by the requirement of Definition 9, the interpreted axioms of $\mathbf{T}$ and those of identity are provable in $\mathbf{T}'$, and all other logical axioms and rules are translated by $\star$ again into the instances of logical axioms and rules (observe that the term $t^\star$ is substitutable for $x^{s_\star}$ iff t is substitutable for x^s). QED

Definition 13 (Faithful interpretations) *The interpretation $\star$ of the theory $\mathbf{T}$ in the theory $\mathbf{T}'$ is* faithful *iff for all formulae φ in the language of $\mathbf{T}$ it holds that $\mathbf{T} \vdash \varphi$ iff $\mathbf{T}' \vdash \varphi^\star$.*

A faithful interpretation $\star$ of $\mathbf{T}$ in itself such that $\varphi^{\star\star} \equiv \varphi$ is called a duality.

Example 14 (Identical interpretation) *If the theory $\mathbf{T}'$ in the language $\mathbf{L}'$ extends the theory $\mathbf{T}$ in the language $\mathbf{L}$, then the identical interpretation of $\mathbf{L}$ in $\mathbf{L}'$ (i.e., $x^\star = x$, $P^\star = P$, and $F^\star = F$ for all sorts and symbols) interprets $\mathbf{T}$ in $\mathbf{T}'$. The interpretation is faithful iff $\mathbf{T}'$ extends $\mathbf{T}$ conservatively.*

The following lemma gives a method how to prove the faithfulness of an interpretation in some cases.

Lemma 15 *Let $\star$ interpret $\mathbf{T}$ in its extension $\mathbf{T}'$ and let $s_\star = s$ for all sorts in $\mathbf{T}$. Let furthermore*

$$\mathbf{T}' \vdash P^\star(F^\star_{s_1}(x_1^{s_1}), \ldots, F^\star_{s_k}(x_k^{s_k})) \leftrightarrow P(x_1^{s_1}, \ldots, x_k^{s_k}) \qquad (3)$$

$$\mathbf{T}' \vdash F^\star_s(y^s) = F^\star(F^\star_{s_1}(x_1^{s_1}), \ldots, F^\star_{s_k}(x_k^{s_k})) \leftrightarrow y^s = F(x_1^{s_1}, \ldots, x_k^{s_k}) \qquad (4)$$

for all function symbols F and predicate symbols P in the language of $\mathbf{T}$ (including the identity predicate).

Then $\mathbf{T}' \vdash \varphi^\star \leftrightarrow \varphi$ for all formulae φ in the language of $\mathbf{T}$ (i.e., all notions in the language of $\mathbf{T}$ are invariant under $\star$).

If furthermore $\mathbf{T}'$ extends $\mathbf{T}$ conservatively, then $\star$ is faithful.

Proof: The first claim is proved straightforwardly by induction on the subformulae of φ. By (3), (4) and Corollary 3 we can assume that $\mathbf{T}' \vdash \psi^\star \leftrightarrow \psi$ holds for all atomic subformulae ψ in φ. Propositional combinations preserve the property $\mathbf{T}' \vdash \psi^\star \leftrightarrow \psi$, since our definition of interpretation leaves all propositional connectives absolute and in the logics under consideration all connectives are extensional w.r.t. provable equivalence. For $\psi \equiv (\forall x^s)\chi$, since $s_\star = s$, its translation $\psi^\star$ is $(\forall x^s)\chi^\star$, and thus $\mathbf{T}' \vdash (\forall x^s)\chi^\star \leftrightarrow (\forall x^s)\chi$ follows from the induction hypothesis $\mathbf{T}' \vdash \chi^\star \leftrightarrow \chi$ by the rules of MTL (similarly for $\exists$).

The claim of faithfulness under conservativity: $\mathbf{T}' \vdash \varphi^\star \leftrightarrow \varphi$ entails ($\mathbf{T}' \vdash \varphi^\star$ iff $\mathbf{T}' \vdash \varphi$), and by conservativity $\mathbf{T}' \vdash \varphi$ iff $\mathbf{T} \vdash \varphi$; thus $\mathbf{T}' \vdash \varphi^\star$ iff $\mathbf{T} \vdash \varphi$. QED

Remark 16 Definition 4 requires that all sorts and symbols occurring in the definitions of $x^\star$, $P^\star$, and $F^\star$ be present in the language $\mathbf{L}'$. Following the usual mathematical practice, we shall not distinguish between a theory and its extensions by conservative definitions. Thus we shall allow giving $x^\star$, $P^\star$, and $F^\star$ by the defining formulae or terms for the needed predicates, functors, and sorts, provided the definitions are conservative.

For the conservative introduction of predicate and function symbols see [7]: the definition of a predicate symbol by an axiom $P(x_1, \ldots, x_k) \leftrightarrow \varphi(x_1, \ldots, x_k)$ is conservative and eliminable for any formula φ, while the introduction of a function symbol $F(x_1, \ldots, x_k)$ by an axiom $\varphi(x_1, \ldots, x_k, F(x_1, \ldots, x_k))$ is conservative on condition that $(\exists x_{k+1})\varphi(x_1, \ldots, x_k, x_{k+1})$ is provable in the theory; the definition is eliminable if the uniqueness of such x_{k+1} is provable in the theory. (In multi-sorted languages, the obvious conditions on the sorts of the arguments must be ensured.)

For the definition of sorts, it is easy (but tedious) to check that a sort s subsumed in a sort t can be introduced by an axiom $(\exists x^s)(x^t = x^s) \leftrightarrow \varphi(x^t)$, which is conservative if the theory proves that $(\exists x^t)\varphi(x^t)$ and that φ is *crisp;* if it is further required that $s \preceq s'$ for any sort s', the conservativity is ensured if the theory further proves $\varphi(x^t) \to (\exists x^{s'})(x^t = x^{s'})$.

The apparatus of relative interpretations is widely applicable in all sorts of formal fuzzy theories. Since Fuzzy Class Theory FCT of [1] is proposed in [2] as a foundational theory for fuzzy mathematics, relative interpretations of various fragments of FCT in itself are of special importance. In Example 17 I give an incomplete list of such interpretations (the details will be given in a separate paper). Some of them (e.g., the upper shift or the relativization) prove important (even if often intuitively obvious) metamathematical properties of FCT, while others codify constructions which either obviate some of the syntactic restrictions of FCT (e.g., the singleton shift), or can be useful in various areas of fuzzy mathematics formalized in FCT (e.g., the "$\times\{0\}$" interpretation, employed in [3]).

Example 17 *The following constructions are important interpretations of* FCT *(or some of its fragments) in* FCT*:*

- Identical interpretations. *Propositional fuzzy logic, classical theory of the identity of individuals, the classical theory of identity of tuples, the theory of fuzzy classes, the theory of fuzzy relations, and monadic Henkin-style higher-order fuzzy logic are all fragments of* FCT *given by a suitable restriction of the language (admitting only some sorts of variables). It can be shown that they can be axiomatized by the axioms of* FCT *restricted to the same language with an additional axiom stating that the sorts for tuples do not exhaust the universal sort of the same order.* FCT *extends these theories conservatively, and thus the identical interpretations of the respective fragments represent all of the above theories faithfully in* FCT.

- Upward shift. *The translation $\sharp$ that consists in raising the order of all variables by* 1 *is an interpretation of* FCT *in itself (since the axioms of* FCT *are invariant under $\sharp$). All definitions and theorems of* FCT *can thus be propagated to all higher orders by iteration of $\sharp$.*

- Relativization. *Restricting all quantifiers to a* crisp *class (resp. its iterated crisp powers in higher orders) is an interpretation of* FCT *in itself. The domain of discourse thus can be arbitrarily chosen from some basic universe (as long as it is crisp).*

- Singleton shift. FCT *does not allow classes to contain elements of different orders (e.g., $\{x, X\}$). Nevertheless, they can be simulated by means of faithful interpretations. It can be shown that the interpretation $\{\cdot\}$ ("singleton shift") which maps x to $\{x\}$ is a faithful interpretation of the theory of identity (which exhausts the relevant features of atomic elements) in the theory of fuzzy classes. The mixed class $\{x, X\}$ thus can be "encoded" by the class $\{x^{\{\cdot\}}, X\} = \{\{x\}, X\}$. (Further adjustments can be made in order to make the backward translation one-to-one and make it work at all levels of the type hierarchy.) Thus by this interpretation, mixed classes of arbitrary orders are available in* FCT.

- Transposition. *Switching all pairs* $\langle x, y\rangle$ *to* $\langle y, x\rangle$ *is a duality in* FCT. *Dual forms of the theorems on fuzzy relations thus need not be proved (e.g.,* $\mathrm{dom}(A \times B) = A$*, follows from* $\mathrm{rng}(A \times B) = B$*).*

- Relational representation of classes. *Fuzzy classes can be represented among fuzzy relations by identifying atomic elements* x *with pairs* $\langle x, 0\rangle$ *(for a fixed element* 0*); any fuzzy class* A *is then identified with the fuzzy relation* $A \times \{0\}$*. This interpretation is employed in [3] for proving hosts of theorems on fuzzy relations and classes at once.*

Remark 18 The interpretations of Example 17 often state an "isomorphism" of some structures in FCT. The need of using interpretations arises primarily from the fact that the notion of isomorphism (not even a bijection) has not yet been developed inside FCT. (Since all notions in FCT are in general fuzzy, this notion would need a careful analysis.) Nevertheless, since FCT is a formal syntactic theory, the metamathematical apparatus of interpretations is very suitable for such tasks, and the "syntactic isomorphisms" obtained by the method of interpretation are usually easier to prove than they would be inside the theory.

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