Johnson System of Parametric Families

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Abstract:

We present a system of parametric unimodal continuous distributions, containing all possible types of score functions of prototypes. We call it a Johnson system since Johnson transformation is used for its construction. The system contains many used probability distributions, some of them in reparametrized forms, with unified meaning of parameters. Three non-Johnson systems are mentioned as well.

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1 Introduction

Distribution of random variable Y with distribution function F is said to be supported by interval $\mathcal{Q} \subseteq \mathbb{R}$ if density $f(x) = dF(x)/dx$ satisfies relation

$$f(x) = \begin{cases} > 0 & \text{if} \quad x \in \mathcal{Q} \\ = 0 & \text{if} \quad x \in \mathbb{R} - \mathcal{Q}. \end{cases}$$

Johnson [1] proposed method for generating distributions of random variable X supported by $\mathcal{Q} = (0, \infty)$ and $\mathcal{Q} = (0, 1)$ by means of transformation $\varphi : \mathbb{R} \to \mathcal{Q}$ defined by

$$\begin{aligned} Y = \varphi^{-1}(X) &= \log X && \text{if} \quad \mathcal{Q} = (0, \infty) \\ Y = \varphi^{-1}(X) &= \log \frac{X}{1 - X} && \text{if} \quad \mathcal{Q} = (0, 1), \end{aligned} \tag{1.1}$$

where Y is a 'prototype' supported by $\mathbb{R}$. He considered the normal [1], Laplace [2] and logistic [3] prototypes. In present paper we use modified transformation (1.1) to a construction of a system of continuous distributions on arbitrary interval support, which is complete from the point of view of behaviour of score functions of 'prototypes' in infinity.

The transformed distributions are generated by mapping $\eta : \mathcal{Q} \to \mathbb{R}$ given by

$$\eta(x) = \begin{cases} x & \text{if} \quad (a, b) = \mathbb{R} \\ \log(x - a) & \text{if} \quad -\infty < a < b = \infty \\ \log \dfrac{(x - a)}{(b - x)} & \text{if} \quad -\infty < a < b < \infty \\ \log(b - x) & \text{if} \quad -\infty = a < b < \infty \end{cases}$$

which is the Johnson transformation adapted to arbitrary interval support. For the sake of simplicity of the formulas, results are presented on 'Johnson's supports' $\mathcal{Q} = (0, \infty)$ and $\mathcal{Q} = (0, 1)$.

2 Parent distributions

2.1 Prototypes

Distribution G supported by $\mathbb{R}$ is called a *prototype*. Denote by g its density and by S its score function

$$S(y) = -\frac{1}{g(y)} \frac{dg(y)}{dy}. \tag{2.1}$$

We choose the simplest prototypes which cover possible types of the behaviour of the score function in infinity:

UE: unbounded, exponentially increasing

UP: unbounded, polynomially increasing

BB: bounded

BR: bounded redescending

UB: unbounded when $x \to -\infty$ and bounded when $x \to \infty$

BU: bounded when $x \to -\infty$ and unbounded when $x \to \infty$.

The selected score functions and corresponding densities of prototypes are given in Table 1. The densities are unimodal. It holds that $g'(0) = 0$ so that

$$S(0) = 0. \tag{2.2}$$

Table 1. Prototype distributions.

type	distribution	$g(y)$	$S(y)$
UE	*	$\frac{1}{K} e^{-\cosh y}$	$\sinh y$
UP	normal	$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$	y
UB	extreme value	$e^{-y} e^{-e^{-y}}$	$1 - e^{-y}$
BU	Gumbel	$e^{y} e^{-e^{y}}$	$e^{y} - 1$
B	logistic	$\frac{e^y}{(1+e^y)^2}$	$\tanh \frac{y}{2}$
BR	Cauchy	$\frac{1}{\pi(1+y^2)}$	$\frac{2y}{1+y^2}$

$K = 2K_0(1)$

2.2 Transformed distributions

The density of transformed distribution $F = G\eta$ is

$$f(x) = g(\eta(x))\eta'(x). \tag{2.3}$$

The densities of distributions from Table 1a transformed to $\mathcal{Q} = (0, \infty)$ and $\mathcal{Q} = (0, 1)$ are given in Table 2 and Table 3, together with the transformed scores of the prototypes,

$$T(x) = S(\eta(x)). \tag{2.4}$$

(2.4) was detected (see [4]) as an important characteristic of transformed distributions. In the same paper it was shown that T can be found without necessity to determine the prototype by the use of formula

$$T(x) = \frac{1}{f(x)} \frac{d}{dx} \left(-\frac{1}{\eta'(x)} f(x) \right). \tag{2.5}$$

From (2.4) and (2.2) it follows that

$$T(\eta^{-1}(0)) = 0, \tag{2.6}$$

which is, together with (2.2), a condition used throughout the paper for checking the correctness of the proposed score functions and their transformed versions.

3 Parametric families

Parametric families on $\mathbb{R}$ are constructed in three steps.

Table 2. Transformed distributions on $(0, \infty)$.

type	distribution	$f(x)$	$T(x)$
UE	GIG	$\frac{1}{Kx} e^{-\frac{1}{2}(x+1/x)}$	$\frac{1}{2}(x - 1/x)$
UP	lognormal	$\frac{1}{\sqrt{2\pi}x} e^{-\frac{1}{2}\log^2 x}$	$\log x$
UB	Fréchet	$\frac{1}{x^2} e^{-1/x}$	$1 - 1/x$
BU	exponential	e^{-x}	$x - 1$
B	log-logistic	$\frac{1}{(1+x)^2}$	$\frac{x-1}{x+1}$
BR	log-Cauchy	$\frac{1}{\pi x(1+\log^2 x)}$	$\frac{2 \log x}{1+\log^2 x}$

Table 3. Transformed distributions on $(0, 1)$.

type	distribution	$f(x)$	$T(x)$
UE		$\frac{1}{Kx(1-x)} e^{-\frac{1}{2}(\frac{x}{1-x}+\frac{1-x}{x})}$	$\frac{x-1/2}{x(1-x)}$
UP	Johnson U_B	$\frac{1}{\sqrt{2\pi}x(1-x)} e^{-\frac{1}{2}\log^2 \frac{x}{1-x}}$	$\log \frac{x}{1-x}$
UB		$\frac{1}{x^2} e^{-(1-x)/x}$	$\frac{x-1/2}{x}$
BU		$\frac{1}{(1-x)^2} e^{-x/(1-x)}$	$\frac{x-1/2}{1-x}$
BB	uniform	1	$x - 1/2$
BR		$\frac{1}{\pi x(1-x)} \frac{1}{1+\log^2 \frac{x}{1-x}}$	$\frac{2 \log \frac{x}{1-x}}{1+\log^2 \frac{x}{1-x}}$

3.1 Introduction of shape parameters

Score functions from Table 1 were provided by two shape parameters $\alpha > 0$ and $\nu > 0$ in such a way that $S(0; \alpha, \nu) = 0$. Results are given in Table 4.

Table 4. System of Johnson prototype families with shape parameters α, ν.

type	$g(z; \alpha, \nu)$	$S(z; \alpha, \nu)$
UE	$\frac{\nu^{\rho/2} z^{\rho-1}}{2K_\rho(\alpha\sqrt{\nu})} e^{-\frac{\alpha}{2}(e^z+\nu e^{-z})}$	$\frac{\alpha}{2}(e^z - \nu e^{-z}) - \rho$
UP	$\frac{1}{2^\lambda \Gamma(\lambda)} e^{-\frac{1}{2}\|z\|^{1+\alpha}}$	$\frac{\alpha+1}{2}$ sgn $z\ \|z\|^\alpha$
UB	$\frac{\alpha^\alpha}{\Gamma(\alpha)} e^{-\alpha} e^{-\alpha e^{-z}}$	$\alpha(1 - e^{-z})$
BU	$\frac{\alpha^\alpha}{\Gamma(\alpha)} e^{\alpha z} e^{-\alpha e^z}$	$\alpha(z - 1)$
BB	$\frac{1}{\nu^\alpha B(\nu\alpha, \alpha)} \frac{e^{\nu\alpha z}}{(e^z+1/\nu)^{[1+\nu]\alpha}}$	$\alpha \frac{z-1}{z+1/\nu}$
BR	$\frac{1}{B(\frac{1}{2}, \alpha-\frac{1}{2})} \frac{1}{(1+z^2)^\alpha}$	$\frac{2\alpha z}{1+z^2}$

B and Γ are the beta and gamma functions, K_ρ is the Bessel function of the third kind, $\rho = \alpha(1 - \nu)/2$ and $\lambda = (\alpha + 2)/(\alpha + 1)$.

The meaning of the introduced parameters α and ν is as follows. For distribution with a symmetric parent, α means the excess. Parameter $\nu \in (0, \infty)$ characterizes the non-symmetry. This is apparent from Fig. 1, showing densities of type BB,

$$g_{BB}(x; 1, \nu) = e^{\nu x}/(e^x + 1/\nu)^{1+\nu},$$

for some values of ν. Finally, parameter α of asymmetric and mutually symmetric distributions UB and BU characterizes both the excess and non-symmetry simultaneously.

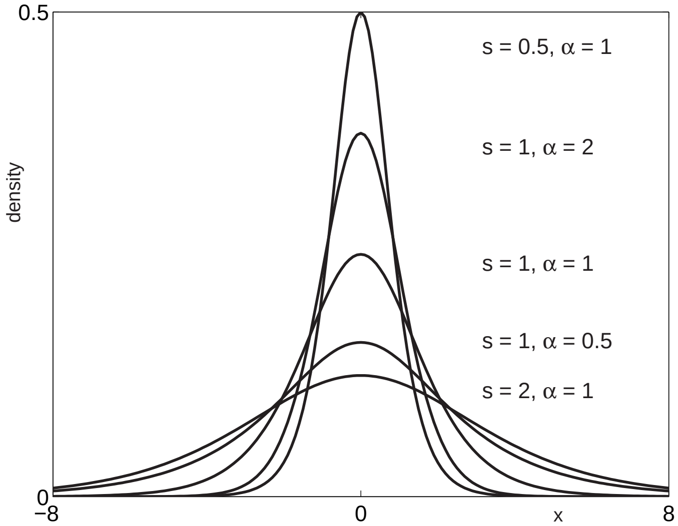

Figure 1. Densities of prototypes of type BB with various values of the parameter of non-symmetry ν.eps

The transformed score functions of prototypes and the densities of transformed distributions on (a, b) are $T(x; \alpha, \nu) = S(\eta(x); \alpha, \nu)$ and $f(x; \alpha, \nu) = g(\eta(x); \alpha, \nu)\eta'(x)$.

3.2 Introduction of location and scale parameters

Let $\mu \in \mathbb{R}$ be the location and $\sigma > 0$ the scale. By setting $z = w$ in Table 2, where

$$w = \frac{y - \mu}{\sigma} \tag{3.1}$$

is a 'pivotal variable on $\mathbb{R}$' (c.f. [5]), one obtains parametric distributions $G_{\mu,\sigma,\alpha,\nu}$ with densities and transformed score functions of the prototype

$$\tilde{g}(y; \mu, \sigma, \alpha, \nu) = \sigma^{-1} g\,(w; \alpha, \nu) \tag{3.2}$$

$$\tilde{S}(y; \mu, \sigma, \alpha, \nu) = -\frac{1}{g(w; \alpha, \nu)} \frac{d}{dy} g(w; \alpha, \nu) = \sigma^{-1} S\,(w; \alpha, \nu)\,. \tag{3.3}$$

$\tilde{g}$ has mode at μ and $\tilde{S}(\mu; \mu, \sigma, \alpha, \nu) = 0$. μ is taken as a measure of central tendency of $G_{\mu,\sigma,\alpha,\nu}$.

3.3 Transformed families

The transformed location of the prototype,

$$t = \eta^{-1}(\mu), \tag{3.4}$$

will be called the *Johnson parameter* of transformed distribution $F_{t,\sigma,\alpha,\nu} = G_{\mu,\sigma,\alpha,\nu}\eta$. Using (2.3) and (3.2), (2.4) and (3.3), the density and transformed score function of $F_{t,s,\alpha,\nu}$ are

$$f(x;t,\sigma,\alpha,\nu) = \sigma^{-1}g\left(\eta(w);\alpha,\nu\right)\eta'(x) \tag{3.5}$$

$$T(x;t,\sigma,\alpha,\nu) = \sigma^{-1}S\left(\eta(w);\alpha,\nu\right) \tag{3.6}$$

where we denoted

$$\eta(w) = \frac{\eta(x)-\eta(t)}{\sigma}. \tag{3.7}$$

(3.7) can be called a 'pivotal variable on $\mathcal{Q}$'. By setting $y = \eta(w)$ in Table 4 and by using relations (3.5) and (3.6) one obtains a system of transformed distributions on arbitrary support. The Johnson parameter (3.4), the explicit form of which is

$$t = \begin{cases} \mu & \text{if} \quad \mathcal{Q} = \mathcal{R} \\ e^{\mu-a} & \text{if} \quad \mathcal{Q} = (a,\infty) \\ \dfrac{be^{\mu}+a}{1+e^{\mu}} & \text{if} \quad \mathcal{Q} = (a,b) \\ e^{b-\mu} & \text{if} \quad \mathcal{Q} = (-\infty,b), \end{cases}$$

can be considered as a measure of the central tendency of transformed distributions [6]. For distributions with support $Q \neq \mathbb{R}$, the scale parameter is often expressed by means of its reciprocal value

$$\beta = 1/\sigma.$$

4 Johnson system of distributions on $(0,\infty)$

On $\mathcal{Q} = (0,\infty)$, it holds that $\eta(x) = \log x$, $\eta'(x) = 1/x$, and the pivotal variable (3.7) is

$$z = \eta(w) = \log(x/t)^{\beta}. \tag{4.1}$$

Let us present some members that occurs in statistical literature.

Type UP. The densities of distributions of this type are

$$f(x;t,s,\alpha) = \frac{1}{2^{\lambda}\Gamma(\lambda)x}e^{-\frac{1}{2}(\log|\frac{x}{t}|^{\beta})^{1+\alpha}}$$

where $\lambda = (\alpha+2)/(\alpha+1)$. Distributions of the 'Johnson triplet' [1, 7] are

$\mathcal{Q}$	distribution	$f(x;t,s,1)$	$T(x;t,s,1)$
$\mathbb{R}$	normal	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu}{s})^2}$	$\frac{x-\mu}{s}$
$(0,\infty)$	lognormal	$\frac{1}{\sqrt{2\pi}x}e^{-\frac{1}{2}\log^2(\frac{x}{t})^{\beta}}$	$\log\left(\frac{x}{t}\right)^{\beta}$
$(0,1)$	Johnson's U_B	$\frac{1}{\sqrt{2\pi}x(1-x)}e^{-\frac{1}{2}\log^2\left(\frac{x(1-t)}{t(1-x)}\right)^{\beta}}$	$\log\left(\frac{x(1-t)}{t(1-x)}\right)^{\beta}$

and the transformed score functions of the prototypes are

$$T(x;t,\beta,\alpha) = \frac{\alpha+1}{2}\left(\log|\frac{x}{t}|^{\beta}\right)^{\alpha}\,\mathrm{sgn}\,(x/t-1)$$

Type UB. The densities of the family with non-symmetric prototypes skewed to the right are

$$f(x;t,\beta,\alpha) = \frac{\beta\alpha^{\alpha}}{\Gamma(\alpha)x}\left(\frac{x}{t}\right)^{-\beta\alpha} e^{-\alpha\left(\frac{x}{t}\right)^{-\beta}}.$$

$f(x;t,\beta,1)$ is the Fréchet distribution. The transformed scores of the prototypes are

$$T(x;t,s,\alpha) = \alpha[1-(x/t)^{-\beta}].$$

Type BU. The family with non-symmetric prototypes skewed to the left, with densities

$$f(x;t,\beta,\alpha) = \frac{\beta\alpha^{\alpha}}{\Gamma(\alpha)x}\left(\frac{x}{t}\right)^{\beta\alpha} e^{-\alpha\left(\frac{x}{t}\right)^{\beta}},$$

is called by [8] the general gamma family. Members of the family are distributions

Weibull	$f(x;t,\beta,1)$
gamma	$f(x;\alpha/\gamma,1,\alpha)$
chi-squared	$f(x;\nu,1,\nu/2)$
Rayleigh	$f(x;a\sqrt{2},1/2,1)$
Maxwell	$f(x;a\sqrt{3},2,3/2)$

and the transformed scores of the prototypes are

$$T(x;t,s,\alpha) = \alpha[(x/t)^{\beta}-1].$$

Type BB. The densities of heavy-tailed distributions of type BB are

$$f(x;t,\beta,\alpha,\nu) = \frac{\beta}{\nu^{\alpha}B(\nu\alpha,\alpha)x}\frac{(x/t)^{\beta\nu\alpha}}{[(x/t)^{\beta}+1/\nu]^{(1+\nu)\alpha}}$$

and transformed score functions of the prototypes are

$$T(x;t,\beta,\alpha,\nu) = \alpha\frac{(x/t)^{\beta}-1}{(x/t)^{\beta}+1/\nu}. \tag{4.2}$$

In [8], a family with densities

$$f_{TB}(x;\gamma,\beta,\varepsilon,\delta) = \frac{\beta}{B(\varepsilon,\delta)x}\frac{(x/\gamma)^{\beta\varepsilon}}{[(x/\gamma)^{\beta}+1]^{\varepsilon+\delta}}$$

is called the transformed beta family. By (2.5), the corresponding transformed scores of the prototypes are

$$T_{TB}(x;\gamma,\beta,\varepsilon,\delta) = \frac{\delta(x/\gamma)^{\beta}-\varepsilon}{(x/\gamma)^{\beta}+1}. \tag{4.3}$$

Since $T_{TB}(\gamma;\gamma,\beta,\varepsilon,\delta)\neq 0$ if $\varepsilon\neq\delta$, the parameter γ is not a Johnson parameter. Comparing (4.3) with (4.2), one obtains relation

$$f_{TB}(x;\gamma,\beta,\varepsilon,\delta)=f(x;\gamma(\varepsilon/\delta)^{1/\beta},\beta,\delta,\varepsilon/\delta).$$

Belonging to this type is the 'beta triplet'

$\mathcal{Q}$	distribution	$f(x;1,1,p,q)$	$T(x;1,1,p,q)$
$\mathbb{R}$	prototype beta	$\frac{1}{B(p,q)}\frac{e^{px}}{(e^x+1)^{p+q}}$	$\frac{qe^x-p}{e^x+q}$
$(0,\infty)$	beta-prime	$\frac{1}{B(p,q)}\frac{x^{p-1}}{(x+1)^{p+q}}$	$\frac{qx-p}{x+q}$
$(0,1)$	beta	$\frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}$	$(p+q)x-p$

some other members of this type are distributions

log-logistic	$f(x;t,\beta,1,1)$
Fisher-Snedecor	$f_{TB}(x;\nu_2/\nu_1,1,\nu_1/2,\nu_2/2)=f(x;1,1,\nu_2/2,\nu_1/\nu_2)$
Burr III	$f(x;k^{1/c},c,1,k)$
Burr XII	$f(x;k^{-1/c},c,k,1/k)$
Lomax	$f_{TB}(x;1,1,1,\alpha)=f(x;1/\alpha,1,\alpha,1/\alpha)$

Type BR. This type contains heavy-tailed distributions with densities

$$f(x;t,s,\alpha)=\frac{1}{B(\frac{1}{2},\alpha-\frac{1}{2})}\frac{1}{[1+\log^2(x/t)^\beta]^\alpha}.$$

Members are log-Cauchy distribution $f(x;t,\beta,1)$ and Student z distribution $f(x;c^{1/2},1,(c+1)/2)$ (c.f. [9]). The transformed scores of the prototypes are redescending functions

$$T(x;t,\beta,\alpha)=\frac{2\alpha\log(x/t)^\beta}{1+\log^2(x/t)^\beta}.$$

Type UE. Belonging to this type is a generalized inverse Gaussian family [7] with densities

$$f_{GIG}(x;p,q,\lambda)=\frac{(p/q)^{\lambda/2}}{2K_\lambda(\sqrt{pq})}x^{\lambda-1}e^{-\frac{1}{2}(px+q/x)},$$

where $p,q,\lambda\in(0,\infty)$. From (2.5) one obtains

$$T_{GIG}(x;p,q,\lambda)=\frac{1}{2}(px-q/x)-\lambda. \tag{4.4}$$

$T_{GIG}(1;p,q,\lambda)=0$ if $\lambda=(p-q)/2$. In this case, (4.4) can be generalized for the Johnson and β parameters. By setting $p=\alpha$ and $q=\nu\alpha$ one obtains a family with densities

$$f(x;t,\beta,\alpha,\nu)=\frac{\nu^{\rho/2}(x/t)^\rho}{2K_\rho(\alpha\sqrt{\nu})x}e^{-\frac{\alpha}{2}\left[(x/t)^\beta+\nu(x/t)^{-\beta}\right]}$$

where $\rho=(\nu+1)\alpha/2$. The transformed score functions of the prototypes are

$$T(x;t,\beta,\alpha,\nu)=\frac{\alpha}{2}\left[(x/t)^\beta-\nu(x/t)^{-\beta}\right]-(1-\nu)\alpha/2.$$

The two frequently used families of type UE, the Wald and inverse Gaussian, was chosen perhaps owing to a simple normalizing constant. Both of them cannot be further generalized for the Johnson and β parameters. A composite distribution belonging to type UE is the Birnbaum-Saunders distribution [7] with transformed score function of the parent distribution in form

$$T(x) = \frac{1}{2}(x - \frac{1}{x}) + \frac{1}{2} - \frac{x}{1+x}. \tag{4.5}$$

(4.5) is rather complex, but it satisfies condition (2.6) and is therefore ready to be generalized for the pivotal variable (4.1).

5 Systems on finite intervals

From Table 1, by using $\eta'(x) = 1/x(1-x)$ instead of $\eta'(x) = 1/x$ and by setting

$$z = \frac{\eta(x) - \eta(t)}{s} = \log\left(\frac{x(1-t)}{t(1-x)}\right)^{\beta}$$

where $\beta = 1/s$, we obtain the Johnson system on $(0, 1)$. A few practically used distributions of this system are listed in 'triplets' in the preceding section.

Let us briefly introduce three non-Johnson systems on finite intervals, originated by transformations

(i)	$\eta : (0, 1) \to \mathbb{R}$	$\eta(x)$	$=$	$-\log(-\log x)$
(ii)	$\eta : (-1, 1) \to \mathbb{R}$	$\eta(x)$	$=$	$\tanh^{-1} x$
(iii)	$\eta : (-\pi/2, \pi/2) \to \mathbb{R}$	$\eta(x)$	$=$	$\tan x$

(i) Set $q = -\log x$, so that $\eta'(x) = 1/xq$. The distributions given in Table 1 can be used as prototypes. By (2.5), the transformed scores of the prototypes are

$$T(x) = 1 - q - xqf'(x)/f(x).$$

The parent distributions can be provided by shape parameters and by 'pivotal quantity on $(0, 1)$', which is in this case $z = \log(t/q)^{1/s}$, where $t = -\log \kappa$. $\kappa = e^{-e^{-\mu}}$ is the image of the location μ of the prototype.

(ii) Since $\tanh^{-1}(x) = \frac{1}{2}\log\frac{x-1}{x+1}$, this system is similar to the Johnson system on $\mathcal{Q} = (-1, 1)$.

(iii) In '$\tan x$' system, $\eta'(x) = \cos^{-2} x$. By (2.5),

$$T(x) = \sin 2x - \cos^2 x f'(x)/f(x). \tag{5.1}$$

In order to obtain simple forms of the densities, simple forms of (5.1) should be chosen. Our proposals are listed in the second column of Table 3. In the third column of the table there are the densities computed by the integration of (5.1), in next columns are the densities and score functions of the prototypes.

Table 5. Parent distributions of 'tan x' system and their prototypes.

type	$T(x)$	$f(x)$	$g(y)$	$S(y)$
UE	$\tan x$	$\frac{1}{\sqrt{2\pi}\cos^2 x}e^{-\frac{1}{2}\tan^2 x}$	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}$	y
BB	$\sin x$	$\frac{1}{2K_1(1)\cos^2 x}e^{-1/\cos x}$	$\frac{1}{2K_1(1)}e^{-\sqrt{1+y^2}}$	$\frac{y}{\sqrt{1+y^2}}$
BR	$\sin 2x$	$1/\pi$	$\frac{1}{\pi}\frac{1}{1+y^2}$	$\frac{2y}{1+y^2}$
$BR-$	$\sin 2x - \cos^2 x$	$\frac{1}{c}e^{x}$	$\frac{1}{c}\frac{1}{1+y^2}e^{\tan^{-1} y}$	$\frac{2y-1}{1+y^2}$
$BR+$	$\sin 2x + \cos^2 x$	$\frac{1}{c}e^{-x}$	$\frac{1}{c}\frac{1}{1+y^2}e^{-\tan^{-1} y}$	$\frac{2y+1}{1+y^2}$

$c = e^{\pi/2} - e^{-\pi/2}$.

In Table 5, type BR is a parent of the Burr XI distribution and the prototype BR_+ is a parent of Pearson IV distribution (cf. [7]). The score function of prototypes BR_+ and BR_- do not satisfy condition (2.2) so that the pertaining densities cannot be provided by the location and scale parameters. Let us put

$$S_{G_+}(y) = \frac{2y}{[1+(y+1/2)^2]}, \qquad S_{G_-}(y) = \frac{2y}{[1+(y-1/2)^2]}.$$

The corresponding densities are

$$g_+(y) = \frac{1}{c[1+(y+1/2)^2]}e^{\tan^{-1}(y+1/2)}$$

and

$$g_-(y) = \frac{1}{c[1+(y-1/2)^2]}e^{-\tan^{-1}(y-1/2)},$$

where $c = e^{\pi/2} - e^{-\pi/2}$. g_+ and g_- are the heavy-tailed asymmetric parents of type BR (see Fig.2), which can be included into prototype distributions of the Johnson system in Table 1.

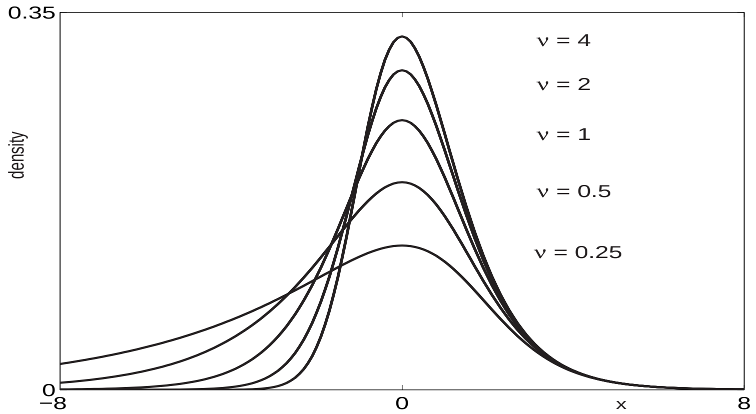

Figure 2. Densities $g_-, g_{0,1,1}$ (Cauchy) and g_+.eps

They can be further provided by shape parameters and finally by pivotal variable on $\mathbb{R}$. Their transformed versions can be provided by 'pivotal quantity on $(-\pi/2, \pi/2)$' which appears to be

$$z = \beta(\tan x - \tan t) = \frac{\beta \sin(x-t)}{\cos x \cos t}.$$

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