Biomechanical Analysis of the Effect of Axial Angle Changes on the Weight-Bearing Total Knee Replacements

Stehlík, J.
2006

Institute of Computer Science
Academy of Sciences of the Czech Republic

Biomechanical analysis of the effect of axial angle changes on the weight-bearing total knee replacements

J. Stehlík, P. Vavřík, J. Daněk, J. Nedoma, I. Hlaváček, F. Denk

Technical report No. 958

May 2006

Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:ics@cs.cas.cz

Institute of Computer Science
Academy of Sciences of the Czech Republic

Biomechanical analysis of the effect of axial angle changes on the weight-bearing total knee replacements

J. Stehlík[1], P. Vavřík[2], J. Daněk[3], J. Nedoma[4],
I. Hlaváček[5], F. Denk[6]

Technical report No. 958

May 2006

Abstrakt:

The aim of this paper is to compare of the biomechanical influences of different grades of valgus deformity after the total knee replacement (TKR) based on the contact model problem with friction, the finite element method and the nonoverlapping domain decomposition method.

Keywords:
Biomechanics, knee, total replacement, valgus deformity

[1]Orthopaedic Department, The Hospital České Budějovice, B. Němcová Str. 585/54, 370 01 České Budějovice, Czech Republic, e-mail: stehlik@nemcb.cz

[2]1st Ortopaedical Department, 1st Medical Faculty of the Charles University Prague, V úvalu 84, 150 00 Prague 5 - Motol, Czech Republic, e-mail: pvavrik@atlas.cz

[3]Centre of Applied Mathematics, University of West Bohemia, Univerzitní 8, 306 14 Pilsen, Czech Republic and Institute of Computer Science, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 2, 182 07 Prague 8, Czech Republic e-mail: danek@kma.zcu.cz

[4]Institute of Computer Science, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 2, 182 07 Prague 8, Czech Republic, e-mail: nedoma@cs.cas.cz

[5]Institute of Computer Science, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 2, 182 07 Prague 8, Czech Republic and Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Prague 1, Czech Republic, e-mail: hlavacek@math.cas.cz

[6]WALTER a.s., Jinonická 329, 158 01 Prague 5 - Jinonice, Czech Republic, e-mail: fDenk@walter.cz

Biomechanical analysis of the effect of axial angle changes on the weight-bearing total knee replacements

Stehlík, J.[1]**, Vavřík, P.**[2]**, Daněk, J.**[3,4]**, Nedoma, J.**[4]**, Hlaváček, I.**[4,5]**, Denk, F.**[6]

[1] Orthopaedic Department, The Hospital České Budějovice, B. Němcová Str. 585/54, 370 01 České Budějovice, Czech Republic e-mail: stehlik@nemcb.cz
[2] 1st Ortopaedical Department, 1st Medical Faculty of the Charles University Prague, V úvalu 84, 150 00 Prague 5 - Motol, Czech Republic e-mail: pvavrik@atlas.cz
[3] Centre of Applied Mathematics, University of West Bohemia, Univerzitní 8, 306 14 Pilsen, Czech Republic e-mail: danek@kma.zcu.cz
[4] Institute of Computer Science, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 2, 182 07 Prague 8, Czech Republic e-mail: nedoma@cs.cas.cz
[5] Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Prague 1, Czech Republic e-mail: hlavacek@math.cas.cz
[6] WALTER a.s., Jinonická 329, 158 01 Prague 5 - Jinonice, Czech Republic e-mail: fDenk@walter.cz

Abstract The aim of this paper is to compare of the biomechanical influences of different grades of valgus deformity after the total knee replacement (TKR) based on the contact model problem with friction, the finite element method and the nonoverlapping domain decomposition method.

Key words Biomechanics, knee, total replacement, valgus deformity

1 Introduction

Division of loads acting on the tibial component after implantation of the total knee replacement is of primary importance in orthopedic surgery. About the pressure ratios in the knee after the total knee replacement (TKR) the soft tissue tension (i.e. capsules of joints, ligaments, muscular insertions) in the vicinity of the replacement and the resulting axial position of the whole limb are determined. These factors are influenced especially by the own technique of implantation and, in a decisive way determines the survival time of the implant. Nonobservance of the balance of both compartments (medial and lateral) or possible overloading of the posterior part of the tibia plate leads to wear out of the polyethylene insert. Asymmetrical overloading leads to the premature abrasion of a plastic polyethylen insert with production of a great amount of polyethylene elements, which instigate a complicated inflammatory reaction leading to loosening metallic components of the total replacement from the bone.

The successful application of the total replacements of the knee joint (TKR) depends on many factors. Namely, it is the optimal axial position of a lower limb. Participating in the resulting axial position of a limb are many factors which must be studied from the clinical and radiological point of view in more details and the respected before the operation. Changes in the form of the distal femur or the proximal tibia forming the knee joint can be the causes of incorrect centering of both components of the knee joint. Therefore, the mechanical axis, i.e. the connection of the centre of the head of the femur with the centre of the incisura intercondylica (area between condyles) and the centre of the talocrural (ankle) joint determines the so-called mechanical axis of the lower limb. After the adjustment of the mechanical axis, the angle with the anatomic femur axis, i.e. the true axis of an extremity which is measured by lines drawn parallel to the shaft of the bone, determines the degree of physiological valgus, in which the resection of the lower femur should be performed. If the correct technique of implantation, especially soft tissue balancing and respect for the exact position of the mechanical axis of extremity are maintained, then the basic conditions for correct biomechanical functioning of the total knee replacement are created.

From the orthopaedic point of view the total knee replacements are studied by [10].

Moreover, the mechanical factors are also an important one. The idea of a total knee prosthesis being a device that transfer the knee joint loads to the bone allows explaining the mechanical factor in terms of the load transfer mechanism. A complex relations exist between this mechanism, the magnitude and direction of the loads, the geometry of the bone-joint prosthesis configuration, the elastic properties of the materials and the physical connections at the material connections. The contact problems in suitable rheology and their finite element approximations are very useful tools for analyzing these relations for knee joint and its artificial replacement [5,7,8, 11,1,9]

2 The model

The idea of a total knee replacement being a device that transfer the knee joint loads to the bone allows explaining the mechanical factor in terms of the load transfer mechanism. The authors of [5,7,8,11,1,9] showed that the contact problems in suitable rheology and their finite element approximations are very useful tools for analyzing deviation of stress-strain fields in the knee joint and its artificial replacement.

Such models are based on the contact problem with friction in elasticity and the finite element method. The presented model problem is formulated as the primary semi-coercive contact problem with given friction and for the numerical solution of the studied problem the nonoverlapping domain decomposition method, discussed in [2], is used.

2.1 The model contact problem with friction and the nonoverlapping domain decomposition method

Let the investigated part of the knee joint occupy a union Ω of bounded domains Ω^ι, $\iota = 1,2,3$ in $\mathbb{R}^N$ ($N = 2$), denoting separate components of the knee joint - the femur (1), the tibia (2) and the fibula (3), with Lipschitz boundaries $\partial\Omega^\iota$. Let the boundary $\partial\Omega = \partial\Omega^1 \cup \partial\Omega^2 \cup \partial\Omega^3$ consist of four disjoint parts such that $\partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c \cup \Gamma_o$. Let $\Gamma_\tau = {}^1\Gamma_\tau \cup {}^2\Gamma_\tau$, where by ${}^1\Gamma_\tau$ we denote the loaded part of the femur and by ${}^2\Gamma_\tau$ the unloaded part of the boundary $\partial\Omega$. By Γ_u we denote the part of the tibial and fibula's boundaries, where we simulate their fixation. The common contact boundary between both joint components Ω^1 and Ω^2 before deformation we denote by $\Gamma_c = \partial\Omega^1 \cap \partial\Omega^2$. By Γ_o we denote the common contact boundary between tibia and fibula, where we assume compliance of bilateral contact condition. Let body forces $\mathbf{F}$, surface tractions $\mathbf{P}$ and slip limits g_c be given. We have the following problem:

Problem ($\mathcal{P}$): find the displacements $\mathbf{u}^\iota$ in all Ω^ι such that

$$\frac{\partial \tau_{ij}\left(\mathbf{u}^\iota\right)}{\partial x_j} + F_i^\iota = 0,\ i,j = 1,2 \ \text{ in } \Omega^\iota,\ \iota = 1,2,3, \tag{1}$$

$$\tau_{ij} n_j = P_i,\ i,j = 1,2 \quad \text{on } {}^1\Gamma_\tau, \tag{2}$$

$$\tau_{ij} n_j = 0,\quad i,j = 1,2 \quad \text{on } {}^2\Gamma_\tau, \tag{3}$$

$$u_i^\iota = 0,\quad i = 1,2 \quad \text{on } \Gamma_u, \tag{4}$$

$$\left\{ \begin{array}{l} u_n^1 - u_n^2 \le 0,\ \tau_n^{12} \le 0,\ \left(u_n^1 - u_n^2\right)\tau_n^{12} = 0, \\ |\boldsymbol{\tau}_t^{12}(\mathbf{u})| \le \mathcal{F}_c |\tau_n^{12}(\mathbf{u})| \equiv g_c, \\ |\boldsymbol{\tau}_t^{12}(\mathbf{u})| < g_c \Rightarrow \mathbf{u}_t^1 - \mathbf{u}_t^2 = 0 \\ |\boldsymbol{\tau}_t^{12}(\mathbf{u})| = g_c \Rightarrow \text{there exists } \kappa \ge 0 \\ \qquad \text{such that } \mathbf{u}_t^1 - \mathbf{u}_t^2 = -\kappa \boldsymbol{\tau}_t^{12}(\mathbf{u}) \end{array} \right. \quad \text{on } \Gamma_c, \tag{5}$$

$$u_n^\iota = 0 \ \text{ and } \ \boldsymbol{\tau}_t^\iota = 0 \quad \text{on } \Gamma_o, \tag{6}$$

where the normal and tangential components of displacement vector $\mathbf{u} = (u_i)$, $i = 1,2$, and stress vector $\boldsymbol{\tau} = (\tau_i)$, $i = 1,2$, are defined as follows:

$u_n = u_i n_i$, $\mathbf{u}_t = \mathbf{u} - u_n\mathbf{n}$, $\tau_n = \tau_{ij} n_j n_i$, $\boldsymbol{\tau}_t = \boldsymbol{\tau} - \tau_n \mathbf{n}$, where $\mathbf{n}$ denotes the outward unit normal to the boundary $\partial\Omega$, and therefore, $\tau_n^k = \tau_n^l \equiv \tau_n^{kl}$, $\boldsymbol{\tau}_t^k(\mathbf{u}) = \boldsymbol{\tau}_t^l(\mathbf{u}) \equiv \boldsymbol{\tau}_t^{kl}(\mathbf{u})$. Moreover, the elastic coefficients c_{ijkl} satisfy the conditions of symmetry $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$ and the condition $0 < c_0^\iota \le c_{ijkl}^\iota \xi_{ij}\xi_{kl} \left|\xi\right|^{-2} \le c_1^\iota < +\infty$, for a.a. $\mathbf{x} \in \Omega^\iota$, $\xi \in \mathbb{R}^4$, $\xi_{ij} = \xi_{ji}, c_0^\iota, c_1^\iota = const. > 0$ independent of $\mathbf{x} \in \Omega^\iota$, $\iota \in \{1,2,3\}$. Let $W = \sqcap_{\iota=1}^3 [H^1(\Omega^\iota)]^2$ be the Sobolev space in the usual sense, let $\|\mathbf{v}\|_W = (\sum_\iota \sum_i \|v_i\|_{1,\Omega^\iota}^2)^{\frac{1}{2}}$. Let us introduce the sets of virtual and admissible displacements

$$V = \{\mathbf{v} \in W \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_u \text{ and } v_n = 0 \text{ on } \Gamma_o\},$$
$$K = \left\{\mathbf{v} \in V \mid v_n^1 - v_n^2 \le 0 \text{ on } \Gamma_c\right\}.$$

Let $c_{ijkl}^\iota \in L^\infty(\Omega^\iota)$, $F_i^\iota \in L^2(\Omega^\iota)$, $P_i \in L^2({}^1\Gamma_\tau)$. Then we have to solve the following variational problem:

Problem $(\mathcal{P}_v)$: find a function $\mathbf{u} \in K$, such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \ge L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K, \tag{7}$$

where

$$\begin{aligned} &a(\mathbf{u},\mathbf{v}) = \textstyle\sum_\iota \int_{\Omega^\iota} c_{ijkl}^\iota e_{ij}\left(\mathbf{u}^\iota\right) e_{kl}\left(\mathbf{v}^\iota\right) d\mathbf{x}, \\ &j(\mathbf{v}) = \textstyle\int_{\Gamma_c} g_c |\mathbf{v}_t^1 - \mathbf{v}_t^2| ds, \\ &L(\mathbf{v}) = \textstyle\sum_\iota \int_{\Omega^\iota} F_i^\iota d\mathbf{x} - \sum_\iota \int_{{}^1\Gamma_\tau \cap \partial\Omega^\iota} P_i^\iota v_i^\iota ds. \end{aligned} \tag{8}$$

Let us define a finite dimensional space V_h by

$$\begin{gathered} V_h = \{\mathbf{v}_h \mid \mathbf{v}_h \in [C(\Omega^1)]^2 \times \cdots \times [C(\Omega^s)]^2, \mathbf{v}_h|_{T_{hi}} \in [P_1(T_{hi})]^2, \forall T_{hi} \in \mathcal{T}_h, \\ v_{hn}(q_i) = 0, q_i \in \Gamma_0; \mathbf{v}_h(q_i) = \mathbf{u}_0(q_i), q_i \in \Gamma_u\} \end{gathered}$$

and a finite dimensional set of admissible displacements

$$K_h = \{\mathbf{v}_h \mid \mathbf{v}_h \in V_h, (v_{hn}^k - v_{hn}^l)(q_i) \le 0, q_i \in \Gamma_c^{kl}, 1 \le k,l \le s\}.$$

The finite element approximation leads to solve the following problem:

Problem $(\mathcal{P}_h)$: find a function $\mathbf{u}_h, \mathbf{u}_h - \mathbf{u}_0 \in K_h$, such that

$$a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + j(\mathbf{v}_h) - j(\mathbf{u}_h) \ge L(\mathbf{v}_h - \mathbf{u}_h) \quad \forall \mathbf{v}_h \in K_h. \tag{9}$$

2.2 The nonoverlapping domain decomposition method

Let us introduce

$$T = \{n \in \{1,\ldots,N\} : \ \bar{\Gamma}_c \cap \partial\bar{\Omega}^n = \emptyset\}$$

the set of all indices of subdomains Ω^n which are not adjacent to a contact, and

$$\vartheta = \{[k,l], \ k,l \in \{1,\ldots,N\} : \partial\bar{\Omega}^k \cap \partial\bar{\Omega}^l \subset \Gamma_c\}$$

represents couples of subdomains in unilateral contact. Suppose that $\Gamma \cap \Gamma_c = \emptyset$. Then for the trace operator $\gamma : [H^1(\Omega^n)]^2 \to [L^2(\partial\Omega^n)]^2$ we have

$$V_\Gamma = \gamma K|_\Gamma = \gamma V|_\Gamma. \tag{10}$$

Let $\gamma^{-1} : V_\Gamma \in V$ be an arbitrary linear inverse mapping satisfying

$$\gamma^{-1}\overline{\mathbf{v}} \ = 0 \quad \text{on } \Gamma_c \quad \forall\, \overline{\mathbf{v}} \in V_\Gamma. \tag{11}$$

Let us introduce restrictions $\overline{R}_n : V_\Gamma \to \Gamma_n$; $L_n : L \to \Omega^n$; $j_i^\iota : j^\iota \to S$; $a_n(.,.) : a(.,.) \to \Omega^n$; $V(\Omega^n) : V \to \Omega^n$ and let

$$V^0(\Omega^n) = \{\mathbf{v} \in V \mid \mathbf{v} = \mathbf{0} \ \text{ on } \ \overline{(\cup_{n=1}^N \Omega^n) \backslash \Omega^n}\}$$

be the space of functions with zero traces on Γ_n where $\Gamma_n = \Gamma \cap \partial\Omega^n$ The algorithm is based on the next theorem and on the use of local and global Schur complements.

Theorem: A function $\mathbf{u}$ is a solution of a global problem (7), if and only if its trace $\overline{\mathbf{u}} = \gamma\mathbf{u}|_\Gamma$ on the interface Γ satisfies the condition

$$\sum_{i=1}^N [a_i(\mathbf{u}_i(\overline{\mathbf{u}}), \gamma^{-1}\overline{\mathbf{w}}) - L_i(\gamma^{-1}\overline{\mathbf{w}})] = 0, \qquad \forall \overline{\mathbf{w}} \in V_\Gamma, \overline{\mathbf{u}} \in V_\Gamma \tag{12}$$

and its restrictions $\mathbf{u}_i(\mathbf{u}) \equiv \mathbf{u}|_{\Omega^i}$ satisfy:
(i) the condition

$$a_i(\mathbf{u}_i(\overline{\mathbf{u}}), \varphi_i) = L_i(\varphi_i), \quad \forall \varphi_i \in V^0(\Omega^i), \ \text{ for } i \in T, \tag{13}$$

(ii) the condition

$$\begin{aligned} a_k(\mathbf{u}_k(\overline{\mathbf{u}}), \varphi_k) + a_l(\mathbf{u}_l(\overline{\mathbf{u}}), \varphi_l) \geq L_k(\varphi_k) + L_l(\varphi_l), \\ \forall \varphi_i \in V^0(\Omega^i),\ i = k, l, \ \text{ for } [k,l] \in \vartheta. \end{aligned} \tag{14}$$

Proof. See [2].

To analyze the condition (12) the **local and global Schur complements** are introduced. Let

$$V_i = \{\gamma \mathbf{v}_{|\Gamma_i} \mid \mathbf{v} \in K\} = \{\gamma \mathbf{v}_{|\Gamma_i} \mid \mathbf{v} \in V\}$$

and define a particular case of the restriction of the inverse mapping $\gamma^{-1}(.)|_{\Omega^i}$ by

$$\begin{cases} Tr_i^{-1} : V_i \to V(\Omega^i), \quad \gamma(Tr_i^{-1}\overline{\mathbf{u}})|_{\Gamma_i} = \mathbf{u}_i,\ i = 1, \ldots, N, \\ a_i(Tr_i^{-1}\overline{\mathbf{u}}_i, \mathbf{v}_i) = 0, \quad \forall \mathbf{v}_i \in V^0(\Omega^i),\ Tr_i^{-1}\overline{\mathbf{u}}_i \in V(\Omega^i),\ \text{for } i \in T. \end{cases} \tag{15}$$

For $[k,l] \in \vartheta$ we complete the definition by the boundary condition (11), i.e.

$$Tr_k^{-1}\overline{\mathbf{u}}_k + Tr_l^{-1}\overline{\mathbf{u}}_l = 0 \text{ on } \Gamma_c. \tag{16}$$

The local Schur complement for $i \in T$ is the operator $\mathcal{S}_i : V_i \to (V_i)^*$ defined by

$$\langle \mathcal{S}_i \overline{\mathbf{u}}_i, \overline{\mathbf{v}}_i \rangle = a_i(Tr_i^{-1}\overline{\mathbf{u}}_i, Tr_i^{-1}\overline{\mathbf{v}}_i) \quad \forall \overline{\mathbf{u}}_i, \overline{\mathbf{v}}_i \in V_i. \tag{17}$$

For subdomains which are in contact we define a **common local Schur complement** for the union $\Omega^k \cup \Omega^l$ (where $[k,l] \in \vartheta$) as the operator $\mathcal{S}_{k,l} : (V_k \times V_l) \to (V_k \times V_l)^* = (V_k)^* \times (V_l)^*$ defined by

$$\begin{aligned} \langle \mathcal{S}_{k,l}(\overline{\mathbf{y}}_k, \overline{\mathbf{y}}_l), (\overline{\mathbf{v}}_k, \overline{\mathbf{v}}_l) \rangle = a_k(\mathbf{u}_k(\overline{\mathbf{y}}_k), Tr_k^{-1}\overline{\mathbf{v}}_k) + a_l(\mathbf{u}_l(\overline{\mathbf{y}}_l), Tr_l^{-1}\overline{\mathbf{v}}_l) \\ \forall (\overline{\mathbf{v}}_k, \overline{\mathbf{v}}_l) \in V_k \times V_l, \end{aligned} \tag{18}$$

where Tr_k^{-1} and Tr_l^{-1} are defined by means of (15) and (16).

The condition (12) can be expressed by means of local Schur complements in the form

$$\sum_{i \in T} \langle \mathcal{S}_i \overline{\mathbf{u}}_i, \overline{\mathbf{v}}_i \rangle + \sum_{[k,l] \in \vartheta} \langle \mathcal{S}_{k,l}(\overline{\mathbf{u}}_k, \overline{\mathbf{u}}_l), (\overline{\mathbf{v}}_k, \overline{\mathbf{v}}_l) \rangle = \sum_{i=1}^{N} L_i(Tr_i^{-1}\overline{\mathbf{v}}_i) \quad \forall \overline{\mathbf{v}} \in V_\Gamma, \tag{19}$$

where $\overline{\mathbf{u}} = \gamma \mathbf{u}_{|\Gamma}$, $\overline{\mathbf{v}}_i = \overline{R}_i \overline{\mathbf{v}}$, $\overline{\mathbf{u}}_i = \overline{R}_i \overline{\mathbf{u}}$. Then we will solve the equation (19) on the interface Γ in the dual space $(V_\Gamma)^*$. We rewrite (19) into the following form

$$\mathcal{S}_0 \overline{\mathbf{U}} + \mathcal{S}_{CON} \overline{\mathbf{U}} = \mathcal{F}, \tag{20}$$

where

$$\begin{aligned} \mathcal{S}_0 &= \sum_{i \in T} (\overline{R}_i)^T \mathcal{S}_i \overline{R}_i, \\ \mathcal{S}_{CON} &= \sum_{[k,l] \in \vartheta} \overline{R}_{k,l}^T \mathcal{S}_{k,l} \overline{R}_{k,l}, \\ \mathcal{F} &= \sum_{i=1}^{N} (\overline{R}_i)^T (Tr_i^{-1})^T L_i \end{aligned} \tag{21}$$

and $\overline{R}_{k,l}(\overline{\mathbf{u}}) = (\overline{R}_k(\overline{\mathbf{u}}), \overline{R}_l(\overline{\mathbf{u}}))^T, \overline{\mathbf{u}} \in V_\Gamma$.

Equation (20) will be solved by **successive approximations**, because the operators $\mathcal{S}_{k,l}$ and therefore $\mathcal{S}_{CON}$ are nonlinear. As a initial approximation $\overline{\mathbf{U}}^0$ we choose the solution of the global primal problem, where the boundary conditions on Γ_c are replaced by the linear bilateral conditions

$$u_{kn} - u_{ln} = 0, \quad \text{on } \Gamma_c. \tag{22}$$

Then we replace the set K by $K^0 = \{\mathbf{v} \in V \mid v_{kn} - v_{ln} = 0 \text{ on } \Gamma_c\}$ and therefore, we solve the following problem

$$\mathbf{u}^0 = \arg\min_{\mathbf{v} \in K^0} \mathcal{L}(\mathbf{v}), \tag{23}$$

where $\mathcal{L}(\mathbf{v}) = \frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v})$ and we set $\overline{\mathbf{U}}^0 = \gamma \mathbf{u}^0|_\Gamma$. The auxiliary problem (23) represents a linear elliptic boundary value problem with bilateral contact and it can be solved by the domain decomposition method again.

The non-linear equation (20) will be solved by successive approximations. We will assume that the approximation $\overline{\mathbf{U}}^{k-1}$ is known and the next approximation $\overline{\mathbf{U}}^k$ we find as the solution of the following linear problem

$$\mathcal{S}_0\overline{\mathbf{U}}^k = \mathcal{F} - \mathcal{S}_{CON}\overline{\mathbf{U}}^{k-1}, k = 1, 2, \ldots \qquad (24)$$

In [2] the convergence of the method of successive approximation (24) to the solution of the original problem (20) in the space $(V_\Gamma)^*$ is proved.

2.3 The models in frontal and sagittal planes

In the next two types of models, the axial angle changes of the weight-bearing total knee replacement in the frontal plane and in the sagittal plane, will be analyzed.

The total replacement of the left knee joint was considered in linkage on the axial deviation. The models are marked as MODEL I, MODEL II, MODEL III for the frontal cross-section, where it is possible to analyze the influence of the axial deviation and as MODEL IV, MODEL V for the sagittal cross-section, which tells only little about the influence of the axial deviation, but it tells something more about the overload of the posterior part of the tibial plate in the sagittal (anteroposterior) direction.

2.3.1 The frontal plain MODEL I corresponds to the angle of the valgus in the resection of the lower end of the femur 5 degree, MODEL II to 7 degree and MODEL III to 9 degree (Fig. 1). The following material parameters are used:

(1) anchoring plate - Ti6A14V: Young's modulus of elasticity $E = 1.15 \times 10^{11}[Pa]$, Poisson constant $\nu = 0.3$,

(2) polyethylene (UHMWPE) insert: Young's modulus of elasticity $E = 3.4 \times 10^8[Pa]$, Poisson constant $\nu = 0.4$,

(3) femoral component - CoCrMo: Young's modulus of elasticity $E = 2.08 \times 10^{11}[Pa]$, Poisson constant $\nu = 0.3$.

(4) cortical bone: Young's modulus of elasticity $E = 1.71 \times 10^{10}[Pa]$, Poisson constant $\nu = 0.25$,

The femorotibial part of the knee joint occupies the region, denoted by $\Omega = \cup_{\iota=1}^s \Omega^\iota$, with boundary denoted by $\partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c$, The boundary $\partial\Omega$ is created by parts 1-2 and 3-4 (Γ_u), where the fibula and tibia are fixed, modelled by the zero displacement (Dirichlet) conditions; by parts 7-8 and 9-10 (Γ_c), which are contact boundaries between collided parts of the femorotibial joint, modelled by the unilateral contacts; by part 11-12 (Γ_o), where the fibula is connected with the tibia, modelled by the bilateral contact; and by part 5-6 (${}^1\Gamma_\tau$), where the load $0.215 \times 10^7[Pa]$ is prescribed;

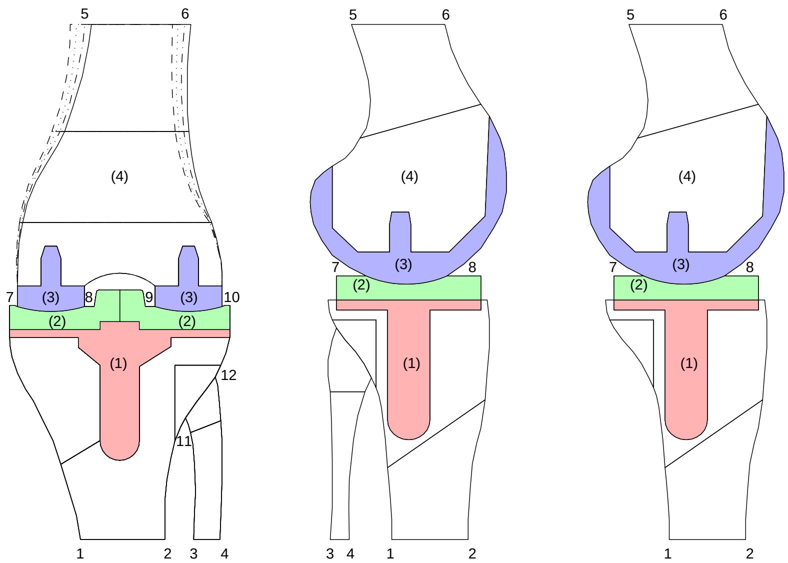

Fig. 1 The models: MODEL I-III, IV and V

on the remaining parts of the boundary $({}^2\Gamma_\tau)$ the femorotibial joint is unloaded.

For computations the following discretization parameters were used:

MODEL I: 13 subdomains of domain decomposition, 3737 nodes, 7132 elements, 31+31+15 unilateral and bilateral contact nodes, 344 interface elements between subdomains of domain decomposition,

MODEL II: 13 subdomains of domain decomposition, 3780 nodes, 7208 elements, 31+31+15 unilateral and bilateral contact nodes, 350 interface elements between subdomains of domain decomposition,

MODEL III: 13 subdomains of domain decomposition, 3763 nodes, 7180 elements, 31+31+15 unilateral and bilateral contact nodes, 34 interface elements between subdomains of domain decomposition.

2.3.2 The sagittal plain In the sagittal plane two types of models of the total knee replacements in linkage on the axial deviation 5, 7, 9 degrees across both condyles (see Fig. 1), where by MODEL IV the cut across outer condyle (on the figure marked by the fibula) and by MODEL V the cut across the inner condyle (on the figure without fibula) and in three variants denoted as A corresponding to 5 degree, B-7degree and C-9degree, were investigated. The sagittal cut tells anything about the influence of the axial deviation, it tells something more about the overload of the posterior part of the tibial plato in the sagittal (anteroposterior) direction, and therefore, it informs us about a relevant strong wear of the polyethylene insert. In both types of models the same material parameters were considered.

The investigated femorotibial area of the knee joint in the sagittal plane occupies the region, we denote it by $\Omega = \cup_{\iota=1}^{s} \Omega^{\iota}$ and its boundary by $\partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c$. The boundary $\partial\Omega$ is created by parts between points 1-2 and 3-4 (Γ_u), where the fibula (${}^1\Gamma_u$) and the tibia (${}^2\Gamma_u$) are fixed; by the part between points 5-6 (${}^1\Gamma_\tau$), where the loading is prescribed; by the part between points 7-8 (Γ_c), which is contact boundary between both collided parts of the femorotibial joint; by part between points 9-10 (Γ_c), where the fibula through the tissues is jointed with the tibia and modelled by the bilateral contact; on the remaining parts of the boundary (${}^2\Gamma_\tau$) the femorotibial joint is unloaded.

The boundary and contact conditions: MODEL IV: Zero displacement (Dirichlet condition) is prescribed between points 1-2 (fixed tibia) and 3-4 (fixed fibula); between points 5-6 the femur is loaded by a loading $1.46 \times 10^6[Pa]$ in the case of MODEL IV-A, $1.52 \times 10^6[Pa]$ in the case of MODEL IV-B and $1.61 \times 10^6[Pa]$ in the case of MODEL IV-C; the unilateral contact condition is between points 7-8 and the bilateral contact condition is between points 9-10.

MODEL IV: Zero displacement vector (Dirichlet condition) is prescribed on the part 1-2 (fixed tibia), on the part 5-6 a loading $1.2 \times 10^6[Pa]$ in the case of MODEL V-A, $1.0 \times 10^6[Pa]$ in the case of MODEL V-B and $0.86 \times 10^6[Pa]$ in the case of MODEL V-C; the unilateral contact condition is between points 7-8 and the bilateral contact condition is between points 9-10; the remaining part of the boundary is unloaded.

For computation the following discretization parameters were used:

MODEL IV: 10 subdomains of domain decomposition, 2867 nodes, 4740 elements, 33 unilateral and bilateral contact nodes, 299 interface elements between subdomains of domain decomposition,

MODEL V: 8 subdomains of domain decomposition, 2474 nodes, 4104 elements, 33 unilateral contact nodes, 290 interface elements between subdomains of domain decomposition.

3 Biomechanical analysis of the axial angle changes on the weight-bearing TKR

The main goal of the paper is to compare the biomechanical influences of different grades of valgus deformity after application of the total knee replacement (TKR).

3.1 The frontal plain

The numerical results are presented in Figs 2-7 for axial deviation 5, 7 and 9 degrees. Fig. 2 characterizes the internal shift of the material points on

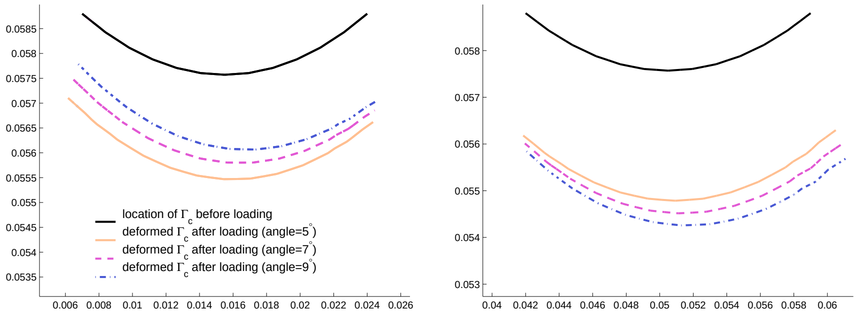

Fig. 2 The internal shift of the material on the contact boundary before and after the deformation for the medial and lateral condyle

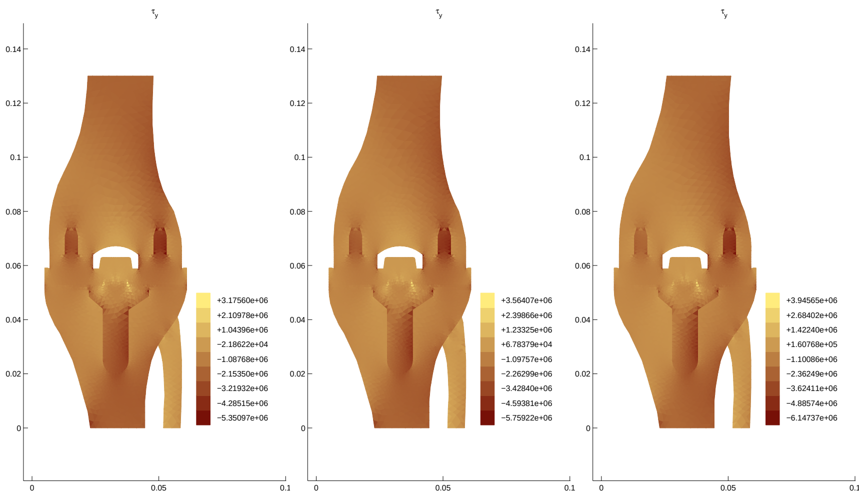

Fig. 3 The vertical component of the stress tensor for MODEL I, II, III

the contact boundary before and after the deformation for the medial and lateral condyle. The vertical and shear components of the stress tensor and the principal stresses in the considered femorotibial part of the lower limb are in Figs 3-5. Normal and tangential components of the displacement and of the stress vectors on the contact boundary between the femoral and tibial components of the knee joint in the area of both condyles are in Figs 6, 7.

3.1.1 Results of computations for 5 degree valgus A model was constructed so that the requirement of maintaining of the mechanical axis with symmetrical division of the load on the whole area. Deformity of the knee joint reveals itself mainly in the femoral part of the joint. The horizontal component of the displacement vector indicates the shifts in the area of the polyethylen insert of the external condyle in its medial part. The vertical component of the displacement reveals itself especially in the considered part of the femur,

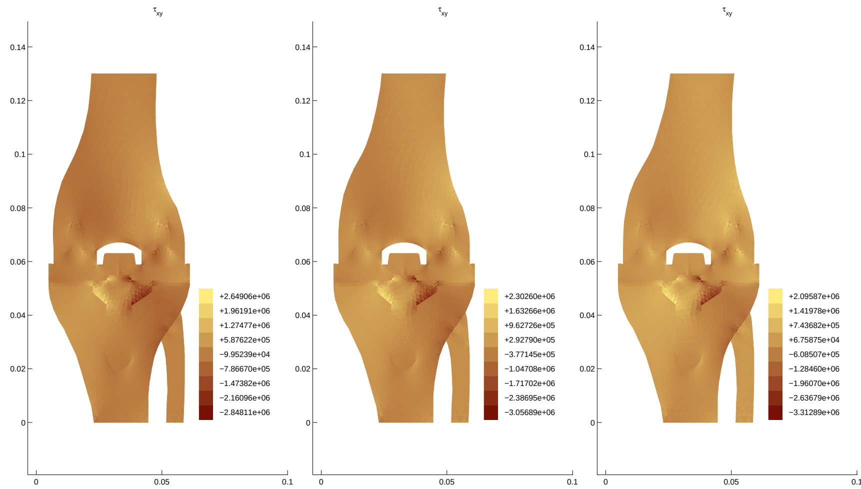

Fig. 4 The shear component of the stress tensor for MODEL I, II, III

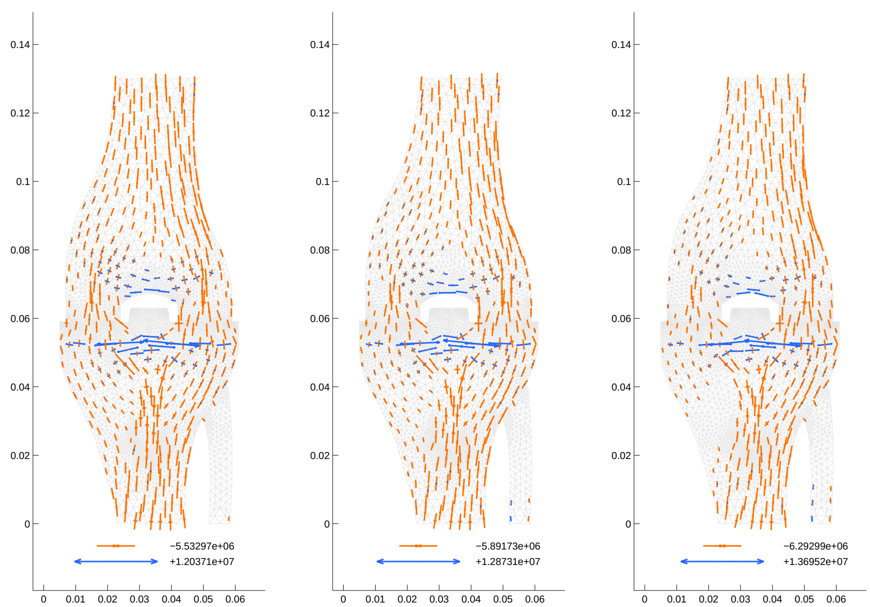

Fig. 5 The principal stresses for MODEL I, II, III

minimally in the area of the fibula. Represented in Figs 3-5 is the division of the vertical component of the stress tensor, the shear and the principle stresses in the area of the artificial replacement of the knee joint. The greatest changes in the horizontal stress component are indicated in the area of the tibial plate. From the vertical component of the stress tensor (Fig. 3) and the principle stresses (Fig. 5) it follows that the stress in the diaphysis

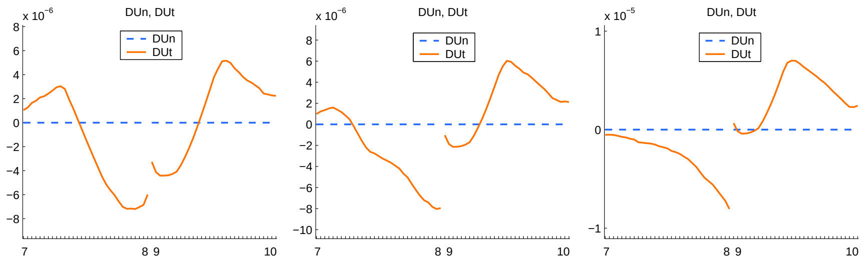

Fig. 6 Normal and tangential components of the displacement on the contact boundary in the area of both condyles for MODEL I, II, III

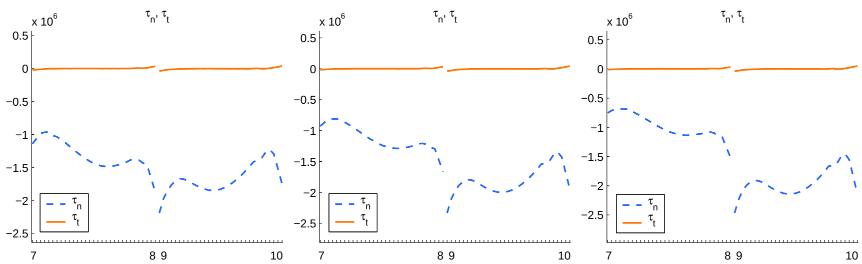

Fig. 7 Normal and tangential components of the stress vectors on the contact boundary in the area of both condyles for MODEL I, II, III

is spread evently, in the area of metaphysis it begins to be separated by the area characterized by pressure, in the area of epiphysis the pressure is transmitted over the fixing elements of the femoral component of the artificial replacement, and it is the external condyle that is more burdened and further transmitted over the polyethylen insert on the tibial plate and over the fixing element of the tibial plato onto tibia. The shear stresses (Fig. 4) and the principle stresses (Fig. 5) indicate the areas characterized by tensile stresses in the area above the incisura intercondylica and in the area of the tibial plato. From the normal and tangential components of the displacement vector on the contact it follows that both components are in tight contact and that the movement on the internal condyle grows in direction to the centre and in the external condyle the tangential component grows from the incisura intercondylica up to the maximum movement that is practically behind the centre of the condyle and then markedly falls to the external rim. The normal component of the stress vector has, on both condyles, similar character in view of the condyle symmetry and a greater load gets over the external condyle. An analysis of the normal contact stresses indicates only the pressure stresses. The tangential contact stresses are near zero.

3.1.2 Results of computations for 7 degree valgus The principle of maintaining the mechanical axis with symmetrical division of the whole surface (area) of the joint has been preserved also in this case. The symmetrical division of forces for both knee compartments is documented also by the saturation of the coloration on the pictures based on the numerical results, especially transmission of the pressure forces over both condyles in case of the principle stresses (Fig. 5). The transfer of the load with maximum in the area of the fixing femoral elements and of the stem of the tibial component is indicated in Figs 3-5. The vertical component of the stress tensor (Fig. 3) indicates a straighten spread of stresses in comparison with the preceding case, which even shear stresses indicate (Fig. 4). A balanced transmission of the load ongoing symmetrically through both parts of the joint is represented in the case of the principle stresses in Fig. 5 by the course of the pressure load. Lowering of the size of the tensile stress is indicated in the area incisura intercondylica. The character of the normal components of the displacement vector does not change, i.e. both knee joint components in an investigated case stay in a tight contact, only the character of the course of the tangential displacement component on the internal condyle changes (Fig. 6). The values of this component change as well. And also a partial straightening of the pressure load on both condyles of the knee joint is indicated. The tangential component of the stress vector changes only a little (Fig. 7).

3.1.3 Results of computations for 9 degree valgus In this model there was effort to preserve the basic condition relevant to the mechanical axis. Horizontal and vertical components of the displacement vector indicate greater changes of deformity in both areas of the polyethylen insert. The components of the stress tensor and the principle stress (Figs 3-5) indicate signs of a knee part overload and increase of transfer (transmission) of the load by an external compartment. The increase load passing through the external compartment is, in addition, emphasized in Fig. 5 by the course of the principle stresses. Numerical results show that also in this case both components of the replacement of the knee joint are in tight contact, the movement in tangential direction grows, which testifies to greater deformity of the joint replacement (Figs 6-7). Normal components of the stress vector on the contact between both femorotibial parts of the knee joint (Fig. 7) indicate greater values of loading (overload) of the external condyle.

Overloading of the posterior part of the tibial plate in the anteroposterior direction in the nonloosing of the soft posterior structures, possibly incorrect inclination of the resection of the proximal tibia is studied in MODEL IV and V. The results display certain overloading of the posterior part of the tibial plate also for the case of a 7 degree deviation, which indicates the possibility of wear of the polyethylene insert TKR.

3.1.4 Valuational remarks From the first analyses of the numerical results it is possible to estimate that the optimal partition of the forces effective on TKR in medio-lateral direction (frontal plane) corresponds to the modification in the 7 degree valgus.

For the horizontal stress components and the principle stresses outweigh the relatively small tensile stresses in the area round the external lateral and internal medial contact parts and in the area incisura intercondylica, otherwise predominantly indicated in femur are pressure stresses, especially in relatively larger distance from the contact boundary and also in tibia. The stress gradients become equal in epiphysis and metaphysis, and diaphysis is already strained evenly. For the vertical stresses predominantly the pressure stresses are observed, the area incisura intercondylica and the medial margin are relieved (reduced) and the stress grows further in the lateral direction, similar situation is observed also in the tibia. The pressure stresses is transmitted over the external lateral and internal medial parts of the contact area. In the course of a vertical component of stress, shear stresses and the principal stresses we can see that concentration of the pressure stresses is more located into the area of the external condyles of femur and tibia, the smaller part across the internal condyles, tensile stresses are located into the area incisura intercondylica.

The normal and tangential components of the displacement and stress vectors on the contact boundaries of both condyles have a testing value for the analysis of the total replacement of the knee joint in dependence on the axial deviation. From the analysis of the normal displacement component DUn we see that both components of the knee joint are constantly in a close contact. The analysis of the tangential component of the displacement vector DUt points at relatively small shifts in both condylar components of the joint, the character of their process is a little different for the studied axial deviation and we observe real mutual shifts of the contralateral points of the contact in the space. Normal and tangential stresses on the contact in both condylar parts of the knee joint characterizes loading relations on the both condylar parts of the knee joint. The pressure is observed in both condyles.

3.2 The sagittal plain

The greatest testifying values have horizontal component of the displacement vector U_x, the normal and tangential components of displacement vector DUn, DUt and normal and tangential stresses τ_n and τ_t on the contact between both components of the knee joint replacement. The contact between both parts of the joint knee replacement is between points 7-8, where the point 7 corresponds to the investigated posterior part of the polyethylene insert. Figs 8, 9 present the horizontal component of the displacement vector U_x, from which the shifts within the bounds approx. $-1.667\times10^{-5}[m]$ - $-1.748 \times 10^{-5}[m]$ - $-1.85 \times 10^{-5}[m]$ in the case of the outer condyle for the axial deviations 5, 7, 9 deg. in the area of the tibial plato and within approx. $-1.38 \times 10^{-5}[m]$ - $-1.15 \times 10^{-5}[m]$ - $-0.92 \times 10^{-5}[m]$ in the case of the inner condyle for the axial deviations 5, 7, 9 deg. in the area of the tibial plato, are observed. From these results it is seen that the polyethylen insert for all axial deviations is press out in the posterior part of the tibial plato, with greater values for the outer condyle than for inner condyle, and that the polyethylene insert is more deformed and worn out.

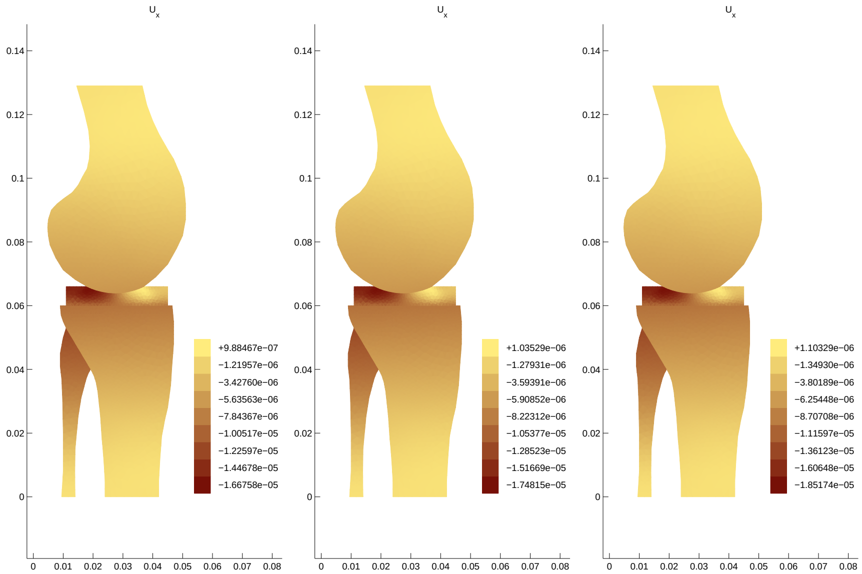

Fig. 8 The horizontal component of the displacement for MODEL IV - A,B,C

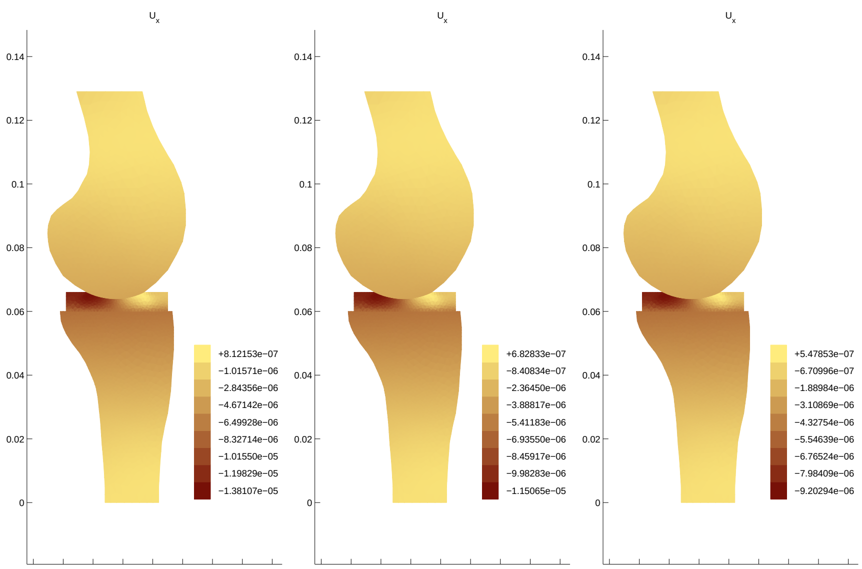

Fig. 9 The horizontal component of the displacement for MODEL V - A,B,C

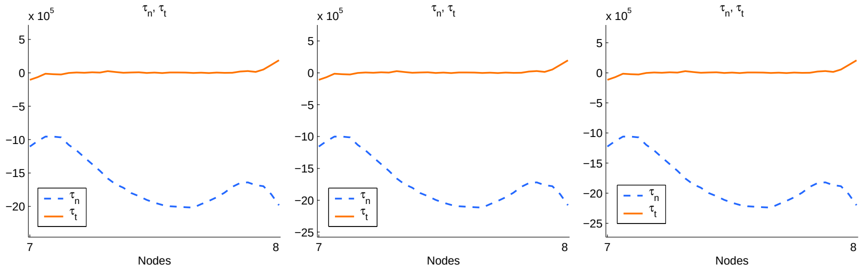

Fig. 10 Normal and tangential components of the stress vectors on the contact boundary for MODEL IV - A,B,C

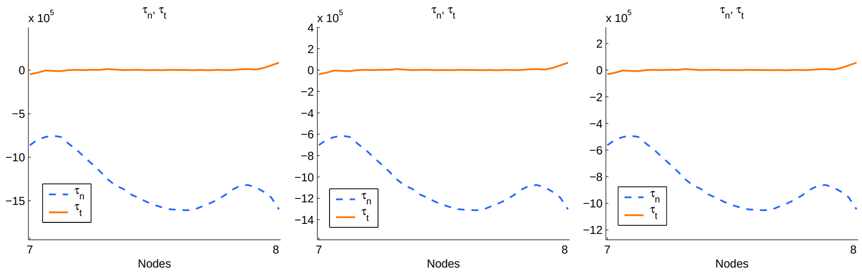

Fig. 11 Normal and tangential components of the stress vectors on the contact boundary for MODEL V - A,B,C

Figs 10, 11 present the normal and tangential contact stresses τ_n and τ_t. Tangential contact stresses are approx. equal to zero, the normal contact stresses on the outer condyle have the values within the bounds approx. $-9.5 \times 10^5 [Pa]$ - $-10. \times 10^5 [Pa]$ - $-10.5 \times 10^5 [Pa]$, and on the inner condyle within the bounds approx. $-6.5 \times 10^5 [Pa]$ - $-6.0 \times 10^5 [Pa]$ - $-5.0 \times 10^5 [Pa]$, in the vicinity of the posterior part of the tibial plato and have the character of pressures, which increase and the next minimum has in the closeness of the front part of the tibial plato. The certain overloading of the posterior part of tibial plato are observed, which leads to the possibility of worn out and resulting wear of polyethylene insert of TKR.

4 Conclusions

Analysis of numerical results deduce optimal distribution of forces operated on total knee replacement in anteroposterior direction and well-balanced transition of forces in anteroposterior direction corresponding to the 7 degree case. Optimal transfer of forces in anteroposterior direction with maximum in the place of amplification of the posterior corticalis of the femur, fixation elements of femoral and/or of tibial stem component is in a good agreement observed in practice. Analysis of numerical results of computations across the lateral and medial condyle in the sagittal plane shows onto certain overloading of the posterior part of the tibial plato, which suggest the possibility of worn out and resulting wear of polyethylene insert of TKR, which is also in a good agreement with observations in orthopaedic surgery.

Acknowledgements The authors was supported by the grant of the Ministry of Industry and Trade of the Czech Republic FT-TA/087.

References

1. Daněk, J., Denk, F., Hlaváček, I., Nedoma, J., Stehlík, J., Vavřík, P. On the stress-strain analysis of the knee replacement, ICCSA'2004, Lecture Notes in Computer Science **3044**, (2004), 456–466.
2. Daněk, J., Hlaváček, I., Nedoma, J. Domain decomposition for generalized unilateral semi-coercive contact problem with given friction in elasticity. Mathematics and Computers in Simulation **68**, (2005), 271–300.
3. Daněk, J., Stehlík, J., Vavřík, P., Nedoma, J., Hlaváček, I., Denk, F. On the effect of axial angle changes on the weight-bearing total knee replacement. In: Proc. 17th IMACS Congress, Session on Mathematical Biomechanics, (2005), Paris.
4. Mann, K.A, Bartel, D.L., Wright, M., Burstein, A.H. Coulomb frictional interfaces in modelling cemented total hip replacements: a more realistic model. J. of Biomechanics **28**(9), (1995), 1067–1078.
5. Nedoma, J. Mathematical Modelling in Biomechanics. Bone- and Vascular-Implant Systems. Habilitation thesis. ICS AS CR, Univ. of West Bohemia, (1993), Prague.
6. Nedoma, J., Bartoš, M., Kestřánek sen, Z, Kestřánek jr, Z., Stehlík, J. Numerical methods for constrained optimization in 2D and 3D biomechanics. Numer. Linear Algebra Appl. **6**, (1999), 557–586.
7. Nedoma, J., Hlaváček, I. On a solution of a semi-coercive contact problem in a non-linear thermo-elastic rheology. Math. Comput. in Simulation **60**, (2002), 117–127.
8. Nedoma, J., Hlaváček, I. , Daněk, J., Vavřík, P., Stehlík, J., Denk, F. Some recent results on a domain decomposition method in biomechanics of human joints. ICCSA'2003, Lecture Notes in Computer Science **2667**, (2003), 587–600.
9. Nedoma, J, Stehlík, J., Bartoš, M., Denk, F., Džupa, V., Fousek, J., Hlaváček, I., Klézl, Z., Květ, I. Biomechanics of Human Joints and Their Replacements. (in Czech , in print).
10. Périé, D., Hobartho, M.C. In vivo determination of contact areas and pressure of the femorotibial joint using non-linear finite element analysis. Classical Biomechanics **13**, (1998), 394–402.
11. Rojek, J., Telega, J.J., Stupkiewicz, S. Contact problems with friction, adhesion, and wear in orthopaedic biomechanics I-II. J.Theor. Appl. Mech. **39**, (2001), 655–677, 679–706.
12. Stehlík, J., Nedoma, J. Mathematical simulation of the function of great human joints and optimal design of their artificial replacements. Technical Report. V-406/407. ÚI AV ČR, (1989), Praha.
13. Vavřík, P., Denk, F., Nedoma, J., Daněk, J., Hlaváček, I. Numerical analysis of the weight-bearing total knee replacement. Realization in pracice - WALTER UNIVERSAL (WU) and WALTER MODULAR (WM). In: Proc. 17th IMACS Congress, Session on Mathematical Biomechanics, (2005), Paris,