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2005

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Technical report No. 952

December 2005

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# Generalizations and Extensions of Lattice-Valued Possibilistic Measures, Part I[1]

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Abstract:

Since the Zadeh's paper [9], which appeared in 1978, possibilistic (or possiblility) measures have been developed into an interesting and worth being investigated alternative tool to quantify and process uncertainty of the kind of randomness, i.e., of the uncertainty classically quantified and processed by the apparata offered by probability theory and mathematical statistics. In spite of the fact that it is the operation of additivity and $\sigma$-additivity which plays the dominating role when processing probability measures and their values, the principal operations applied in the theory of possibilistic measures are those of supremum and infimum, which can be defined also in mathematical structures built over sets of non-numerical elements, in particular, in partially ordered sets and in complete lattices. Consequently, possibilistic measures with non-numerical values and, in particular, lattice-valued possibilistic measures can be defined, seem to be worth being investigated in more detail, and also this work is devoted to this goal. Our attention will be focused to various extensions and generalizations of lattice-valued possibilistic measures and mutual relations among them. This research report presents the first part of a longer work dealing with the problems introduced above, the second part is expected to follow the next year.

Keywords:
$\sigma$-field of sets, ample field, partially ordered set, (complete) lattice, monotone measure, real-valued possibilistic measure, lattice-valued possibilisitc measure, partial possibilistic measure, inner (lower) and outer (upper) possibilistic measures, measurability in the Lebesgue sense, approximations of lattice-valued possibilistic measures, completions of partial lattice-valued possibilistic measures

[1]This research was partially supported by the grant No. IAA100300503 of GA AS CR and by the Institutional Research Plan AVOZ10300504 "Computer Science for the Information Society: Models, Algorithms, Application"

# 1 The Phenomenon of Uncertainty — from a Non-Reflected Matter of Evidence to a Challenge

During the archaic times, for each human being just a very close and narrow horizon restricted the part of the world around which one was able to observe, to understand, to reflect and to answer reasonably the challenges coming from. As a matter of fact, the space inside this horizon was just that of immediate and elementary emipirical evidence offered by human senses and only very slowly and in a very small degree extended and generalized due to one's daily repeated experience. So, uncertainty was nothing else than an implicit common attribute intrinsically penetrating all phenomena coming from beyond one's horizon and influencing, if not directly attacking, one's life and often even menacing one's own ;physical existence. In the enemial world beyond the horizon everything was possible and nothing was certain, everything could happen and, having happened, could penetrate the horizon and could emerge in the world surrounded by this horizon. In other words, for an archaic human being every day was a sequence of surprises, much more often unpleasant or even dangerous than pleasant and agreable ones.

Nevertheless, with the ages going on, individual experience combined and shared with the experience of parents, ancestors and other members of the tribe in general, enabled to distinguish, step by step, the degrees of levels of surprising or, to introduce the term which will be perhaps the most important and frequent one in what follows, the degrees of uncertainty related to various phenomena occurring in one's everyday life. So, the events offering periodical daily and seasonal changes, the rise and the decline of the Sun, the cycle of Moon's phases, the regular changes of year's seasons, have been rather soon taken as regularities the repeating of which is reasonable to expect contrary to, say, the changes of the weather which are charged by a non-negligible portion of uncertainty still in our days. Continually improving manual skillfulness, as well as the tools used, one was able to reduce the degree of uncertainty related to various operations applied when hunting, farming or preparing the food. And again, with hundreds and thousands of years going on, this rather vague and relative classification of various phenomena according to the degrees of uncertainty connected with them, step by step shifted towards their qualitative distinguishing — some phenomena were taken as regular or governed by regularity (later on called the Laws of God(s) and still later the Laws of Nature). The other phenomena remained to be taken as uncertain or random, being then called rather events than phenomena. In what follows, we will also take profit of this linguistic difference, using the term "events" for phenomena charged by a non-negligible portion of uncertainty to the quantification and processing of which all our further effort will be focused.

As a matter of fact, we have to go as back as to those pre-historic times, if we want to find and discover when and why both the groups to which the phenomena under consideration were separated taking their evident classification on the axis ranging from "well, good, desirable, ..." to "bad, awful, undesirable, ..." with regularity, certainty, law, ... taken as positive and desirable attributes and uncertainty, irregularity, randomness, ... as negative ones. The reason seems to be very simple: the cases when a pre-historic man or woman could predict and with a degree of certainty high enough expect the results of his/her action were taken by them as much more favorable when compared with possible uncertain results charged by randomness which could bring much more often a danger, loss or disaster, in common, something bad and undesirable, than an unexpected profit or advantage. Indeed, the human being under consideration was very far from the position of a modern age or even contemporary hazard game player who hopes to win a great amount of money conditioned by a risk to loss a small amount given as the stake. And there were just the scientists of 20th century who discovered that processes and procedures purposely charged by uncertainty may offer qualitatively better results than deterministic procedures even when solving problems having in their nature nothing in common with uncertainty and randomness (let us recall probabilistic algorithms or simulated annealing). Turning back to the idea of individual horizon limiting the position of every human being in the world around, much more often enemial than friendly, we could say what any regularity, certainty or causality which one became acquainted extended, in a sense, the space in front of one's horizon incorporating into this space also the consequences of one's activities made according to the discovered laws, regularities and dependences. It is why, till our days, as positive results and achievements of scientific effort are taken just the pieces of knowledge expressing some causalities, dependences and implications like "it is true that if A holds then B holds as well", not the negative results like "it is not true that...", even if it is in spite of the formal emancipation of negative statements as true assertions concerning the negation of the original assertion, as done by Aristotle. Let us recall the strange and extraordinary position of the deepest, but in the negative form expressed results of the antigue geometry: a common unit of length for side and diagonal of a square does not exist, it is impossible to construct a square equal to a given circle in its size, trisection of angle is impossible, .... Many antigue mathematicians and philosophers could not prevent themselves from a non-negligible feelings of fear when considering and analyzing such results.

For the reasons very briefly sketched above, the phenomenon of uncertainty was taken as something undesirable or even enemial in all fields of human activity and the aim was to suppress or as least to restrict in the most possible degree. However, this degree was very different even in the antigue times, and this fact has influenced the reasoning on uncertainty till our days. Among the branches proving remarkably great successes when fighting again uncertainty two very important were celestial mechanics and linguistic argumentation — the art of speech and discussion. As already mentioned the regularities governing the moves of Sun and Moon without any reference to uncertainty were known at the very rise of the human civilization and the oldest models, sometimes very sophisticated, aiming to explain and to predict at the same level of certainty all what happens at the heavens are also several thousand year old. Also the antigue Greek schools of the art of discourse aiming to eliminate the phenomenon of uncertainty from the process of human reasoning are well known and their successes are quoted and acknowledged still in our days. The more important were these fields of reasoning, the stronger the remaining portion of uncertainty was felt as a challenge for further effort of the most excellent philosophers and scientists to eliminate completely the uncertainty from these domains. As far as the reasoning and argumentation is concerned, remember the famous Socrates' dialogues (written by Plato) and dealing with problems when uncertainty emerges when using the common non-formalized language in everyday human communication. For the case of celestial mechanics the problem with explication and, consequently, prediction of solar and lunar eclipses or with the moves of planets on the heavens are well known (let us recall that in Greek "planet" denotes a "wandering" or rather "erring" celestial body). In other terms: the leading spirits of human civilization stood at the position that the domains like reasonable and rational argumentation or celestial mechanics were governed by laws working beyond any impact of uncertainty and can be also described and used without any reference to this undesirable phenomenon; it was just a matter of human effort to seek for and to reach such an explanation, description and application.

Both these domains of uncertainty can be taken as good examples of two qualitatively different phenomena, but just the second one will be presented and analyzed, in more detain, in what follows. The first case is that dealing with vagueness, impreciseness, lack of crisp notions, in common and using the modern term, with fuzziness. Many notions used in common language admit uncertainty, doubts and hesitation, when applied to particular cases, hence, when still using them, we do so with a non-negligible portion of subjectivity which cannot be fully avoided in principle: remember the famous Plato's dialogues in which Socrates analyses adjectives like "good" or substantives like "justice", so arriving at very serious and unsolvable problems. Another well-known and philosophically very deep problem of this kind is that of the Sorites' paradox of heap: a single corn is not a heap and joining or removing a corn does not change "non-heap" to heap or vice versa. Perhaps the best known and the most widely spread formal mathematical apparatus to describe and process this kind of uncertainty is that of fuzzy sets, based on the idea that a given individual entity (element) can belong to the given set only partially, to some degree. Applying this idea to the membership relation between a particular sentence of a formalized language and the set of true (deducible, resp.) sentences we arrive at the so called fuzzy logics of various kinds and types. However, as stated already above, we purposely leave vagueness, fuzziness, lack of crispness and related issues of uncertainty beyond the horizon of our further reasoning focusing our attention, in what follows, to another type of uncertainty, perhaps covered by the notion of randomness.

## 2 From Uncertainty as Randomness to Uncertainty as Measure

The kind of uncertainty to be described and analyzed in this chapter consists in the lack of the subject's ability to predict whether a phenomenon will occur or has occurred before immediately observing it. However, at the level of observation, no uncertainty, as far as the interpretation and identification of the observation is concerned, is supposed to enter the scene. E.g., a pre-historic man was not able to predict the eclipse of Sun or Moon, but when observing the Sun or Moon disc he could see and say whether this disc is or is not eclipsed (covered by the shadow) at the moment. The same is the case when tossing a dice in such a way that the resulting side is covered by a hand or a cup. Before removing the hand or cup we are not able to forecast the resulting number, so that the result is uncertain in this case, but when doing so, we can see and read the result, the number of points on the upper side of the dice, without any uncertainty in the sense of ambiguity, vagueness, or lack of crispness. In textbooks and monographs on probability theory and mathematical statistics great number of examples of various nature illustrating this kind of uncertainty can be found and, as a rule, this phenomenon is called randomness and the events governed by such uncertainty are called random events.

Very often, coin tossing is taken as the most simple example of a procedure influenced by randomness. Let us consider another situation when coin and uncertainty are related to each other. During an archeological research, an antigue or medieval coin has been dug out, but it is damaged by erosion or other impacts, so that neither the specialists are able to say, surely and without any doubts, which coin (from a list of known types of coins listed in a catalogue, say) it is and neither which is its head and its tail side. This uncertainty can be taken in two qualitatively different senses. First, we may assume that the dug out coin is, as a matter of fact, a copy of just one particular coin from the catalogue so that one of its sides is, at least from the Platonist (God-like) point of view, its head, the other being its tail. However, being hidden in the soil and subjected to various influences, the coin passed something like the tossing process and we do not know which side is the head and which is the tail. The difference when compared with the coin tossing consists in the fact that, to be able to observe and identify the result, much more difficult operations than a simple removing the hand or the cup are necessary. Sometimes these operations are possible and the "tossed" side of the coin can be identified (a more detailed investigation of the dug coin by microscope or other technical tool, consultation with a more skilled and experiences specialist, ... sometimes it is not possible (the "cup" covering the "tossed" coin is too big and hard to be removed by the individual powers of the experimenting subject). In every case, however, under this interpretation the uncertainty is caused by the limited subject's cognitive and empirical powers, the reality "as such" remains to be crisp and definite.

The other approach to the phenomenon of uncertainty related to the dug out coin in question can be illustrated from the position of a collector of, or businessman dealing with, old coins. From this point of view the dug out damaged coin is not a particular coin from the list as such, it is some other object which can be taken as a copy of a coin from this list only to a certain degree. Consequently, a side of this dug out coin is not the head (the tail, resp.), it is some new entity which is related to the head (the tail, resp.) of a coin from the list in the catalogue only by similarity relation, it is the head (the tail, resp.) of a particular coin from the list only to a certain degree. This "degree of similarity" of the dug out object to a coin from the list or, to take profit of the terms of fuzzy sets, the "degree of membership" of this object to the fuzzy set of coins of a particular type, can be, at least under some circumstances, expressed even numerically, e.g., by the relation of the amount which a collector is to pay (or a seller can obtain) for the dug out "coin-like object" related to the financial value of the perfect (non-damaged, crisp) coin of the type in question. The same is the basic idea of the laws and bank rules according to which damaged banknotes are changed for the perfect ones. E.g., if the relative size of the damaged, destroyed, or even missing area of a particular banknote does not reach 25%, the banknote is taken as fully valid, if the relative size of this area reaches or exceeds 75%, the banknote is totally invalid, and in the remaining cases the degree of validity of the damaged banknote (the degree of membership of this piece of paper to the fuzzy class of banknotes of the type in question), and the amount payed by the bank for this damaged copy decreases linearly from 100%

to 0% of its full value with the relative non-damaged size of the banknote in question decreasing from 75% to 25%.

A more systematic and mathematically based scientific investigation and analysis of the phenomenon of uncertainty in the sense of randomness originated in 18th century and was motivated by hazard games like coin or dice tossing, sampling at random from urns containing black and white (and perhaps still other colours possessing) balls in various proportions and sampled according to various, more or less complicated rules. Recall the most simple cases of sampling with of without removing. the sampled ball into the urn, or more sophisticated urn schemata with hierarchically ordered systems of many urns with consecutive sampling. This historical background of the mathematical probability theory, at least in its elementary combinatorial setting, is still conserved in the terminology used till our days. Also the names of the most famous mathematicians building the foundations of mathematical probability theory (Laplace, de Moivre, d'Alembert, Poisson, Gauss, ...) are still explicitly demonstrated in the names of the most important notions, theorems and other results achieved when developing probability theory as a fully valuable and precisely formalized branch of theoretical mathematics. The readers interested in the history, but primary in the philosophical and methodological problems emerging with the times passing, are recommended to consult, e.g. [18] or [22] as interesting sources of further information and details.

From the point of view of further mathematical formalization and possible semantigue interpretation of the phenomenon of uncertainty in the sense of randomness, leading to the well-known axiomatic setting of probability theory as done by Kolmogorov in 1932 [28], the most important notion offered by the combinatorial probability theory was that of elementary random event. Elementary random events are the atoms of the field of random events which may occur when realizing the random sampling procedure under consideration. Hence, as the result of each sample just one elementary random event occurs and each non-elementary random event is fully and completely defined by a subset of the set of all elementary random events under consideration. E.g., in the case of dice tossing the elementary random events are the particular positive integers from 1 to 6, so that every non-elementary random event, e.g., "number greater than 4", "odd number", and so on are uniquely defined by the corresponding subsets of the set $\{1,2,\ldots,6\}$ of integers. If the set elementary random events is finite or countable, and if the probability values ascribed to particular elementary random events are non-negative and their sum (or series) over all elementary random events gives one, the values of probabilities for each subset of the set of all elementary random events, hence, also for each random event, can be easily and uniquely defined taking the sum (the series, resp.) of probability values ascribed to all elementary random events contained in the random event in question (in the terms of probability theory, these elementary random events are called favorable to the random event in question).

Let us denote by $\Omega$ the set of all possible elementary random events which can result when realizing the random sample in question, let $\omega$ denote particular elementary random events, i.e., elements of $\Omega$, let $\mathcal{P}(\Omega)$ denote the system of all subsets of $\Omega$ (the power-set over $\Omega$). The most important feature of this model consists in the fact that given the elementary random event $\omega_0$ resulting from a random sample, the occurrence or non-occurrence of any random event $A \subset \Omega$ is completely determined by this value $\omega_0 - A$ occurs iff $\omega_0 \in A$ holds (let us recall that we take into consideration only classical crisp subsets of $\Omega$ with no fuzziness entering the scene). E.g., in the case of dice tossing, when $\Omega = \{1, 2, \ldots, 6\}$ if $\omega_0 = 4$ then the random event "even number" occurs, the random event "number smaller than 3" does not occur, the random event "number greater than 2" occurs, etc. Hence, from the formalized mathematical point of view, elementary random events play the role of hidden parameters in models defined by functional deterministic dependences, in other terms, our model enables to reduce randomness to the lack of full (complete, precise) knowledge as far as the value of the hidden parameter is concerned. From the philosophical point of view, this approach to uncertainty and randomness presents the mechanistic and deterministic point of view, inspired by the great successes of Newtonian physics and astronomy in 18th century and introduced into probability theory by Laplace [17], [18], [22]. From the methodological point of view, however, this approach enables to embed probability theory into the standard theoretical mathematics as a particular application of the theory of real-valued functions and of measure theory. This paradigm is applied also in the Kolmogorov axiomatic probability theory [38] and will be accepted also here, hence, randomness is nothing else than our lack of complete knowledge of the actual values of hidden parameters in processes described by functional dependences.

Having agreed with the idea to restrict our reasoning to the cases when random event can be identified with the set of "favourable" values of a hidden parameter within the framework of a functional model of the problem and environment under investigation, let us declare explicitly also our second basic paradigmatic assumption accepted in all what follows below. This assumption reads that the degree or amount of uncertainty (i.e., randomness in our case) contained in, or ascribed to, the random event in question is defined by a mapping which ascribes a value to the subset of $\Omega$ identified with this random event (hence, defined by a set function with the definition domain contained in the power-set $\mathcal{P}(\Omega)$ but nor, in general, identical with $\mathcal{P}(\Omega)$). The values taken by his set function need not be, in general, of numerical nature (as a matter of fact, the greatest portion of our attention, in what follows, will be focused to uncertainty degrees with non-numerical values), but the mappings under consideration will be supposed to posses at least the perhaps most general property of the intuitive notion of size of geometric objects and, in particular, sets, and it is the property of monotonicity with respect to set inclusion. Hence, if $\varphi$ is a mapping ascribing to random events over the system $\Omega$ of elementary random events, i.e., to (some) subsets of $\Omega$, degrees of randomness in a space $T$ of possible values equipped by a partial binary relation $\leq$ admitting to understand the relation $t_1 \leq t_2,\ t_1, t_2 \in T$, as "$t_1$ is smaller than or equal to $t_2$" (e.g., if $\leq$ defines a partial ordering in $T$), then for every $A, B \subset \Omega$ such that $A \subset B$ holds and $\varphi$ is defined for both $A, B$, the relation $\varphi(A) \leq \varphi(B)$ should hold true. Such uncertainty degrees are often called *$\mathcal{T}$-valued monotone measures*, where $\mathcal{T} = \langle T, \leq \rangle$, and in all what follows we will limit ourselves to investigation of uncertainty taken as randomness and investigated within the framework in which such an uncertainty can be defined and processed as $\mathcal{T}$-valued monotone measure for appropriate, real-valued or non-numerical, set $T$ of partially ordered uncertainty (randomness) degrees.

Let us note that perhaps the notion "fuzzy measure" is used more often than "monotone measure", but the adjective "fuzzy" in this case does not meet its standard meaning and intuition behind as it is in the case of fuzzy sets, so that the adjective "monotone" will be preferred below.

# 3 Additive and Maxitive Real-Valued Monotone Measures

Having agreed with the idea to quantify degrees of uncertainty (in the sense of randomness) by sizes of certain subsets of the universe $\Omega$ of elementary random events, let us note that the common intuition behind the notion of size of geometric objects was sketched, for the first time, by Euclides [43] in his well-known eight principles for size. Let us introduce them, for the reader's convenience, also here, not trying to translate them verbally, but rather to catch the spirit of each of these principles.

1. Two (values of) sizes being equal to a third size, also these two values are equal to each other.
2. If equal sizes are joined with equal sizes, also the resulting sizes are equal to each other.
3. If equal sizes are removed from equal sizes, also the remaining sizes are equal to each other.
4. If equal sizes are joined with inequal sizes, also the resulting sizes are not equal to each other.
5. Two sizes being equal to each other and multiplying both of them twice, the resulting sizes are also equal to each other.
6. Two sizes being equal to each other and dividing both of them to their halves, also the resulting sizes are equal to each other.
7. If two geometric objects mutually cover each other, their sizes are equal.
8. The whole object is greater than its part.

Hence, using the terms to be introduceds below in this chapter, we could say that the Euclides' idea of size of geometrical objects meets the demands of monotone measure, shifting the notion of size toward the additive measures, but this shift is far from being done consecutively. So, wanting to begin our more formalized reasoning on uncertainty and, in particular, randomness quantification and processing with real-valued uncertainty degrees, we introduce, first of all, the notion of real-valued monotone measure.

**Definition 3.1** *Let $\Omega$ be a nonempty set, let $\mathcal{A}$ be a system of subsets of $\Omega$. A mapping $\varphi$ which takes $\mathcal{A}$ into the unit interval $[0,1]$ of real numbers ($\varphi : \mathcal{A} \to [0,1]$, in symbols) is called* (real-valued) monotone measure *on $\mathcal{A}$, if the inequality $\varphi(A) \leq \varphi(B)$ holds for each $A, B \in \mathcal{A}$ such that $A \subset B$. The* inner measure *$\varphi_\star$ and the* outer measure *$\varphi^\star$ induced by the monotone measure $\varphi$ on the power-set $\mathcal{P}(\Omega)$ are defined by*

$$\varphi_\star(B) = \bigvee \{\varphi(A) : A \subset B, A \in \mathcal{A}\} \tag{3.1}$$

$$\varphi^\star(B) = \bigwedge \{\varphi(A) : A \supset B, A \in \mathcal{A}\} \tag{3.2}$$

*for each $B \subset \Omega$, here $\bigvee(\bigwedge$, resp.), denotes the standard supremum (infimum, resp.) in $[0,1]$ with respect to the linear ordering $\leq$ of real numbers, and the conventions $\bigvee \emptyset = 0, \bigwedge \emptyset = 1$ apply for the empty subset of $[0,1]$. The monotone measure $\varphi$ on $\mathcal{A}$ is called* normalized*, if $\varphi(\emptyset) = 0$ and/or $\varphi(\Omega) = 1$ supposing that $\emptyset \in \mathcal{A}$ and/or $\Omega \in \mathcal{A}$.*

Let us note that this definition is (purposely) conceived at a very general level to that, e.g., if $\mathcal{A}$ contains only nonepmty proper subsets of $\Omega$ no pair of them being in the relation of set inclusion, then *each* mapping $\varphi : \mathcal{A} \to [0,1]$ is a normalized real-valued monotone measure on $\mathcal{A}$. Even the case $\mathcal{A} = \emptyset$ is not excluded, so that the "mapping" with the empty domain is also a normalized real-valued monotone measure (due to the conventions concerning $\Lambda\emptyset$ and $\bigvee \emptyset$ both $\varphi_\star$ and $\varphi^\star$ are defined for every $B \subset \Omega, \varphi_\star(B) = 0$ and $\varphi^\star(B) = 1$ for each such $B$). As can be easily seen, if $\varphi$ is a monotone measure on $\mathcal{A}$, then both $\varphi_\star$ and $\varphi^\star$ are monotone measures conservatively extending $\varphi$ from $\mathcal{A}$ to $\mathcal{P}(\Omega)($ i.e., $\varphi_\star(A) = \varphi(A) = \varphi^\star(A)$ for every $A \in \mathcal{A}$). If $\{\emptyset, \Omega\} \subset \mathcal{A}$ holds and $\varphi$ is a normalized real-valued monotone measure on $\mathcal{A}$, then both $\varphi_\star(A)$ and $\varphi^\star(A)$ are normalized real-valued monotone measures on $\mathcal{P}(\Omega)$.

We will not go in more detail, here and now, as far as real-valued monotone measures are concerned. Later on, we introduce a more general notion of monotone measures taking their values in (complete, as a rule) lattices. As the interval [0,1] with its standard linear ordering obviously meets the conditions imposed on complete lattices, our more general results achieved below will cover also real-valued monotone measures, these later ones being picked up and investigated in particular in all cases when such an effort will be felt as reasonable and useful.

The size of field used for food plantation can be quantitatively measured by the quantity of corn harvested every year and the size of an army can be measured by the number of soldiers. On the other side, however, the size of field can be defined either by the qualitative fact whether there is a source of water on this field or by the quantitative characteristic giving the distance of the field from the nearest water source or flow. Analogously, the quality of an army can be given by the individual personal qualities of the best soldier of this army which could represent it in the case when a respective battle is replaced by the personal combat of the representatives of the enemial armies (remember Achilleus and Hector before the walls of Troya).

As can be easily seen when re-considering the Euclidean postulates concerning the sizes of geometric objects and sketched above, the first interpretation of (or intuition behind) the notion of sizes of sets prevailed and led to the notion of additive measure. Let us introduce here the definition of additive measure in such a way that not only values in [0,1], but all non-negative values including the joined value $+\infty$ are admitted, on the other side, we simplify our definition in such a way that the system $\mathcal{A}$ of subsets of $\Omega$ for which the measure is defined is supposed to be appropriately structured.

**Definition 3.2** *Let $\Omega$ be a nonempty set. A nonempty system $\mathcal{A}$ of subsets of $\Omega$ is called a* field*, if for each $A, B \in \mathcal{A}$ also $\Omega - A$ and $A \cup B$ are sets from $\mathcal{A}$ (consequently, due to de Morgan rules, also $A \cap B = \Omega - ((\Omega - A) \cup (\Omega - B))$ belongs to $\mathcal{A}$). The system $\mathcal{A}$ is called a $\sigma$-*field*, if for each $A, A_1, A_2, \ldots \in \mathcal{A}$ also $\Omega - A$ and $\bigcup_{i=1}^{\infty} A_i$ (hence, also $\bigcap_{i=1}^{\infty} A_i$) are sets from $\mathcal{A}$. The system $\mathcal{A}$ is called an* ample field*, if for each $A \in \mathcal{A}$ and each nonempty $\mathcal{A}_0 \subset \mathcal{A}$ also $\Omega - A$ and $\bigcup \mathcal{A}_0 (= \bigcup_{A \in \mathcal{A}_0} A)$ (hence, also $\bigcap \mathcal{A}_0 = \bigcap_{A \in \mathcal{A}_0} A$) are sets from $\mathcal{A}$.*

Obviously, each field ($\sigma$-field, ample field) $\mathcal{A}$ contains the empty set $\emptyset$ and the space $\Omega$. Indeed, take no matter which $A \in \mathcal{A}$, so that $\Omega - A$ is in $\mathcal{A}$ and $A \cup (\Omega - A) (\bigcup_{i=1}^{\infty} A_i, A_1 = A, A_i = \Omega - A$ for all $i \geq 2$, $\bigcup \mathcal{A}_0$ for $\mathcal{A}_0 = \{A, \Omega - A\}) = \Omega$ is also in $\mathcal{A}$, the same being valid for $\emptyset = \Omega - \Omega$.

Fields, $\sigma$-fields and ample fields will be the most often used systems of subsets of the universe of discourse (the set of all elementary random events under consideration, to recall our primary motivation) $\Omega$ in what follows.

**Definition 3.3** *Let $\Omega$ be a nonempty set, let $\mathcal{A}$ be a field of subsets of $\Omega$. A mapping $\mu : \mathcal{A} \to [0, \infty] = [0, \infty] \cup \{\infty\}$ is a* (finitely) additive measure *on $\mathcal{A}$, if $\mu(\emptyset) = 0$ and*

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{3.3}$$

*hold for each $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$ (usual conventions applied when operating with the value $\infty$ are adopted). If $\mathcal{A}$ is a $\sigma$-field and the relations $\mu(\emptyset) = 0$,*

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) \tag{3.4}$$

*for each sequence $A_1, A_2, \ldots$ of mutually disjoint sets from $\mathcal{A}$ are valid (in the sense that for each such sequence the series on the right-hand side of (3.4) is defined and the identity (3.4) holds), then $\mu$ is a $\sigma$*-additive measure *on $\mathcal{A}$.*

Let us note that if there is at least one nonempty $A \in \mathcal{A}$ such that $\mu(A)$ is positive but finite, the condition $\mu(\emptyset) = 0$ is superfluous. Indeed, $A \cap \emptyset = \emptyset$, so that $\mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset) = \mu(A)$ holds due to (3.3) so that $\mu(\emptyset) = 0$ follows.

Finally, (*finitely*) *additive* ($\sigma$-*additive*, resp.) *probability measure* on $\mathcal{A}$ is a normalized additive ($\sigma$-additive, resp.) measure on $\mathcal{A}$, i.e., measure for which $\mu(\Omega) = 1$. Let us refer to the excellent monograph [20] on measure theory, perfectly and in more detail analyzing all the notions and relations concerning measure theory.

Probability measure and probability theory will not be the main tool for uncertainty (randomness) quantification and processing to which our attention in what follows will be focused, but in every case it will be the tool with which our alternative, possibilistic and lattice-valued measures will be thoroughly compared and confronted. It is why we take as useful to introduce the definition of probability measure explicitly.

**Definition 3.4** *Let $\Omega$ be a nonempty set, let $\mathcal{A}$ be a nonempty $\sigma$-field of subsets of $\Omega$. A mapping $P : \mathcal{A} \to [0, 1]$ is called a $\sigma$*-additive probability measure *on $\mathcal{A}$, if $P(\Omega) = 1$ and*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \tag{3.5}$$

*holds for each sequence $A_1, A_2, \ldots$ of mutually disjoint sets $A_i$ from $\mathcal{A}$. As in the case of $\sigma$-additive measure (Definition 3.3), (3.5) is to say that for all sequence $A_1, A_2, \ldots$ of mutually disjoint sets from $\mathcal{A}$ the series $\sum_{i=1}^{\infty} P(A_i)$ is defined and the equality in question holds. The triple $\langle \Omega, \mathcal{A}, P \rangle$ is called* probability space.

Obviously, each probability measure $P$ on a $\sigma$-field $\mathcal{A} \subset \mathcal{P}(\Omega)$ is a normalized monotone measure on $\mathcal{A}$. Indeed, if $A, B \in \mathcal{A}$ are such that $A \subset B$ holds, then $B - A = B \cap (\Omega - A)$ is in $\mathcal{A}$ and $A \cap (B - A) = \emptyset$, so that $P(B) = P(A) + P(B - A) \geq P(A)$ holds.

Set functions, perhaps obeying the conditions imposed on monotone measures but, in general, non-additive in the sense of finite additivity or $\sigma$-additivity introduced above, are investigated under the common name "non-additive measures" (cf. [12], e.g.). The latest and rather artificially created adjective "maxitive" defines the subclass of non-additive measures for which the values ascribed to particular sets are bound with the value ascribed to the union of the sets in question rather by the operation of maximum or supremum (no matter whether these sets are mutually disjoint or not) than by that of addition or series (with the condition of mutual disjointness imposed). Postponing some reasoning about the origins and appropriateness of the adjective till an appropriate place below, let us introduce the notion of (real-valued) possibilistic (or possibility measure) in its as general setting as possible.

**Definition 3.5** *Let $\Omega$ be a nonempty set, let $\mathcal{A}$ be a nonempty system of subsets of $\Omega$. A mapping $\Pi : \mathcal{A} \to [0,1]$ is a (*real-valued normalized) possibilistic measure *on $\mathcal{A}$, if $\Pi(\emptyset) = 0$ and/or $\Pi(\Omega) = 1$ supposing that $\emptyset \in \mathcal{A}$ and/or $\Omega \in \mathcal{A}$, and if $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ for every $A, B, A \cup B \in \mathcal{A}$, here and below $\vee(\bigvee$, resp.) denotes the supremum operation defined in $[0,1]$ by the standard linear ordering of real numbers. Possibilistic measure $\Pi$ on $\mathcal{A}$ is* finitely complete*, if $\Pi(\bigcup \mathcal{A}_0) = \bigvee\{\Pi(A) : A \in \mathcal{A}_0\}$ holds for* each finite *$\mathcal{A}_0 \subset \mathcal{A}$ such that $\bigcup \mathcal{A}_0 (= \bigcup_{A \in \mathcal{A}_0} A$ is in $\mathcal{A})$. Possibilistic measure $\Pi$ on $\mathcal{A}$ is complete, if $\Pi(\bigcup \mathcal{A}_0) = \bigvee\{\Pi(A) : A \in \mathcal{A}_0\}$ holds for* each *nonempty $\mathcal{A}_0 \subset \mathcal{A}$ such that $\bigcup \mathcal{A}_0 \in \mathcal{A}$ holds.*

As can be easily seen, a possibilistic measure on $\mathcal{A}$ need not be,. in general, finitely complete. Indeed, take $\mathcal{A} = \{\emptyset, A_1, A_2, A_3, \Omega\}$ where $A_1, A_2, A_3$ are such that $A_1 \cup A_2 \cup A_3 = \Omega$, but no of the unions $A_1 \cup A_2, A_2 \cup A_3, A_1 \cup A_3$ is in $\mathcal{A}$ (e.g., $A_1, A_2, A_3$ define a disjoint covering of $\Omega$ by nonempty sets). Setting $\Pi(\emptyset) = \Pi(A_1) = \Pi(A_2) = \Pi(A_3) = 0, \Pi(\Omega) = 1$, we can easily check that $\Pi$ is a possibilistic measure on $\mathcal{A}$, as for all the trivial unions $\emptyset \cup A_i, A_i \cup \Omega, \emptyset \cup \Omega$, which are in $\mathcal{A}$ the condition imposed on $\Pi$ holds, but $\Pi$ is not finitely complete, as $A_1 \cup A_2 \cup A_3 = \Omega$, but $1 = \Pi(\Omega) \neq \Pi(A_1) \vee \Pi(A_2) \ \vee \Pi(A_3) = 0$.

Obviously, each possibilistic measure on $\mathcal{A}$ is a monotone measure on $\mathcal{A}$. Indeed, if $A, B \in \mathcal{A}$ are such that $A \subset B$ holds, then $A \cup B = B$ is in $\mathcal{A}$, so that

$$\Pi(B) = \Pi(A \cup B) = \Pi(A) \vee \Pi(B) \geq \Pi(A) \tag{3.6}$$

easily follows.

# 4 Most Elementary Properties of Real-Valued Possibilistic Measures

Perhaps the most simple way towards an elementary possibilistic measure may be as follows. Let $\Omega$ be the set of all elementary random events describing a sample, procedure or decision making under uncertainty which is supposed to be of the kind of randomness, hence, which can be formally described and processed using the idea of hidden parameter the value of which is not known. Let $E \subset \Omega$ be a sure event in $\Omega$, hence, due to some more knowledge (e.g., obtained due to some further experiments, observations or consultations with experts) we know that the actual value of $\omega$ is in $E$. Given a random event $A \subset \Omega$, it is quite natural and compatible with the common meaning of the adjective in question, to call the random event $A$ as *possible*, if $A \cap E \neq \emptyset$, i.e., if it is not avoided that the actual value of $\omega$ is in $A$. Using an appropriate mathematical notation, we may define a mapping $\Pi : \mathcal{P}(\Omega) \to [0,1]$ such that $\Pi(A) = 1,$ if $A \cap E \neq \emptyset, \Pi(A) = 0,$ if $A \cap E = \emptyset$ (as the set $E$ contains the actual value $\omega \in \Omega$, $E$ is nonempty). As can be easily checked, this mapping defines a real-valued possibilistic measure on $\mathcal{P}(\Omega)$.

This idea can be easily applied to the more general case when the sure event $E$ is replaced by a fuzzy set with the characteristic function $\pi_E$, hence, for every $\omega \in \Omega, \pi_E(\omega)$ defines the degree with which $\omega$ is in $E$. Consequently, we can say that a random event $A \subset \Omega$ is *possible to the degree at least* $\alpha, 0 \leq \alpha \leq 1$, if it has a nonempty intersection with the subset of those $\omega \in \Omega$, which belong to the sure event $A$ to the degree at least $\alpha$, in symbols, if

$$A \cap \{\omega \in \Omega : \pi_E(\omega) \geq \alpha\} \neq \emptyset. \tag{4.1}$$

It is quite natural and intuitive to ask which is the supremum of the values $\alpha$ for which (4.1) holds, as can be easily seen, it is the value

$$\Pi(A) = \bigvee\{\pi_E(\omega) : \omega \in A\}. \tag{4.2}$$

Under the condition that there exists $\omega \in \Omega$ such that $\pi_E(\omega_0) = 1$ (which replaces the condition $E \neq \emptyset$ imposed in the case of crisp set $E$), and applying the convention that $\bigvee \emptyset = 0$ for the empty subset of [0,1], so that $\Pi(\emptyset) = 0$, we obtain easily that $\Pi$ is a real-valued normalized possibilistic measure on $\mathcal{P}(\Omega)$. Due to the elementary properties of the supremum operation on $\langle [0,1], \leq \rangle$ this possibilistic measure is obviously complete.

To conclude, each mapping $\pi : \Omega \to [0,1]$ such that $\bigvee_{\omega \in \Omega} \pi(\omega) = 1$ defines uniquely a complete possibilistic measure $\Pi$ on $\mathcal{P}(\Omega)$, applying (4.2) (if $\Omega$ is infinite, the condition $\bigvee_{\omega \in \Omega} \pi(\omega) = 1$ is more weaken than that strictly demanding the existence of some $\omega_0 \in \Omega$ such that $\pi(\omega_0) = 1$). The mapping $\pi$ is called, in this context, the *possibilistic distribution* inducing or defining the possibilistic measure $\Pi$ on $\mathcal{P}(\Omega)$. Each complete possibilistic measure on $\mathcal{P}(\Omega)$ is defined by the uniquely defined possibilistic distribution $\pi$ on $\Omega$; set simply $\pi(\omega) = \Pi(\{\omega\})$ for every $\omega \in \Omega$ (let us recall that $\{\omega\}$ denotes the singleton in $\mathcal{P}(\Omega)$ — the subset of $\Omega$ containing just the element $\omega \in \Omega$). If $\Omega$ is finite, the 1-1 relation between possibilistic distributions and possibilistic measures trivially holds also for possibilistic measures on $\mathcal{P}(\Omega)$. as each of them is obviously complete, but for infinite spaces $\Omega$ this need not be the case. Indeed, let $\Omega$ be infinite and let $\Pi : \mathcal{P}\Omega \to [0,1]$ be such that $\Pi(A) = 0$, if $A$ is empty or finite, and $\Pi(A) = 1$ for infinite $A \subset \Omega$. Then $\Pi$ is, obviously, a possibilistic measure on $\mathcal{P}(\Omega)$ which is not complete and which is not defined by any possibilistic distribution on $\Omega$. To cover also this case, the following formalized definition of real-valued possibilistic measure over the system of all subsets of a universe of discourse seems to be appropriate for our purposes.

**Definition 4.1** *Let $\Omega$ be a nonempty set, let $\mathcal{P}(\Omega)$ denote the system of all subsets of $\Omega$. A mapping $\Pi : \mathcal{P}(\Omega) \to [0,1]$ is called a (*real-valued)possibilistic measure *on $\mathcal{P}(\Omega)$, if $\Pi(\emptyset) = 0, \Pi(\Omega) = 1$ and $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ for every $A, B \subset \Omega$. The possibilistic measure $\Pi$ on $\mathcal{P}(\Omega)$ is called* complete*, if $\Pi(\bigcup \mathcal{A}) = \bigvee \{\Pi(A) : A \in \mathcal{A}\}$ holds for each $\emptyset \neq \mathcal{A} \subset \mathcal{P}(\Omega)$.*

For each $A \subset \Omega$, either $\Pi(A) = 1$ or $\Pi(\Omega - A) = 1$, as the relation $\Pi(A) \vee \Pi(\Omega - A) = \Pi(\Omega) = 1$ holds, but the possibility that $\Pi(A) = \Pi(\Omega - A) = 1$ holds simultaneously is not excluded. This can be interpreted in such a way that of two contradictory events, one at least is completely possible. Further, when one event is taken as possible, this does not prevent the contrary event from being so as well, which is consistent with the semantigues of judgements of possibility that commit their makers very little. More generally, for every finite system $\mathcal{A} \subset \mathcal{P}(\Omega)$ such that $\bigcup \mathcal{A} = \Omega$ there exists $A \in \mathcal{A}$ such that $\Pi(A) = 1$. Consequently, for each $A \subset \Omega$ the inequality $\Pi(A) + \Pi(\Omega - A) \geq 1$ holds, more generally, for each (not necessarily finite) system $\mathcal{A}$ of subsets of $\Omega$ there exists finite or countable subsystem $\mathcal{A}_0 \subset \mathcal{A}$ such that the inequality $\sum_{A \in \mathcal{A}_0} \Pi(A) \geq 1$ holds.

As shown above, possibilistic measures can be taken as a specific case of (normalized real-valued) monotone measures on $\mathcal{P}(\Omega)$ supposing that the inequality

$$\Pi(A \cup B) \geq \Pi(A) \vee \Pi(B), \tag{4.3}$$

obviously valid for each such monotone measure, is strenghened to equality (i.e., this equality is imposed on $\Pi$ as a new axiomatic demand). A dual idea would be to strenghen the inequality

$$\sigma(A \cap B) \leq \sigma(A) \wedge \sigma(B), \tag{4.4}$$

also trivially valid for each (normalized real-valued) monotone measure $\sigma$ on $\mathcal{P}(\Omega)$, to the equality, so obtaining the so called *(normalized real-valued) necessity measures* on $\mathcal{P}(\Omega)$. The most simple case of such necessity measure is the mapping $\sigma_E : \mathcal{P}(\Omega) \to \{0,1\}$ such that $\sigma_E(A) = 1$ if $E \subset A$ holds, and $\sigma_E(A) = 0$ otherwise, here $E$ is the sure event (subset of $\Omega$) and $A$ is also a subset of $\Omega$. As can be easily seen, a set function $\sigma$ on $\mathcal{P}(\Omega)$ satisfies (4.4) with the equality if and only if the set function $\Pi$ on $\mathcal{P}(\Omega)$, defined by

$$\Pi(A) = 1 - \sigma(\Omega - A) \tag{4.5}$$

for each $A \subset \Omega$, is a (normalized real-valued) possibilistic measure on $\mathcal{P}(\Omega)$. Dually, if $\Pi$ is a (normalized real-valued) possibilistic measure on $\mathcal{P}(\Omega)$, the mapping defined by

$$\sigma(A) = 1 - \Pi(\Omega - A) \tag{4.6}$$

for every $A \subset \Omega$, obviously defines a (normalized real-valued) necessity measure on $\mathcal{P}(\Omega)$. In particular, $\sigma(A) = 1$ iff $\Pi(\Omega - A) = 0$, hence, an event $A$ is necessary iff its negation (complement) is possible, i.e., possible to the degree 0, what agrees with the semantigues of the adjectives "possible" and "necessary" common in modal logics. Given a possibilistic distribution $\pi$ on $\Omega$, the necessity measure $\sigma$ on $\mathcal{P}(\Omega)$, defined when using (4.5) and the possibilistic measure $\Pi$ induced by $\pi$, can be obtained also directly through $\pi$, setting

$$\sigma(A) = \bigwedge\{1 - \pi(\omega) : \omega \in \Omega - A\}; \tag{4.7}$$

let us recall that $\bigwedge$ stands for the standard infimum in the unit interval [0,1] of reals. For $A = \Omega$ the convention concerning the empty subset of [0,1] applies, so that $\sigma(\Omega) = \bigwedge \emptyset = 1$, for $A = \emptyset$ we obtain easily that

$$\sigma(\emptyset) = \bigwedge\{1 - \pi(\omega) : \omega \in \Omega\} = 1 - \bigvee\{\pi(\omega) : \omega \in \Omega\} = 1 - 1 = 0; \tag{4.8}$$

As expected, dually to the properties of possibilistic measures, we obtain for each necessity measure $\sigma$ and each $A \subset \Omega$, that

$$\sigma(A) \wedge \sigma(\Omega - A) = \sigma(A \cap (\Omega - A)) = \sigma(\emptyset) = 0, \tag{4.9}$$

so that either $\sigma(A)$ or $\sigma(\Omega - A)$ (or both) equal to zero.

The following relations binding together the possibilistic and necessity measures mutually induced by (4.5) and (4.6) can be easily proved and are valid for every $A \subset \Omega$.

$$\Pi(A) \geq \sigma(A), \tag{4.10}$$

$$\text{if } \sigma(A) > 0, \text{ then } \Pi(A) = 1, \tag{4.11}$$

$$\text{if } \Pi(A) < 1, \text{ then } \sigma(A) = 0, \tag{4.12}$$

$$\sigma(A) + \sigma(\Omega - A) \leq 1. \tag{4.13}$$

Possibilistic measures can be used as a base for integration in a way analogous to that in which standard (probability) measures are used when defining integrals and expected values in the classical measure and probability theory. Even if these matters are to be analyzed, in more detail, later on when investigating non-numerical and, in particular, lattice-valued possibilistic measures, let us introduce the basic ideas just now, in the more simple case of normalized real-valued possibilistic measures.

**Definition 4.2** *A mapping $\lambda$ which takes the Cartesian product $[0,1] \times [0,1]$ into $[0,1]$ is called a (*normalized real-valued) $t$-seminorm *on $[0,1]$, if*

$$\lambda(x, 1) = \lambda(1, x) = x \text{ for each } x \in [0,1], \tag{4.14}$$

*and, for each , $x_1, y_1, x_2, y_2 \in [0,1]$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$ holds,*

$$\lambda(x_1, y_1) \leq \lambda(x_2, y_2) \tag{4.15}$$

*holds. If, moreover, $\lambda$ is symmetric and associative, i.e., if*

$$\lambda(x, y) = \lambda(y, x) \tag{4.16}$$

*and*

$$\lambda(x, \lambda(y, z)) = \lambda(\lambda(x, y), z) \tag{4.17}$$

*hold for each $x, y, z \in [0,1]$, then $\lambda$ is called a (*normalized real-valued) $t$-norm *on [0,1].*

The most simple and well-known $t$-norms on [0,1] are the infimum (i.e., minimum) $x \wedge y$ and the product $xy$, moreover, for each $t$-norm $\lambda$ on [0,1] the inequality $\lambda(x,y) \leq x \wedge y$ holds for each $x, y \in [0,1]$, so that the minimum is the "largest" $t$-norm on [0,1]. Indeed, for every $x, y \in [0,1]$ the inequalities $\lambda(x,y) \leq \lambda(x,1) = x$ and, similarly, $\lambda(x,y) \leq y$ easily follow, hence, $\lambda(x,y) \leq x \wedge y$ holds as well.

**Definition 4.3** *Let $\Omega$ be a nonempty set, let $\Pi$ be a complete (normalized real-valued) possibilistic measure on $\mathcal{P}(\Omega)$, let $f : \Omega \to [0,1]$ be any mapping, let $\lambda$ be a $t$-norm on $[0,1]$. The Sugenȯ integral of $f$ over $\Omega$ and with respect to the possibilistic measure $\Pi$ on $\mathcal{P}(\Omega)$ is denoted by $\oint_\Omega f d\Pi$ and defined as*

$$\oint_\Omega f d\Pi = \bigvee_{x \in [0,1]} \lambda[x, \Pi(\{\omega \in \Omega : f(\omega) \geq x\})]. \tag{4.18}$$

Hence, introducing $\lambda$ as an explicit parameter of $\oint_{\Omega,\lambda} f d\Pi$, we obtain that for each $t$-norm $\lambda$ the inequality

$$\oint_{\Omega,\lambda} f d\Pi \leq \bigvee_{x \in [0,1]} [x \wedge \Pi(\{\omega \in \Omega : f(\omega) \geq x\})] = \oint_{\Omega,\wedge} f d\Pi \tag{4.19}$$

is valid. In the particular case when $f$ is the characteristic function (identifier) of a subset $A \subset \Omega$, i.e., $f(\omega) = I_A(\omega) = 1,$ if $\omega \in A$ and $I_A(\omega) = 0,$ if $\omega \in \Omega - A$, we obtain easily that

$$\oint_{\Omega,\lambda} I_A d\Pi = \bigvee_{x \in [0,1]} \lambda[x, \Pi(\{\omega \in \Omega : I_A(\omega) \geq x\})] = \lambda[1, \Pi(\{\omega \in \Omega : I_A(\omega) = 1\})] = \lambda[1, \Pi(A)] = \Pi(A), \tag{4.20}$$

as it is the case in standard measure and integration theory.

# 5 Possibilistic Measures Induced by Random Variables

When considering two alternative mathematical tools for uncertainty quantification and processing, namely the probability and the possibilistic measures, the questions concerning the mutual relations of the two models are quite legitimate, important and worth being explicitly stated and answered. As a surveyal discussion concerning these matters the reader can consult [15], but we will also touch this question several times in what follows. In this chapter let us restrict our reasoning to introduce perhaps the most simple way in which real-valued possibilistic measures can be defined.

In Chapter 3 above we recalled the notion of probability space, defined as a triple $\langle \Omega, \mathcal{A}, P \rangle$, where $\Omega$ is a nonempty set of *elementary random events*, $\mathcal{A}$ is a nonempty $\sigma$-field of subsets of $\Omega$ the elements of which are called *random events*, and $P : \mathcal{A} \to [0,1]$ is a normalized $\sigma$-additive measure on $\mathcal{A}$ called *probability measure*. What is processed by probability measures, and probability theory in general, are just certain sets of elementary random events and classes of such sets, so that the problem how to decide, given a particular elementary random event $\omega_0 \in \Omega$ and a set $A \subset \Omega$, whether the membership relation $\omega_0 \in A$ does or does not hold, need not be and is not, as a matter of fact, considered and reflected within the framework of the axiomatic probability theory.

Nevertheless, let us focus our attention, in the rest of this chapter, just to this decision problem supposing that it is not only far from being trivial, but that it is solvable only in some cases and only in a partial and negative sense. Namely, given a proper subset $A$ of $\Omega$, for no $\omega \in A$ we are able to decide that this is the case, the only we are able to do is that for *some* (but not for all, in general) elements $\omega \in \Omega - A$ we can decide that this $\omega$ indeed is not in $A$, i.e., that it is not an elementary random event favorable to (in favor of) $A$. In symbols, for each $A \subset \Omega$ there exists a subset $B_A \subset \Omega - A$ (obviously empty for $\Omega = A$ and perhaps also for some other $A \subset \Omega$) such that, for every $\omega \in B_A$, we are able to decide (check, verify) that $\omega \in A$ does not hold. All $\omega \in \Omega - B_A$ are called *elementary random events possibly favorable to $A$ with respect to $B_A$*. The inclusion $A \subset \Omega - B_A$ is obvious, hence, each elementary random event favorable to $A$ is also possibly favorable to $A$ w.r. to $B_A$, so that our application of the adjective "possible" meets the common intuition behind this word. Referring the reader to [35] as far as a more general mathematical formalization of this model is concerned, let us limit ourselves to a rather simple particular case how to choose the sets $B_A$ given $A \subset \Omega$. The notion of real-valued random variable in its classical setting will be of use for our purposes and it is, hence, worth being re-called explicitly.

**Definition 5.1** *Let $\langle \Omega, \mathcal{A}, P \rangle$ be a probability space, let $R = (-\infty, \infty)$ be the space of all real numbers, let $\mathcal{B}$ be the system of all Borel subsets of $R$, i.e., $\mathcal{B}$ is the minimal $\sigma$-field of subsets of $R$ containing all semi-intervals $(-\infty, a]$ for each $a \in R$. Hence, $\langle R, \mathcal{B} \rangle$ is the well-known Borel line. A mapping $X : \Omega \to R$ is called a* (real-valued) random variable *defined on $\langle \Omega, \mathcal{A}, P \rangle$, if it is $\mathcal{A} - \mathcal{B}$-measurable, i.e., if the inverse image of each Borel set is in $\mathcal{A}$, in symbols, if the inclusion*

$$\{\{\omega \in \Omega : X(\omega) \in B\} : B \in \mathcal{B}\} \subset \mathcal{A} \tag{5.1}$$

*is valid. As a matter of fact, the inclusion*

$$\{\{\omega \in \Omega : X(\omega) \leq a\} : a \in R\} \subset \mathcal{A} \tag{5.2}$$

*is suffient. In the definition of $\mathcal{B}$ $(-\infty, a]$ can be replaced by $(-\infty, a]$ and in (5.2) $X(\omega) \leq a$ can be replaced by $X(\omega) < a$. The function $F_X : R \to [0, 1]$, defined for each $x \in R$ by*

$$F_X(x) = P(\{\omega \in \Omega : X(\omega) \leq x\}) \tag{5.3}$$

*is called the* distribution function of the random variable $X$.

Let $R^\star = [-\infty, \infty]$ be the real line enriched by elements $-\infty$ and $\infty$, let $\leq$ be the standard linear ordering on $R$ extended to $R^\star$ in the usual sense, so that $-\infty < x < \infty$ holds for every $x \in R$, let $\bigvee$ denote the supremum and $\bigwedge$ the infimum in $\langle R^\star, \leq \rangle$. Let $X$ be a random variable defined on a fixed probability space $\langle \Omega, \mathcal{A}, P \rangle$. The system $\mathcal{S}$ (or $\mathcal{S}_X$, to explicitate the role of $X$) of subsets of $\Omega$, defined by $\mathcal{S} = \{Sx : -\infty \leq x \leq \infty\}$, where $S_x = \{\omega \in \Omega : X(\omega) \leq x\}$ for each $-\infty < x < \infty$, $S_{-\infty} = \emptyset$ (the empty subset of $\Omega$) and $S_\infty = \Omega$ is called the *classification system induced by random variable* $X$. Obviously, for each $-\infty \leq x_1 \leq x_2 \leq \infty$, the inclusion $S_{x_1} \subset S_{x_2}$ holds, so that, for each $A \subset \Omega$, the value

$$i(A)(= i(A, \mathcal{S}) \text{ or } i(A, X)) = \bigwedge \{x \in R^\star : A \subset S_x\} \tag{5.4}$$

is defined. As can be easily proved, the relation

$$S_{i(A)} = \bigcap_{A \in S_x} S_x \tag{5.5}$$

is valid for each $A \subset \Omega$, so that $S_{i(A)}$ is the minimum (w.r. to set inclusion) set of elementary random events from $\Omega$ possibly favorable to $A$ w.r. to the classification system $\mathcal{S}$.

The value $i(A, \mathcal{S})$ can be easily explicitly computed. Indeed,

$$\begin{aligned} i(A, \mathcal{S}) &= \bigwedge \{-\infty \leq x \leq \infty : A \subset S_x\} = \\ &= \bigwedge \{-\infty \leq x \leq \infty : A \subset \{\omega \in \Omega : X(\omega) \leq x\}\} = \\ &= \bigwedge \{-\infty \leq x \leq \infty : (\forall \omega \in A)(X(\omega) \leq x)\} = \\ &= \bigwedge \{-\infty \leq x \leq \infty : \bigvee(A, X) \leq x\} = \bigvee(A, X), \end{aligned} \tag{5.6}$$

where $\bigvee(A, X) = \bigvee_{\omega \in A} X(\omega)$ denotes the supremum of values taken by $X$ on $A$. Set, for any $A \subset \Omega$,

$$\Pi(A, \mathcal{S}) = P(S_{i(A,\mathcal{S})}) = P(S_{\bigvee(A,X)}) = P(\{\omega \in \Omega : X(\omega) \in \bigvee(A, X)\}) = F_X(\bigvee(A, X)), \tag{5.7}$$

where $F_X : R \to [0,1]$ is the distribution function of the random variable $X$ defined by (5.3). E.g., if the supremum value of $X$ is reached on $A$, i.e., if $\bigvee(A,X) = \bigvee(\Omega,X)$ under our notation, then $\Pi(A,\mathcal{S}) = 1$. As will be shown and analyzed below, the mapping $\Pi : \mathcal{P}(\Omega) \to [0,1]$, defined by (5.7), meets the demands imposed on complete normalized real-valued possibilistic measures.

**Theorem 5.1** *Let $\langle \Omega, \mathcal{A}, P \rangle$ be a probability space, let $X$ be a real-valued random variable defined on $\langle \Omega, \mathcal{A}, P \rangle$, let $\Pi : \mathcal{P}(\Omega) \to [0,1]$ be the mapping defined by (5.7). Then $\Pi$ is a complete possibilistic measure on $\mathcal{P}(\Omega)$.*

**Proof 5.1** *The constraints for $\emptyset$ and $\Omega$ can be easily verified. Indeed,*

$$\Pi(\emptyset) = P\left(\left\{\omega \in \Omega : X(\omega) \le \bigvee(\emptyset, X)\right\}\right) = P(\{\omega \in \Omega : X(\omega) \le -\infty\}) = P(\emptyset) = 0 \qquad (5.8)$$

*applying the convention according to which $\bigvee\{X(\omega) : \omega \in \emptyset\} = -\infty$, and*

$$\Pi(\Omega) = P\left(\left\{\omega \in \Omega : X(\omega) \le \bigvee(\Omega, X)\right\}\right) = P(\Omega) = 1 \qquad (5.9)$$

*Let $\mathcal{A}$ be a nonempty system of subsets of $\Omega$. An easy calculation yields that, for $\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$,*

$$\begin{aligned}
\Pi\left(\bigcup \mathcal{A}\right) &= P\left(\left\{\omega \in \Omega : X(\omega) \le \bigvee\left(\bigcup \mathcal{A}, X\right)\right\}\right) = \\
&= P\left(\left\{\omega \in \Omega : X(\omega) \le \bigvee_{A \in \mathcal{A}}\left(\bigvee(A,X)\right)\right\}\right) = \\
&= P\left(\bigcup_{A \in \mathcal{A}}\left\{\omega \in \Omega : X(\omega) \le \bigvee(A,X)\right\}\right) = \\
&= \bigvee_{A \in \mathcal{A}} P\left(\left\{\omega \in \Omega : X(\omega) \le \bigvee(A,X)\right\}\right) = \bigvee_{A \in \mathcal{A}} \Pi(A), \qquad (5.10)
\end{aligned}$$

*as the sets $\{\omega \in \Omega : X(\omega) \le \bigvee(A,X)\}$ are nested w.r. to the standard linear ordering of the values $\bigvee(A,X), A \in \mathcal{A}$ as real numbers. The assertion is proved.*

A weakened version of Theorem 5.1 can be proved also in the case of more general classification systems than those defined by a random variable $X$. Let $\langle \Omega, \mathcal{A}, P \rangle$ be a probability space. A system $\mathcal{S} = \{S_t : -\infty \le t \le \infty\}$ is called a *general classification system* over $\langle \Omega, \mathcal{A}, P \rangle$, if $\mathcal{S} \subset \mathcal{A}$, $S_{-\infty} = \emptyset$, $S_\infty = \Omega$, and if the inclusion $S_{t_1} \subset \mathcal{S}_{t_2}$ holds for each $-\infty < t_1 \le t_2 \le \infty$, i.e., if the system $\mathcal{S}$ of subsets of $\Omega$ is nested w.r. to the standard linear ordering in the real line. The mapping $\Pi$ (or $\Pi(., \mathcal{S})$, to explicitate the role of $\mathcal{S}$) is defined, for each $A \subset \Omega$, by

$$\Pi(A) = P(S_{i(A)}), \qquad (5.11)$$

where

$$i(A) = \bigwedge\{t \in (-\infty, \infty) : A \subset S_t\}. \qquad (5.12)$$

Obviously, $i(A)$ and $\Pi(A)$ are defined for each $A \subset \Omega$, so that $\Pi$ takes $\mathcal{P}(\Omega)$ into $[0,1]$. The general classification system $\mathcal{S}$ is called *continuous from above*, if the relation

$$S_t = \bigcap\{S_s : -\infty < s \le \infty, s > t\} \qquad (5.13)$$

holds for each $-\infty \le t \le \infty$.

**Theorem 5.2** *Let $\langle \Omega, \mathcal{A}, P \rangle$ be a probability space, let $R^\star = [-\infty, \infty]$, let $\mathcal{S} = \{S_t : t \in R^\star\}$ be a continuous from above general classification system. Then $\Pi$ is a possibilistic measure on $\mathcal{P}(\Omega)$.*

**Proof 5.2** *If $A = \emptyset$, the inclusion $A \subset S_t$ holds for each $t \in R^\star$, so that $i(A) = -\infty$, $S_{i(A)} = \emptyset$ and $\Pi(\emptyset) = P(\emptyset) = 0$. If $A = \Omega$, then $A \subset S_t$ holds iff $S_t = \Omega$, so that, due to the continuity from above, we obtain that*

$$S_{i(A)} = \bigcap\{S_t : t > i(A)\} = \Omega, \tag{5.14}$$

*as $S_t = \Omega$ holds for each $t > i(A)$, if $A = \Omega$. Hence, $\Pi(\Omega) = P(\Omega) = 1$.*

*Let us prove that, for each $A, B \subset \Omega$, $i(A \cup B) = i(A) \vee i(B)$. Indeed, for both $C = A, B$, the inequality*

$$i(A \cup B) = \bigwedge\{t \in R^\star : A \cup B \subset S_t\} \geq \bigwedge\{t \in R^\star : C \subset S_t\} = i(C), \tag{5.15}$$

*hence, also the inequality $i(A \cup B) \geq i(A) \vee i(B)$ obviously hold. Suppose, that the strict inequality is the case, i.e., that $i(A \cup B) > t_0 \geq i(A) \vee i(B)$ holds for some $t_0$. Then $A \subset S_{t_0}$ and $B \subset S_{t_0}$ follows, so that $A \cup B \subset S_{t_0}$ and $i(A \cup B) \leq t_0$ results — a contradiction. So, $i(A \cup B) = i(A) \vee i(B)$ and we may suppose, without any loss of generality, that $i(A \cup B) = i(A)$. Consequently, $i(A) \geq i(B)$, $S_{i(A)} \supset S_{i(B)}$ and*

$$\Pi(A) = P(S_{i(A)}) \geq P(S_{i(B)}) = \Pi(B) \tag{5.16}$$

*follows. Hence,*

$$\Pi(A \cup B) = P(S_{i(A \cup B)}) = P(S_{i(A) \vee i(B)}) = P(S_{i(A)}) = \Pi(A) = \Pi(A) \vee \Pi(B). \tag{5.17}$$

*The assertion is proved.*

Let us consider the situation when we have at our disposal two classification systems $\mathcal{S}_x$ and $\mathcal{S}_y$, induced by real-valued random variables $X$ and $Y$, both defined on the same probability space $\langle \Omega, \mathcal{A}, P \rangle$. Our aim is to take profit of both these classification systems in order to specify the set of elementary random events possibly favorable to a given subset $A \subset \Omega$ and using as mathematical tools only the possibilistic measures $\Pi_X$ and $\Pi_Y$ induced by the random variables under consideration.

Let us try to approach this problem applying, at a higher level, the idea that not only the actual elementary random events are recognizable only and in the negative sense, but that the same is the case with possibly favorable elementary random events. Hence, given a subset $A \subset \Omega$ and a random variable $X$ defined on the probability space $\langle \Omega, \mathcal{A}, P \rangle$, we are not able to decide that some $\omega \in \Omega$ is possibly favorable to $A \subset \Omega$ with respect to the classification system $\mathcal{S}_X$, only for *some* (but not for all, in general) elementary random events which are not possibly favorable to $A$ we are able to decide (check, verify) that this is indeed the case, i.e., that they are not possibly favorable to $A$.

So, let $\langle \Omega, \mathcal{A}, P \rangle$ be a probability space, let $X$ and $Y$ be real-valued random variables defined on $\langle \Omega, \mathcal{A}, P \rangle$, let $\mathcal{S}_X$ and $\mathcal{S}_Y$ be the classification systems induced by $X$ and $Y$, let $A \subset \Omega$. Recalling, that $\bigvee(A, X)$ denotes the supremum of the values taken by $X$ on $A$, denote by $A^X$ the set

$$A^X = \left\{\omega \in \Omega : X(\omega) \leq \bigvee(A, X)\right\}, \tag{5.18}$$

so that the inclusion $A \subset A^X$ trivially holds. Hence, due to the notations used and results obtained above,

$$\Pi_X(A) = \Pi_{\mathcal{S}_X}(A) = P(A^X) \tag{5.19}$$

holds for every $A \subset \Omega$. Repeating the same consideration with $A$ replaced by $A^X$ and with $X$ replaced by $Y$, we obtain that

$$(A^X)^Y = \left\{\omega \in \Omega : Y(\omega) \leq \bigvee(A^X, Y)\right\}, \tag{5.20}$$

and

$$\Pi_Y(A^X) = P((A^X)^Y) = P\left(\left\{\omega \in \Omega : Y(\omega) \leq \bigvee(A^X, Y)\right\}\right). \tag{5.21}$$

The value $\Pi_Y(A^X)$ will be denoted by $\Pi_{XY}(A)$ and will be called the *second-level possibilistic measure induced on $\mathcal{P}(\Omega)$ by the ordered pair $\langle X, Y\rangle$ of random variables.* Obviously, $\Pi_{XY}(A)$ is defined for each $A \subset \Omega$, but the justification of the adjective "possibilistic" remains to be proved.

**Theorem 5.3** *Let $X, Y$ be random variables defined on a probability space $\langle \Omega, \mathcal{A}, P\rangle$, let $\Pi_{XY} : \mathcal{P}(\Omega) \to [0,1]$ be defined by (5.21). Then $(i)\Pi_{XY}$ is a possibilistic measure on $\mathcal{P}(\Omega), (ii)\Pi_{XX}(A) = \Pi_X(A)$ for every $A \subset \Omega$ and $(iii)$ the inequalities $\Pi_{XY}(A) \geq \Pi_X(A),\ \Pi_{XY}(A) \geq \Pi_Y(B)$ hold for any $A \subset \Omega$.*

**Proof 5.3** *For each $A, B \subset \Omega$ we obtain that*

$$\begin{aligned}(A \cup B)^X &= \left\{\omega \in \Omega : X(\omega) \leq \bigvee(A \cup B, X)\right\} = \\ &= \left\{\omega \in \Omega : X(\omega) \leq \bigvee(A, X) \vee \bigvee(B, X)\right\} = \\ &= \left\{\omega \in \Omega : X(\omega) \leq \bigvee(A, X)\} \cup \{\omega \in \Omega : X(\omega) \leq \bigvee(B, X)\right\} \\ &= A^X \cup B^X.\end{aligned} \tag{5.22}$$

*As $\Pi_Y$ is a possibilistic measure on $\mathcal{P}(\Omega)$ (cf. Theorem 5.1), we obtain that*

$$\Pi_{XY}(A \cup B) = \Pi_Y((A \cup B)^X) = \Pi_Y(A^X \cup B^X) = \Pi_Y(A^X) \vee \Pi_Y(B^X) = \Pi_{XY}(A) \vee \Pi_{XY}(B) \tag{5.23}$$

*and $(i)$ is proved. As can be easily seen, $\bigvee(A^X, X) = \bigvee(A, X)$ holds for each $A \subset \Omega$, so that*

$$\Pi_{XX}(A) = P\left(\left\{\omega \in \Omega : X(\omega) \leq \bigvee(A^X, X)\right\}\right) = P\left(\left\{\omega \in \Omega : X(\omega) \leq \bigvee(A, X)\right\}\right) = \Pi_X(A) \tag{5.24}$$

*and $(ii)$ is also proved.*

*For each $A \subset \Omega$ the inclusions $A \subset A^X,\ A \subset A^Y$, hence, also the inclusion $A^X \subset (A^X)^Y$ obviously hold, so that the inequality $\bigvee(A^X, Y) \geq \bigvee(A, Y)$ and the inclusion*

$$\left\{\omega \in \Omega : Y(\omega) \leq \bigvee(A, Y)\right\} \subset \left\{\omega \in \Omega : Y(\omega) \leq \bigvee(A^X, Y)\right\} \tag{5.25}$$

*easily follow. So, we obtain that the inequality*

$$\begin{aligned}\Pi_{XY}(A) &= P\left(\left\{\omega \in \Omega : Y(\omega) \leq \bigvee(A^X, Y)\right\}\right) \geq \\ &\geq P\left(\left\{\omega \in \Omega : Y(\omega) \leq \bigvee(A, Y)\right\}\right) = \Pi_Y(A)\end{aligned} \tag{5.26}$$

*holds. The inclusion $A^X \subset (A^X)^Y$ yields that the inequality*

$$\Pi_{XY}(A) = P((A^X)^Y) \geq P(A^X) = \Pi_X(A) \tag{5.27}$$

*is also valid and $(iii)$ holds. The proof is completed.*

Let us note that the proof of the relation (5.22) above cannot be extended to infinite systems $\mathcal{A}_0$ of subsets of $\Omega$. In this case, only the inclusion

$$\bigcup_{A \in \mathcal{A}_0} \left\{\omega \in \Omega : X(\omega) \leq \bigvee(A, X)\right\} \subset \left\{\omega \in \Omega : X(\omega) \leq \bigvee_{A \in \mathcal{A}_0} \bigvee(A, X)\right\}, \tag{5.28}$$

hence, the inclusion

$$\bigcup_{A \in \mathcal{A}_0} A^X \subset \left(\bigcup \mathcal{A}_0\right)^X \tag{5.29}$$

and the inequality

$$\bigvee_{A \in \mathcal{A}_0} \Pi_{XY}(A) \leq \Pi_{XY}\left(\bigcup \mathcal{A}_0\right) \tag{5.30}$$

can be proved. Indeed, if $\mathcal{A}_0$ is infinite, it is possible that for some $\omega_0 \in \Omega$ the equality $X(\omega_0) = \bigvee_{A \in \mathcal{A}_0} \bigvee(A, X)$, but also the inequalities $X(\omega_0) > \bigvee(A, X)$ for each $A \in \mathcal{A}_0$ hold together. Modifying our definition of $\Pi_X(A)$ by setting $\Pi_X(A) = P(\{\omega \in \Omega : X(\omega) < \bigvee(A, X)\})$ (instead of $\leq \bigvee(A, X)$, as defined above) we would solve the problem of completeness for $\Pi_{XY}$, but the classification system $\mathcal{S}_X$ would not be continuous from above, so that our former constructions and assertions would be menaced.

Recalling the semantics behind the idea of possibly favorable elementary random events we can easily see that the best approximation of a set $A \subset \Omega$ by possibly favorable elementary random events induced by two random variables $X, Y$ defined on the probability space $\langle \Omega, \mathcal{A}, P \rangle$ would be the intersection $A^X \cap A^Y$, and the most reasonable and intuitive numerical quantification of this set would be the probability value $P(A^X \cap A^Y)$. A problem is that this reasoning and formalization cannot be embedded into the framework of possibilistic measures and nested classification systems induced by the random variables in question. The next example illustrates the case when the discrepancy between the values $P(A^X \cap A^Y)$ and $\Pi_{XY}(A)$ (or $\Pi_{XY}(A)$) is the most remarkable.

Let $\Omega = R = (-\infty, \infty)$, let $\mathcal{B} = \mathcal{A}$ be a system of all Borel subsets of $R$, so that $\langle \Omega, \mathcal{A} \rangle$ is the Borel line $\langle R, \mathcal{B} \rangle$, let $P$ be a probability measure on $\mathcal{A}$ Let $X$ be the identity on $\mathcal{R}$, so that $X(\omega) = \omega$ for each $\omega \in R$, let $A = [a, b]$ be a closed interval of real numbers such that $a < b$ holds. As can be easily observed, $\bigvee(A, X) = b$ and $A^X = \{\omega \in \Omega : X(\omega) \leq b\} = (-\infty, b]$, so that $\Pi_X(A) = P(A^X) = P((-\infty, b])$. For $Y$ we obtain that $\bigvee(A, Y) = \bigvee\{-\omega : a \leq \omega \leq b\} = -a$, so that $A^Y = \{\omega \in \Omega : -\omega \leq -a\} = \{\omega \in \Omega : \omega \geq a\} = [a, \infty)$, and $\Pi_Y(A) = P([a, \infty))$. In this case, $A^X \cap A^Y = (-\infty, b] \cap [a, \infty) = [a, b] = A$, so that the set $A$ is completely defined by the intersection of the sets of elementary random events possibly favorable to $A$ w.r. to $X$ and w.r. to $Y$. On the other side, when using only the second-level possibly favorable elementary random events, i.e., the sets $(A^X)^Y$ and/or $(A^Y)^X$, all the information concerning the event $A$ disappears. Indeed,

$$\begin{aligned} \bigvee(A^X, Y) &= \bigvee\{-\omega : \omega \in (-\infty, b]\} = \infty, \\ \bigvee(A^Y, X) &= \bigvee\{-\omega : \omega \in [a, \infty)\} = \infty, \end{aligned} \tag{5.31}$$

so that

$$\begin{aligned} (A^X)^Y &= \{\omega \in \Omega : Y(\omega) \leq \infty\} = \{\omega \in \Omega : -\omega \leq \infty\} = \Omega = \\ &= \{\omega \in \Omega : \omega \leq \infty\} = \{\omega \in \Omega : X(\omega) \leq \infty\} = (A^Y)^X, \end{aligned} \tag{5.32}$$

hence, $\Pi_{XY}(A) = \Pi_{YX}(A) = 1$.

The mapping $\Pi_{XY} : \mathcal{P}(\Omega) \to [0, 1]$ need not be, in general, commutative in $X$ and $Y$, i.e., the sets $(A^X)^Y$ and $(A^Y)^X$, as well as the values $\Pi_{XY}(A)$ and $\Pi_{YX}(A)$, may differ for some $A \subset \Omega$. Indeed, let us consider the following example.

Let $\Omega = [0, 1]$, let $\mathcal{A}$ be the system of all Borel subsets of $[0, 1]$, let $P$ be the uniform probability measure on $\mathcal{A}$, so that $P([a, b]) = b - a$ for each interval in $[0, 1]$. Let $X(\omega) = \omega$ for each $\omega \in \Omega$, let $Y(\omega) = 2\omega$, if $0 \leq \omega \leq \frac{1}{2}$ holds and $Y(\omega) = 2 - 2\omega$, if $\frac{1}{2} \leq \omega \leq 1$ holds. Hence, the graph of $Y$ defines a triangle shape in $[0, 1] \times [0, 1]$ with the tops in $\langle 0, 0 \rangle, \langle 0, 1 \rangle$ and $\langle \frac{1}{2}, 1 \rangle$. Let $A = [0, b]$ with $b < \frac{1}{2}$. Then

$$\bigvee(A, X) = \bigvee\{\omega : \omega \in [0, b]\} = b, \tag{5.33}$$

so that $A^X = [0, b] = A$, and

$$\bigvee(A, Y) = \bigvee(A^X, Y) = \bigvee\{2\omega : \omega \leq b\} = 2b, \tag{5.34}$$

as $b < \frac{1}{2}$ holds. Consequently,

$$(A^X)^Y = A^Y = [0, b] \cup [1 - b, 1], \tag{5.35}$$

as $A^Y = \{\omega \in \Omega : Y(\omega) \leq 2b\} = [0, b] \cup [1 - b, 1]$, so that $\Pi_{XY}(A) = P((A^X)^Y) = = P([0, b] \cup [1 - b, 1]) = 2b < 1$. On the other side,

$$\bigvee(A^Y, X) = \bigvee\{\omega : \omega \in [0, b] \cup [1 - b, 1]\} = 1, \tag{5.36}$$

so that $(A^Y)^X = \{\omega \in \Omega : \omega \leq 1\} = \Omega$ and the inequality

$$\Pi_{YX}(A) = P(\Omega) = 1 > 2b = \Pi_{XY}(A) \tag{5.37}$$

is obvious.

The construction of the sets $(A^X)^Y$ and the values $\Pi_{XY}(A)$ can be easily generalized to a finite sequence $X_1, X_2, \ldots, X_n$ of random variables defined on the probability space $\langle\Omega, \mathcal{A}, P\rangle$, under consideration. Given $A \subset \Omega$, we define $A^{X_1}$ as above and we denote by $A^{X_1 X_2}$ the set $(A^{X_1})^{X_2}$, also defined above. By induction, having already defined $A^{X_1 \ldots X_{n-1}}$, we set $A^{X_1 \ldots X_n} = (A^{X_1 \ldots X_{n-1}})^{X_n}$ and $\Pi_{X_1 \ldots X_n}(A) = P(A^{X_1 \ldots X_n})$ for each $A \subset \Omega$. As can be easily seen the mapping $\Pi_{X_1 \ldots X_n}$ defines a possibilistic measure on $\mathcal{P}(\Omega)$ (not necessarily complete) and the inequality $\Pi_{X_1 \ldots X_{n-1}}(A) \leq \Pi_{X_1 \ldots X_n}(A)$ holds for each $A \subset \Omega$ and each $n \geq 2$.

# 6 Preliminaries on Sets and Structures over Them

In all this text, when using the notion of set and when processing sets in various ways and constructions, we will always be able to do so within the framework of the common naive set theory with the foundations of which the reader is supposed to be familiar. To avoid possible troubles with inconsistencies or other formal difficulties, all sets under consideration will be supposed to be subsets of various universes of discourse and the existence of such universes will be always explicitly assumed, most often by saying "let $X$ be a set".

Following [10], let us begin with the notion of power-set and ample field. Given a set $X, \mathcal{P}(X)$ denotes the system of all subsets of $X$ (including the empty set $\emptyset$), which is called the *power set* (over $X$), in symbols,

$$\mathcal{P}(X) = \{A : A \subset X\}. \tag{6.1}$$

Obviously, for each $A \in \mathcal{P}(X)$ and for each nonempty system $\mathcal{A}$ of subsets of $X$ (i.e., for each $\mathcal{A} \subset \mathcal{P}(X)$), the sets $X - A$, $\bigcup \mathcal{A}(= \bigcup_{A \in \mathcal{A}} A)$ and $\bigcap \mathcal{A}(= \bigcap_{A \in \mathcal{A}} A)$ are in $\mathcal{P}(X)$. Picking up these three properties as axiomatic demands imposed on a system of subsets of $X$, we arrive at the notion of ample field: a system $\mathcal{R}$ of subsets of $X$ is called an *ample field* (over $X$), if for each $A \in \mathcal{A}$ and each nonempty $A_0 \subset \mathcal{R}$ also the sets $X - A$ and $\bigcup \mathcal{A}_0(X - A$ and $\bigcap \mathcal{A}_0$, resp.) are in $\mathcal{R}$ (as a matter of fact, due to de Morgan rules both the sets $\bigcup \mathcal{A}_0$ and $\bigcap \mathcal{A}_0$ are in $\mathcal{R}$ in both the cases).

Still weakening the demands imposed on system $\mathcal{R}$ of subsets of $X$, the well-known notions of field and $\sigma$-field result. A system $\mathcal{A}$ of subsets of $X$ is called a *field* (over $X$), if for each $A, B \in \mathcal{A}$ also the sets $X - A$ and $A \cup B$ (hence, also $A \cap B$) are in $\mathcal{A}$. If the set $\bigcup_{i=1}^{\infty} A_i$ is in $\mathcal{A}$ for each sequence $A_1, A_2, \ldots$ of sets from a field $\mathcal{A}$ over $X$, then $\mathcal{A}$ is called a *$\sigma$-field* or *countable field* (over $X$), also in this case the relation $\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$ immediately follows due to de Morgan rules.

Each ample field over $X$ is also a $\sigma$-field over $X$ and each $\sigma$-field over $X$ is also a field over $X$, as can be immediately observed, moreover, for each nonempty field $\mathcal{A}$ over $X$ the relation $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$ hold (if $A \in \mathcal{A}$, then $X - A \in \mathcal{A},\ X = (X - A) \cup A,$ and $\emptyset = X - X$ are in $\mathcal{A}$).

Fields and $\sigma$-fields play an important role when defining and processing standard real-valued measures and, in particular, probabilities. The reason is simple: the main numerical operations applied when processing values of measures of probabilities taken as set functions are those of addition and series calculations and these operations can be defined only when the number of items added is finite or, under some more conditions imposed, countable. When shifting our attention to possibilistic measures, the main operation over uncertainty degrees is that of supremum which can be defined also for uncountable sets of items. It is why ample fields as definition domains on which possibilistic measures are defined can be taken as a slight generalization if compared with the case when the possibilistic measure is supposed to be defined or the whole power-set over the set of elementary random events under consideration, with the greatest part of technical simplifications resulting in the case of the power-set in question taken as domain of possibilistic measure still being applicable in the case of ample fields. These problems are analyzed, in more detail, in [14], the following survey of the most elementary properties of ample fields is borrowed from [10].

Let $X$ be a set, let $\mathcal{R}$ be an ample field over $X$. This ample field is called *proper*, if the set $X$ is nonempty and if both $\emptyset$ and $X$ are in $\mathcal{R}$. Obviously, the system $\{\emptyset, X\}$ defines the *smallest* (when referring to cardinality) or the *roughest* (when referring to the ability to separate different elements of $X$ from each other) ample field over $X$, on the other side, the *largest* or the *finest* ample field over $X$ is obviously the power-set $\mathcal{P}(X)$. In what follows, all ample fields under consideration will be assumed to be proper, if not explicitly stated otherwise.

The *atom* of $\mathcal{R}$ containing the element $x$ of $X$ is denoted by $[x]_{\mathcal{R}}$ and is defined by

$$[x]_{\mathcal{R}} = \bigcap\{A : x \in A,\ A \in \mathcal{R}\}. \tag{6.2}$$

As can be easily seen, each $[x]_{\mathcal{R}}$ is in $\mathcal{R}$ and the system $\{[x]_{\mathcal{R}} : x \in X\}$ defines a partition (disjoint covering) of $X$. If $\mathcal{R} = \{\emptyset, X\}$, then $[x]_{\mathcal{R}} = X$ for every $x \in X$, if $\mathcal{R} = \mathcal{P}(X)$, then $[x]_{\mathcal{R}} = \{x\}$ for every $x \in X$, so that atoms can be taken as generalizations of singletons. Consequently, for any $x \in X$ and $A \in \mathcal{R}$, $x \in A$ holds iff $[x]_{\mathcal{R}} \subset A$.

A subset $E$ of $X$ is called $\mathcal{R}$-*measurable* (or simply *measurable*, if no misunderstanding menaces), if $E \in \mathcal{R}$ (let us recall the case of measure of probability theory, when a subset of the universe of elementary random events is called measurable simply when belonging to the field or $\sigma$-field on which the measure or probability in question is defined). It follows easily that $E$ is in $\mathcal{R}$ iff $E = \bigcup_{x \in E}[x]_{\mathcal{R}}$.

Consider an arbitrary system $\mathcal{E}$ of subsets of $X$, i.e., an arbitrary $\mathcal{E} \subset \mathcal{P}(X)$. The intersection of any family of ample fields being again an ample field, we obtain that the system

$$\mathcal{T}(\mathcal{E}) = \bigcap\{\mathcal{R} : \mathcal{E} \subset \mathcal{R},\ \mathcal{R} \text{ is an ample field over X}\} \tag{6.3}$$

is an ample field over $X$, called the *ample field generated* by $\mathcal{E}$ (again, recall the idea of minimal $\sigma$-field over a system of sets in measure and probability theory). The mapping $\mathcal{T}$ can be regarded as an operator on $\mathcal{P}(\mathcal{P}(X))$ which defines a closure operator on $\mathcal{P}(X)$ in the sense that for any $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{P}(X)$, $\mathcal{E} \subset \mathcal{T}(\mathcal{E})$, $\mathcal{T}(\mathcal{T}(\mathcal{E})) = \mathcal{T}(\mathcal{E})$, and $\mathcal{T}(\mathcal{E}_1) \subset \mathcal{T}(\mathcal{E}_2)$ holds supposing that $\mathcal{E}_1 \subset \mathcal{E}_2$. Moreover, $\mathcal{T}(\mathcal{E}) = \mathcal{E}$ iff $\mathcal{E}$ is an ample field over $X$.

This notion can be used to introduce product ample fields. If we consider the universes $X_1, X_2$ equipped with the respective ample fields $R_1$ and $\mathcal{R}_2$, then the *product ample field* of $R_1$ and $\mathcal{R}_2$ is the ample field $R_1 \times \mathcal{R}_2$ over $X_1 \times X_2$, defined as

$$\mathcal{R}_1 \times \mathcal{R}_2 = \mathcal{T}(\{A_1 \times A_2 : A_1 \in \mathcal{R}_1 A_2 \in \mathcal{R}_2\}). \tag{6.4}$$

For the atoms of $\mathcal{R}_1 \times \mathcal{R}_2$ we obtain, that for any pair $\langle x_1, x_2\rangle \in X_1 \times X_2$

$$[\langle x_1, x_2\rangle]_{\mathcal{R}_1 \times \mathcal{R}_2} = [x_1]_{\mathcal{R}_1} \times [x_2]_{\mathcal{R}_2}, \tag{6.5}$$

what confirms the interpretation of an atom as a generalization of *a* singleton. These ideas and results can be immediately and routinely extended to products of more than two ample fields. Finally, given an ample field $\mathcal{R}$ oven a universe $X$, we can define a mapping $p_{\mathcal{R}} : \mathcal{P}(X) \to \mathcal{P}(X)$, setting for each $A \subset X$

$$p_{\mathcal{R}}(A) = \bigcup_{x \in A}[x]_{\mathcal{R}}. \tag{6.6}$$

This mapping can be called the *closure operator in* $X$ *associated with the ample field* $\mathcal{R}$.

The most elementary ideas and results dealing with ample fields are closely related to the notion of *rough sets*, introduced and investigated by $Z$. Pawlak in [41], and can be defined and processed in an alternative way using the apparatus of rough sets. Given a nonempty set $X$, a binary relations

$\equiv$ on $X$, i.e., a subset $\equiv\, \subset X \times X$, is called an *equivalence (relation) on* $X$, if it is *reflexive*, i.e., $x \equiv x$ holds for any $x \in X$, *symmetric*, i.e., if $x_1 \equiv x_2$ holds for some $x_1, x_2 \in X$, then $x_2 \equiv x_1$ holds as well, and *transitive*, i.e., for each $x_1, x_2, x_3 \in X$ such that $x_1 \equiv x_2$ and $x_2 \equiv x_3$ hold together, also $x_1 \equiv x_3$ holds. The pair $\langle X, \equiv \rangle$, where $X$ is a nonempty set and $\equiv$ is an equivalence relation on $X$, is called the *rough set induced by* $\equiv$ *on* $X$. For each $x \in X$, the equivalence class $[x]_\equiv$ is defined by

$$[x]_\equiv = \{y \in X : y \equiv x\} \tag{6.7}$$

and the factor-space $X| \equiv$ is defined as the set of all equivalence classes induced in $X$ by $\equiv$, hence

$$X| \equiv = \{[x] \equiv : x \in X\}. \tag{6.8}$$

Given an ample field $\mathcal{R}$ over $X$ and introducing a binary relation $\equiv_\mathcal{R}$ on $X$ in such a way that $x_1 \equiv_\mathcal{R} x_2$ holds iff $[x_1]_\mathcal{R} = [x_2]_\mathcal{R}, sx_1, x_2 \in X$, we obtain easily that $\equiv_\mathcal{R}$ is an equivalence relation on $X$ and atoms $[x]_\mathcal{R}$ are nothing else than the equivalence classes $[x]_\equiv$. Consequently, instead of investigating ample fields and atoms, we can investigate the factor-space $X| \equiv_\mathcal{R}$ and the power-set of all subsets of this factor-space, what enables to take profit of many of the simplifications applicable in the case when the set functions (possibilistic measures, in what follows) are defined for all subsets of the universe of discourse.

# 7 Partial Orderings, Lattices and Complete Lattices

All the standard classical mathematical models for uncertainty quantification and processing, including, e.g., probability measures or membership functions of fuzzy sets, were based on the idea to quantify degrees of uncertainty by real numbers, most often from the unit interval [0,1], taking profit of the rich mathematical structures over these reals and of powerful mathematical apparata developed in order to process them.

On the other side, however, when the application of standard mathematical models for uncertainty quantification and processing is to be justified, reasonable and senseful, very detailed knowledge and quantifications concerning the input data charged by uncertainty are necessary, e.g., all the input probabilities or degrees of membership (values of membership functions in fuzzy sets) must be known either absolutely precisely, or within very narrow tolerance intervals, so that to obtain such input data is often beyond the user's abilities in practical applications. Hence, alternative mathematical models for uncertainty quantification and processing with uncertainties taking their degrees in some non-numerical sets and structures would be of use and their investigation is well-motivated and deserves a high support (perhaps the first work in this direction is [19] by J. A. Goguen, dealing with fuzzy sets defined by appropriate non-numerical membership functions).

Perhaps the most general and still non-trivial demands imposed on non-numerical uncertainty degrees reads that at least some pairs of these degrees should be comparable with each other as far as their sizes are concerned, i.e., for at least some pairs of uncertainty degrees a relation like "is greater than", "is at least as great as", or something like this should be defined. As these intuitive demands are mathematically formalized by the notions of partial orderings and partially ordered sets, let us begin with their definitions.

**Definition 7.1** *Let $T$ be a nonempty set. A binary relation $\leq$ on $T$ (i.e., a subset $\leq\, \subset T \times T$) is called a* pre-ordering *on $T$, if it is a reflective and transitive, i.e., if (i) $t \leq t$ holds for each $t \in T$ and (ii) for each $t_1, t_2, t_3 \in T$ such that $t_1 \leq t_2$ and $t_2 \leq t_3$ hold , $t_1 \leq t_3$ holds as well. If the pre-ordering $\leq$ on $T$ is antisymmetric, i.e., for each $t_1, t_2 \in T$, $t_1 \leq t_2$ and $t_2 \leq t_1$ hold simultaneously only when $t_1 = t_2$, then the relation $\leq$ is called a* partial ordering *on $T$ and the pair $\mathcal{T} = \langle T, \leq \rangle$ is called* partially ordered set *(on $T$ or with the support $T$).*

Two remarks are perhaps worth being introduced explicitly. First, when denoting the relation of partial ordering, we use purposely the same symbol $\leq$ as for the standard linear ordering on real line $(-\infty, \infty)$. The reason is that this symbol is easy to process typographically and to combine it with other standard mathematical symbols. Let us hope that no misunderstanding menaces, describing the situation explicitly in any case of possible confusion.

Second remark concerns the fact each pre-ordering on $T$ defines an equivalence relation on $T$ such that this pre-ordering induces a partial ordering on the resulting factor-space. Indeed, given a pre-ordering $\leq$ on $T$, set $t_1 \equiv t_2$ for each $t_1, t_2 \in T$ such that $t_1 \leq t_2$ and $t_2 \leq t_1$ hold together. Obviously, $\equiv$ defines an equivalence relation on $T$, so that the factor-space $T| \equiv$ can be defined. Setting, for each $[t_1]_\equiv, [t_2]_\equiv \in T| \equiv$, $[t_1] \leq^\star [t_2]$ iff $t_1 \leq t_2$ holds (the choice of representatives of the equivalences classes $[t_1]_\equiv$ and $[t_2]_\equiv$ is obviously irrelevant) we can easily prove that $\leq^\star$ defines a partial ordering on the factor-space $T| \equiv$.

The most simple, but also the most important for our purposes, operations enabling to process elements and subsets of partially ordered sets are those of *supremum* and *infimum*.

**Definition 7.2** *Let $\mathcal{T} = \langle T, \leq \rangle$ be a partially ordered (p.o., abbreviately) set, let $S$ be a nonempty subset of $T$. If there exists $t_0 \in T$ such that*

*(i) $s \leq t_0$ holds for each $s \in S$ and*

*(ii) for each $t_1 \in T$ such that $s \leq t_1$, holds for each $s \in S$, $t_0 \leq t_1$ holds as well,*

*then $t_0$ is called the* supremum *of $S$ and denoted by $\bigvee_{s \in S} s$ or simply $\bigvee S$. If there exists $t_0 \in T$ such that*

*(iii) $t_0 \leq s$ holds for each $s \in S$ and*

*(iv) for each $t_1 \in T$ such that $t_1 \leq s$ holds for each $s \in S$, $t_1 \leq t_0$ holds as well,*

*then $t_0$ is called the* infimum *of $S$ and denoted by $\bigwedge_{s \in S} t$ or simply by $\bigwedge S$.*

If $S = \{t_1, t_2\} (S = \{t_1, t_2, \dots, t_n\},\ S = \{t_1, t_2, \dots\}, resp.)$, we write often $t_1 \vee t_2 (t_1 \vee t_2 \dots \vee t_\infty, \bigvee_{i=1}^n t_i, t_1 \vee t_2 \vee \dots, \bigvee_{i=1}^\infty t_i$, resp.) for $\bigvee S$ and $t_1 \wedge t_2 (t_1 \wedge t_2 \dots \wedge t_n,\ \bigwedge_{i=1}^n t_i,\ t_1 \wedge t_2 \wedge \dots,\ \bigwedge_{i=1}^n t_i$, resp.) for $\bigwedge S$. What is important: in general, neither $\bigvee S$ nor $\bigwedge S$ need not be defined for some $S \subset T$, but if they are defined, they are defined uniquely. Supposing that $\bigvee T$ and/or $\bigwedge T$ are defined, also $\bigvee S$ and/or $\bigwedge S$ can be defined by convention for $S = \emptyset$, setting $\bigvee \emptyset = \bigwedge T$ and/or $\bigwedge \emptyset = \bigvee T$.

When processing partially ordered sets at their most general level as just introduced, we would be put into serious troubles as we have to state explicitly, for each statement declared and proved, for which subsets of $T$ their suprema and/or infima are supposed to be defined, leaving aside and perhaps treating later separately the situations when this is not the case. A standard way how to avoid such difficulties is to restrict our reasoning to such particular cases of partially ordered sets, where the existence of some suprema and infima is postulated a priori. Pursuing this idea in a mathematically formalized way, we arrive at the notions of semilattices and lattices of various kinds.

**Definition 7.3** *Let $\mathcal{T} = \langle T, \leq \rangle$ be a p.o. set. $\mathcal{T}$ is called*

*(i) an* upper semilattice*, if $t_1 \vee t_2$ is defined for any $t_1, t_2 \in T$,*

*(ii) a* lower semilattice*, if $t_1 \wedge t_2$ is defined for any $t_1, t_2 \in T$,*

*(iii) a* lattice*, if it is an upper semilattice and a lower semilattice.*

*$\mathcal{T} = \langle T, \leq \rangle$ is called*

*(iv) a* complete upper semilattice*, if $\bigvee S$ is defined for any $\emptyset \neq S \subset T$*

*(v) a* complete lower semilattice*, if $\bigwedge S$ is defined for any $\emptyset \neq S \subset T$*

*(vi) a* complete lattice*, if it is a complete upper semilattice and a complete lower semilattice, i.e., to express it explicitly in this case playing the most important role in what follows, if $\bigvee S$ and $\bigwedge S$ is defined for each $\emptyset \neq S \subset T$, hence, also $\bigvee \emptyset$ and $\bigwedge \emptyset$ can be defined using the conventions mentioned above. In this case, the element $\bigvee T$ will be denoted by $\mathbf{1}_\mathcal{T}$ and called the* unit *or the* maximum element *of the complete lattice $\mathcal{T}$, the element $\bigwedge T$ will be denoted by $\mathbf{0}_\mathcal{T}$ and called the* zero *or the* minimum *element of $\mathcal{T}$.*

*A complete lattice* $\mathcal{T} = \langle T, \leq \rangle$ *is called* completely distributive with respect to supremum *(or* Brouwerian, *cf. [2]), if for any* $t \in T$ *and* $S \subset T$ *the identity*

$$t \wedge \left(\bigvee S\right) = \bigvee_{s \in S} (t \wedge s) \tag{7.1}$$

*holds. Dually, a complete lattice* $\mathcal{T} = \langle T, \leq \rangle$ *is called* completely distributive w.r. to infimum, *if*

$$t \vee \left(\bigwedge S\right) = \bigwedge_{s \in S} (t \vee s) \tag{7.2}$$

*holds for each* $t \in T$ *and* $S \subset T$.

A complete lattice $\mathcal{T} = \langle T, \leq \rangle$ is called *completely distributive*, if it is completely distributive with respect to sumpremum as well as with respect to infimum.

Both the particular complete lattices $\langle [0,1], \leq \rangle$ and $\langle \mathcal{P}(X), \subset \rangle$, $X \neq \emptyset$, are well-known to be completely distributive, but, in general, this need not be the case, just the inequalities

$$t \wedge \left(\bigvee S\right) \geq \bigvee_{s \in S} (t \wedge s),\ t \vee (\bigwedge S) \leq \bigwedge_{s \in S} (t \vee s) \tag{7.3}$$

can be proved. Indeed, $t \wedge (\bigvee S) \geq t \wedge s$ and $t \vee (\bigwedge S) \leq t \vee s$ hold for each $s \in S$, so that (7.3) easily follows. However, the inequalities inverse to (7.3) do not hold in general, as the following simple example demonstrates.

Let $T = \{0, t_1, t_2, t_3, 1\}$, let $0 < t_i < 1$, $i = 1,2,3$, be the only pairs of elements of $T$ between which the relation $\leq$ applies, hence, no $t_i, t_j, i \neq j$, are mutually comparable by $\leq$. The resulting supremum and infimum operations read as follows: $0 \vee t_i = t_i = 1 \wedge t_i$, $0 \wedge t_i = 0$, $1 \vee t_i = 1$ for each $i = 1,2,3$, and for each $S \subset T$ such that $S$ contains at least two elements $t_i, t_j, i \neq j$, $\bigwedge S = 0$ and $\bigvee S = 1$. For every $i = 1,2,3$, set $C_i = \{s \in T : s \wedge t_i = 0\}$ and $C_i^\star = \{s \in T : s \vee t_i = 1\}$. In particular, $C_1 = \{0, t_2, t_3\}$ and $C_1^\star = \{t_2, t_3, 1\}$, so that $\bigvee C_1 = t_2 \vee t_3 = 1$ and $\bigwedge C_1^\star = t_2 \wedge t_3 = 0$. Hence

$$t_1 \wedge \left(\bigvee C_1\right) = t_1 \wedge 1 = t_1 = t_1 \vee \left(\bigwedge C_1^\star\right) = t_1 \vee 0. \tag{7.4}$$

But,

$$\bigvee_{s \in C_1} \{t_1 \wedge s\} = \bigvee \{t_1 \wedge 0,\ t_1 \wedge t_2,\ t_1 \wedge t_3\} = \bigvee \{0,0,0\} = 0 \neq t_1 = t_1 \wedge \left(\bigvee C_1\right), \tag{7.5}$$

and

$$\bigwedge_{s \in C_1^\star} \{t_1 \vee s\} = \bigwedge \{t_1 \vee t_2,\ t_1 \vee t_3,\ t_1 \vee 1\} = \bigwedge \{1,1,1\} = 1 \neq t_1 = t_1 \vee \left(\bigwedge C_1^\star\right) \tag{7.6}$$

Analogous inequalities for $t_2$ and $t_3$ are obvious. Consequently, the complete lattice $\langle T, \leq \rangle$ is not completely distributive.

Complete lattice is perhaps the most specific structure still covering both the most often used structures in which uncertainty degrees take their values: the unit interval of real numbers with their standard linear ordering, and the system of (all or some) subsets of a universe partially ordered by set inclusion — this last case is, as a matter of fact, identical with complete Boolean algebra. It is just the operation of complement which is defined in both these structures in different ways incompatible with each other — as the substraction $1 - .$ in $\langle [0,1], \leq \rangle$ and as the set complement $X - .$ in $\langle \mathcal{P}(X), \subset \rangle$. Therefore, it will be useful for our further purposes to define the notion of (pseudo-)complement within the framework of complete lattices.

**Definition 7.4** *Let* $\mathcal{T} = \langle T, \leq \rangle$ *be a complete lattice. For every* $t \in T$, *its* (pseudo-)complement $t^c$ *is defined by*

$$t^c = \bigvee \{s \in T : s \wedge t = \mathbf{0}_{\mathcal{T}}\}. \tag{7.7}$$

*The complete lattice* $\mathcal{T}$ *is called* semi-Boolean, *if* $t^c \wedge t = \mathbf{0}_{\mathcal{T}}$ *holds for each* $t \in T$ *and it is called* Boolean-like, *if, moreover,* $t^c \vee t = \mathbf{1}_{\mathcal{T}}$ *for every* $t \in T$.

As can be easily seen, the intuition hidden behind the definition of (pseudo-)complement by (7.3) is much more close to the idea of set complement, where the set $X - A$ can be taken as the union of all subsets of $X$ which are disjoint with $A$, than to the numerical operation of abstraction in [0,1]. Indeed, if $\mathcal{T} = \langle [0,1], \leq \rangle$, then $x^c = \bigvee \{y \in [0,1] : x \wedge y = 0\} = 0$, if $x > 0$, and $0^c = 1$, so that $x^c \wedge x = 0$ for each $x \in [0,1]$, but $x^c \vee x < 1$ for every $0 < x < 1$, hence, the complete lattice $\langle [0,1], \leq \rangle$ is semi-Boolean but not Boolean-like. For the complete lattice $\langle \mathcal{P}(X), \subset \rangle$ (pseudo-)complements obviously are identical with the standard set complements, i.e., $A^c = X - A$ for every $A \subset X$ and the complete lattice $\langle \mathcal{P}(X), \subset \rangle$ is Boolean-like.

The above mentioned set-theoretic intuition behind the notion of (pseudo-)complement immediately involves the idea to define this notion in a dual way, i.e., to set, for each $t \in T$

$$t^d = \bigwedge \{s \in T : s \vee t = \mathbf{1}_{\mathcal{T}}\}. \tag{7.8}$$

If $\mathcal{T} = \langle \mathcal{P}(X), \subset \rangle$ (as a matter of fact, if $\mathcal{T}$ is Boolean-like), then $t^d = t^c$ for every $t \in T$, i.e., $A^d = A^c = X - A$ for every $A \subset X$. If $\mathcal{T} = \langle [0,1], \leq \rangle$, then $x^d = 1$ fore every $x < 1$, so that $x^d \neq x^c$ for every $0 < x < 1$. In order to simplify the situation we have decided, rather arbitrarily, to take (7.7) as the definition of (pseudo-)complement in all our further considerations. Because of an important role of the operation of (pseudo-)complement in what follows, a number of properties of this notion will be introduced, and related assertions proved, at the relevant places in the following chapters.

During our investigations concerning complete lattices and their special cases we have arrived very close to the notion of Boolean algebra, as a matter of fact, a Boolean-like Brouwerian complete lattice is a Boolean algebra. Boolean algebras can be taken as an immediate abstraction and generalization of the power-sets over a universe of discourse with the standard operations of union, intersection and complement and with the relation of set inclusion defined at the secondary level. So, Boolean algebra is defined by a set on which two binary and one unary operations are defined with such properties axiomatically imposed that they can be interpreted as those of supremum (set union), infimum (set intersection) and complement. There are many equivalent axiomatic systems giving a mathematical formalization to this intuition behind, let us introduce that one from [42], conserving also the notation which emphasizes the set-theoretic interpretation and intuition behind.

**Definition 7.5** Boolean algebra *is a quadruple $\langle \mathcal{B}, \cup, \cap, - \rangle$, where $\mathcal{B}$ is a nonempty set, $\cup$ and $\cap$ are binary operations taking $\mathcal{B} \times \mathcal{B}$ into $\mathcal{B}$ (i.e., ascribing elements of $\mathcal{B}$ to each pair of such elements), and $-$ is a unary operation taking $\mathcal{B}$ into $\mathcal{B}$, so that the following five axioms are valid for every $A, B, C \in \mathcal{B}$:*

*(i)* $A \cup B = B \cup A,\ A \cap B = B \cap A,$

*(ii)* $A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C,$

*(iii)* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$

*(iv)* $(A \cap B) \cup B = B,\ (A \cup B) \cap B = B,$

*(v)* $(A \cap -A) \cup B = B,\ (A \cup -A) \cap B = B.$

As can be easily proved, $A \cap -A = B \cap -B$ and $A \cup -A = B \cup -B$ holds foe each $A, B \in \mathcal{B}$, so that, setting $\mathbf{0}_{\mathcal{B}} = A \cap -A$ and $\mathbf{1}_{\mathcal{B}} = A \cup -A$, the elements $\mathbf{0}_{\mathcal{B}}$ and $\mathbf{1}_{\mathcal{B}}$ are uniquely defined and are called the *zero* (or the *minimum*) and the *unit* (or the *maximum*) element of the Boolean algebra $\langle \mathcal{B}, \cup, \cap, - \rangle$. Moreover, when introducing a binary relation $\leq$ on $\mathcal{B}$ in such a way that $A \leq B$ holds iff $A \cap B = A$, we can easily prove that $A \leq B$ holds iff $A \cup B = B$ and that $\leq$ is a partial ordering on $\mathcal{B}$ with $\mathbf{0}_{\mathcal{B}}$ and $\mathbf{1}_{\mathcal{B}}$ as the zero and unit elements, i.e., that $\mathbf{0}_{\mathcal{B}} \leq A \leq \mathbf{1}_{\mathcal{B}}$ holds for each $A \in \mathcal{B}$.

# 8 Triangular Semi-Norms and Triangular Norms on Complete Lattices

Even if we have decided, for a number of reasons some of them being very briefly sketched above, to abandon the idea of taking profit of all the rickes of operations over the unit interval of reals when quantifying and processing uncertainties, we still would like to have at our disposal more operations than those of supremum and infimum supposing that we want to process the uncertainties in question within the framework of complete lattices. Instead of thinking over and introducing various more or less sophisticated particular operations on complete lattices, let us define a rather broad class of such binary operations called *triangular semi-norms* and *triangular norms*. If not stated explicitly otherwise, we will suppose that all the complete lattices under consideration are proper in the sense that their support contains at least two elements and the partial ordering relation is nontrivial in the sense that $\mathbf{0}_{\mathcal{T}} \neq \mathbf{1}_{\mathcal{T}}$ *holds.*

**Definition 8.1** *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice. A binary operation $\lambda$ on $T$ (i.e., a mapping which takes $T \times T$ into $T$) is called a* triangular seminorm ($t$-seminorm) *on $\mathcal{T}$, if the following conditions hold:*

*(i) for each $t \in T$, $\lambda(\mathbf{1}_{\mathcal{T}}, t) = \lambda(t, \mathbf{1}_{\mathcal{T}}) = t$,*

*(ii) for each $s_1, s_2, t_1, t_2 \in T$ such that $s_1 \leq s_2$ and $t_1 \leq t_2$ hold simultaneously, also $\lambda(s_1, t_1) \leq \lambda(s_2, t_2)$ holds.*

*A* triangular norm ($t$-norm) *$\lambda$ on $\mathcal{T}$ is a $t$-seminorm on $\mathcal{T}$ which is, furthermore, associative and commutative, i.e., such that for each $t_1, t_2, t_3 \in T$ the relations*

*(iii)* $\lambda(t_1, \lambda(t_2, t_3)) = \lambda(\lambda(t_1, t_2), t_3)$,

*(iv)* $\lambda(t_1, t_2) = \lambda(t_2, t_1)$

*hold.*

Obviously, for each complete lattice $\mathcal{T} = \langle T, \leq \rangle$ the infimum operation $\bigwedge$, induced by $\leq$ on $T$, is the only idempotent $t$-norm on $T$, i.e., such that $t \wedge t = t$ holds for each $t \in T$. Moreover, infimum is the maximal $t$-seminorm (and $t$-norm) on $T$ in the sense that the relation $\lambda(s,t) \leq s \wedge t$ holds for each $t$-seminorm $\lambda$ on $T$ and each $s, t \in T$. Indeed, applying (ii) and (i), we obtain that for every $s, t \in T$, $\lambda(s,t) \leq \lambda(s, \mathbf{1}_{\mathcal{T}}) = s$, $\lambda(s,t) \leq \lambda(\mathbf{1}_{\mathcal{T}}, t) = t$, hence, $\lambda(s,t) \leq s \wedge t$ follows.

A $t$-seminorm $\lambda$ on $\langle T, \leq \rangle$ is called *completely distributive with respect to supremum*, if for each $t \in T$ and each $\emptyset \neq S \subset T$ the identities $\lambda(t, \bigvee S) = \bigvee_{s \in S}(\lambda(t,s))$ and $\lambda(\bigvee S, t) = \bigvee_{s \in S}(\lambda(s,t))$ are valid. Dually, $\lambda$ is called *completely distributive w.r. to infimum*, if for each $t \in T$ and each $\emptyset \neq S \subset T$ the identities $\lambda(t, \bigwedge S) = \bigwedge_{s \in S}(\lambda(t,s))$ and $\lambda(\bigwedge S, t) = \bigwedge_{s \in S}(\lambda(s,t))$ are valid. The triple $\langle T, \leq, \lambda \rangle$, where $\langle T, \leq \rangle$ is a complete lattice and $\lambda$ is a $t$-(semi)norm on $\langle T, \leq \rangle$ which is completely distributive w.r. to supremum is called a *complete lattice with $t$-(semi)norm*. Of course, a complete lattice with $t$-norm $\langle T, \leq, \wedge \rangle$ is a complete Brouwerian lattice.

Let $\langle T, \leq, \lambda \rangle$ be a complete lattice with $t$-seminorm $\lambda$, let $s, t \in T$. An element $\alpha \in T$ is called a *left-inverse for $\lambda$ of $s$ w.r. to $t$*, iff $\lambda(\alpha, s) = t$, and a *right-inverse for $\lambda$ of $s$ w.r. to $t$*, iff $\lambda(s, \alpha) = t$. For a $t$-norm $\lambda$ on $\langle T, \leq \rangle$ left inverses and right inverses coincide due to the commutativity of $\lambda$ and they are simply called *inverses*. As can be easily proved, if $\lambda$ is a $t$-seminorm on $\langle T, \leq \rangle$ and $s, t \in T$ are such that $t \leq s$ does not hold, there are no left- and right inverses for $\lambda$ of $s$ w.r. to $t$, consequently, for the same $s$ and $t$ but with $\lambda$ being a $t$-norm on $\langle T, \leq \rangle$ there are no inverses for $\lambda$ w.r. to $t$.

So, a $t$-seminorm $\lambda$ on $\langle T, \leq \rangle$ is called *weakly left-invertible*, if for any $s, t \in T$ such that $t \leq s$ holds there exists a left-inverse for $\lambda$ of $s$ w.r. to $t$. The definition of weak right-invertibility is completely similar. A $t$-norm $\lambda$ on $\langle T, \leq \rangle$ is called *weakly invertible*, iff for any $s, t \in T$ such that $t \leq s$ holds there exists an inverse for $\lambda$ of $s$ w.r. to $t$.

In the rest of this chapter we assume that $\langle T, \leq \rangle$ is a complete lattice, $\nu$ is a $t$-seminorm on $\langle T, \leq \rangle$ and $\lambda$ is a $t$-norm on $\langle T, \leq \rangle$. As proved in [6], there exists an important relation between the notion of a weak (left- and right-)inverse and the order-theoretic notion of *residuation* (cf. [2]). Let $s$ and $t$ be any elements of $T$. The *left-residual* $s \lhd_\nu t$ *for* $\nu$ *of* $s$ *by* $t$ is defined by

$$s \lhd_\nu t = \bigvee \{u : u \in T,\ \nu(u,t) \leq s\}. \tag{8.1}$$

For the right-residual $t \rhd_\nu s$ for $\nu$ of $s$ by $t$ a similar definition can be given. For the $t$-norm $\lambda$ the notions of left- and right-residual coincide and are simply called residual and denoted by $s\triangle_\lambda t$ for $\lambda$ of $s$ by $t$, hence, $s\triangle_\lambda t = s \lhd_\lambda t = t \rhd_\lambda s$. In [6], the following propositions are proved.

**Lemma 8.1** *Let $s, t$ be elements of $T$.*

*(i) If the equation $\nu(u,s) = t$ in $u$ admits a solution, then $t \lhd_\nu s$ is the greatest solution w.r. to the order relation $\leq$, and if the equation $\nu(s,u) = t$ in $u$ admits a solution, then $s \rhd_\nu t$ is the greatest solution w.r. to the order relation $\leq$.*

*(ii) If the equation $\lambda(u,s) = t$ in $u$ admits a solution, then $t\triangle_\lambda s$ is the greatest solution w.r. to the order relation $\leq$.*

**Lemma 8.2** *(i) $\nu$ is weakly left-invertible iff, for all $s, t \in T$, the implication*

$$s \geq t \Rightarrow \nu(t \lhd_\nu s, s) = t \tag{8.2}$$

*holds, and $\nu$ is weakly right-invertible iff, for all $s, t \in T$, the implication*

$$s \geq t \Rightarrow \nu(s, s \rhd_\nu t) = t \tag{8.3}$$

*holds*

*(ii) $\lambda$ is weakly invertible iff, for all $s, t \in T$, the implication*

$$s \geq t \Rightarrow \lambda(t\triangle_\lambda s, s) = t \tag{8.4}$$

*holds.*

As an example, let us consider the unit interval [0 ,1] of real numbers with their standard ordering relation $\leq$. The structure $\langle < 0, 1 >, \leq \rangle$ is a complete chain, moreover, it is a complete Brouwerian chain, i.e., $\langle [0,1], \leq, min \rangle$ is a complete chain with $t$-norm *min*, where *min* is the standard minimum in $\langle [0,1], \leq \rangle$. As can be easily verified, for each $x, y \in [0,1]$

$$x\triangle_{min} y = sup\{z : z \in [0,1],\ \min\ (z,y) \leq x\} = 1,\ \text{if } y \leq x, \tag{8.5}$$

$$x\triangle_{min} y = x,\ \text{if } y > x. \tag{8.6}$$

Alternatively, we may consider the algebraic product operator $x$ on [0,1] which clearly defines a triangular norm on $\langle [0,1], \leq \rangle$ that is completely distributive w.r. to supremum. In other words, the structure $\langle [0,1], \leq, \times \rangle$ is a complete chain with $t$-norm, and for any $x, y \in [0,1]$ we obtain that

$$x\triangle_\times y = sup\{z : z \in [0,1], z \times y \leq x\} = 1, \tag{8.7}$$

if $y \leq x$, and

$$x\triangle_t imesy = x/y, \tag{8.8}$$

if $y > x$. Moreover,

$$(x\triangle_\times y)xy = \ \min\ (x,y), \tag{8.9}$$

which implies that $\times$ is also weakly invertible.

As another simple example let us consider the power-set $\mathcal{P}(X)$ of all subsets of a nonempty set $X$, partially ordered by the relation $\subset$ of set inclusion. The structure $\langle \mathcal{P}(X), \subset \rangle$ is obviously a complete lattice with infimum defined by the set conjuction $\cap$ which is a $t$-norm on $\langle \mathcal{P}(X), \subset \rangle$. For each $A, B \subset X$ we obtain that

$$A \triangle_{\cap} B = \bigcup \{C : C \subset X,\ C \cap B \subset A\} = B^C \cup A, \tag{8.10}$$

in particular, $A \triangle_{\cap} B = X$ iff $B \subset A$ holds. Indeed, if this is not the case, i.e., if there exists $x \in B - A$, then for each $C \subset X,\ x \in C$ we obtain that $x \in C \cap B$, hence, the inclusion $C \cap B \subset A$ does not hold.

# 9 Lattice-Valued Possibilistic Measures and Integration - the Most Elementary Notions

More or less copying [10], let us introduce here the notion of possibilistic measure with values in a complete lattice, and the definition of integration with respect to such possibilistic measures. In the following chapters, both these notions will serve as a starting point for their various extensions, modifications and weakenings, referring to [10] when some more sophisticated properties of these measures and integrals are concerned.

**Definition 9.1** *Let $\Omega$ be a nonempty set, let $\mathcal{A}$ be an ample field of subsets of $\Omega$, let $\mathcal{T}\langle T, \leq \rangle$ be a complete lattice with the zero element $\mathbf{0}_{\mathcal{T}}(= \bigwedge T)$ and the unit element $\mathbf{1}_{\mathcal{T}}(= \bigvee T)$. A mapping $\Pi : \mathcal{A} \to T$ is called a $\mathcal{T}$-*(valued) possibilistic (or possibility) measure on $< \langle Om, \mathcal{A} \rangle$*, if for any system $\mathcal{R} \subset \mathcal{A}$ of subsets of $\Omega$ the relation*

$$\Pi \left( \bigcup \mathcal{R} \right) = \bigvee \{\Pi(A) : A \in \mathcal{R}\} \tag{9.1}$$

*holds. The $\mathcal{T}$-possibility measure $\Pi$ on $\langle \Omega, \mathcal{A} \rangle$ is called* normalized*, if $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$, here $\bigcup \mathcal{R} = \bigcup_{A \in \mathcal{R}} A$. The triple $\langle \Omega, \mathcal{A}, \Pi \rangle$ is called $\mathcal{T}$-*possibilistic (or possibility) space.

Using different terms we could say that $\Pi$ is a possibilistic measure on $\langle \Omega, \mathcal{A} \rangle >$, if it is a complete join-morphism between the complete lattices $\langle \mathcal{A}, \subset \rangle$ and $\langle T, \leq \rangle$. When supposing (9.1) to hold, the case when $\mathcal{R} = \emptyset$ is not excluded, hence, by convention, $\bigcup \mathcal{R} = \emptyset$ and $\bigvee \{\Pi(A) : A \in \emptyset\} = \mathbf{0}_{\mathcal{T}}$ so that the relation $\Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}$ follows (more often it is introduced as an independent axiom imposed on $\Pi$).

**Definition 9.2** *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\langle \Omega, \mathcal{A}, \Pi \rangle$ be a $\mathcal{T}$-possibilistic space. A mapping $\pi : \Omega \to T$ is the $\mathcal{T}$-*possibilistic distribution on $\Omega$ induced by $\Pi$*, if for each $t \in T$*

$$\{\omega \in \Omega : \pi(\omega) = t\} \in \mathcal{A} \tag{9.2}$$

*holds, and if for each $A \in \mathcal{A}$ the identity*

$$\Pi(A) = \bigvee \{\pi(\omega) : \omega \in A\} \tag{9.3}$$

*is valid.*

*Obviously, such a $\mathcal{T}$-possibilistic distribution is defined uniquely and for every $\omega \in \Omega$ the relation $\pi(\omega) = \Pi([\omega]_{\mathcal{A}})$ holds.*

Lattice-valued possibilistic measures are generalizations towards more general domains and codomains of Zadeh's real-valued possibilistic measures [46], Wang's fuzzy contactibilities [44], the possiblistic measures studied by Wang and Klir [45], and the possibilistic measures introduced in [3]. For a more detailed discussion on these generalizations let us refer to [7], [5], [3] or to the surveyal work [10]. The introduction of lattice-valued possibilistic measures can be justified as follows. Using a complete lattice as a codomain (i.e., as the domain in which possibility degrees take their value) allows us to model potential incomparability of possibility degrees as far as their sizes are concerned, and for instance to associate possibilistic measures with the $\langle T, \leq \rangle$ - fuzzy sets introduced by Goguen in [19], in order to represent more general forms of linguistic uncertainty [8], [46].

When abandoning the idea of possibilistic measure as a total function defined on the power-set $\mathcal{P}(\Omega)$, i.e., for each $A \subset \Omega$, in favour of possibilistic measure as a partial function, the choice of ample field as the most appropriate structure on which possibilistic measures should be defined can be briefly justified as follows. Actually, we could call a mapping from an arbitrary subsystem $\mathcal{E} \subset \mathcal{P}(\Omega)$ to $T$ a $\langle T, \leq \rangle$ - possibilistic measure, if it is extendable to a $\langle T, \leq \rangle$ - possibilistic measure on $\mathcal{P}(\Omega)$. As proved in [3], this extendability is equivalent to the extendability to a $\langle T, \leq \rangle$ - possibilistic measure on $\tau(\mathcal{E})$, where $\tau(\mathcal{E})$ denotes the smallest ample field which includes $\mathcal{E}$. Therefore, ample fields arise naturally as domains of possibilistic measures. Generally speaking, if we want to define a possibilistic measure on $\mathcal{P}(\Omega)$ we have to be more specific than if we want to define one on any other ample field $\mathcal{A} \subset \mathcal{P}(\Omega)$, since, e.g., the atoms of $\mathcal{P}(\Omega)$ constitute a refinement of the atoms of $\mathcal{A}$. So, at least in principle, we want to be able to be as nonspecific as possible, and introduce possibilistic measures rather on ample fields than on power-set.

Later on in this work we will investigate various generalizations of lattice-valued possibilistic measures, either in the sense that their definition domain is not necessarily an ample field but simply a nonempty system of subsets of $\Omega$, or in the sense that the structure over the set of values, i.e., $\mathcal{T} = \langle T, \leq \rangle$, need not be a complete lattice, but only a weaker structure, e.g., a partially ordered set. However, for the moment let us return our attention to the $\mathcal{T}$-valued possibilistic measures in the sense of Definition 9.1. and let us define the notion of possibilistic integral with respect to such possibilistic measures, still copying [10].

In [9], De Cooman and Kerre argued in favour of the idea that a generalization of Sugeno's fuzzy integral, the so called semi-normed fuzzy integral, is ideally suited for combination with $\langle T, \leq \rangle$ - valued possibilistic measures. In [10], De Cooman explores this idea in more detail and shows that these semi-normed fuzzy integrals can be used to give a consistent and unifying account of possibility theory. The most important points of the argumentation from [10] and the resulting formulas can be briefly sketched as follows.

**Definition 9.3** *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\nu$ be a t-semi-norm on $\langle T, \leq \rangle$, let $\langle \Omega, \mathcal{A}, \Pi \rangle$ be a $\mathcal{T}$-possibilistic space such that $\pi$ denotes the $\mathcal{T}$-possibilistic distribution induced by $\Pi$. Let $A$ be an $\mathcal{A}$-measurable subset of $\Omega$ (i.e., $A \in \mathcal{A}$), let $h : \Omega \to T$ be an $\mathcal{A}$-measurable mapping (i.e., $\{\omega \in \Omega : h(\omega) = t\} \in \mathcal{A}$ for each $t \in T$). Then the $\langle T, \leq, \nu \rangle$ -* possibilistic (or possibility) integral of $h$ on $A$ with respect to $\Pi$ *is defined by*

$$(\nu) \int_A h d\Pi = \bigvee \{\nu(h(x)\pi(x)) : x \in A\}. \tag{9.4}$$

In particular, if $t \in T$ and $\underline{t}$ is the constant mapping on $\Omega$, i.e., $\underline{t}(\omega) = t$ for each $\omega \in \Omega$, then obviously

$$(\nu) \int_A \underline{t} d\Pi = \nu(t, \Pi(A)), \tag{9.5}$$

hence, for $t = \mathbf{1}_{\mathcal{T}}$,

$$(\nu) \int_A \mathbf{1}_{\mathcal{T}} d\Pi = \Pi(A). \tag{9.6}$$

Let $\chi_A$ be the $\mathcal{T}$-valued characteristic function (identifier) of the set $A \subset \Omega$, i.e., $\chi_A(\omega) = \mathbf{1}_{\mathcal{T}}$, if $\omega \in A$, $\chi_A(\omega) = \mathbf{0}_{\mathcal{T}}$, $\omega \in \Omega - A$, let $\wedge$ be the infimum in $\langle T, \leq \rangle$, let $(h \wedge \chi_A)(\omega) = h(\omega) \wedge \chi_A(\omega)$ for each $\omega \in \Omega$. Then we can easily prove that

$$(\nu) \int_A h d\Pi = (\nu) \int_\Omega (h \wedge \chi_A) d\Pi. \tag{9.7}$$

Consequently,

$$(\nu) \int_\Omega \chi_A d\Pi = (\nu) \int_A \mathbf{1}_{\mathcal{T}} d\Pi = \Pi(A). \tag{9.8}$$

Moreover, let $\{h_j : j \in J\}$ be a system of $\mathcal{A}$-measurable mappings each of them taking $\Omega$ into $T$ (i.e., a system of $\mathcal{T}$-fuzzy variables on $\langle \Omega, \mathcal{A} \rangle$ or a system of $\mathcal{T}$-fuzzy subsets of $\Omega$), let $\bigvee_{j \in J]} h_j : \Omega \to T$ be defined by $(\bigvee_{j \in J} h_j)(\omega) = \bigvee\{h_j(\omega) : j \in J]\}$ for each $\omega \in \Omega$. Then

$$(\nu) \int_X \left( \bigvee_{j \in J} h_j \right) d\Pi = \bigvee\{(\nu) \int_X h_j d\Pi : j \in J\}, \tag{9.9}$$

cf. [10] for the proof. Hence, denoting by $\mathcal{G}_{\mathcal{T}}^{\mathcal{A}}(\Omega)$ the set of all $\mathcal{T}$-fuzzy variables on $\langle \Omega, \mathcal{A}, \rangle$, and setting, for each $h \in \mathcal{G}_{\mathcal{T}}^{\mathcal{A}}(\Omega)$,

$$\Pi_\nu(h) = (\nu) \int_\Omega h d\Pi = \bigvee\{\nu(h(\omega), \pi(\omega)) : \omega \in \Omega\}, \tag{9.10}$$

(9.9) yields that $\Pi_\nu$ is a complete join-morphism between the complete lattices $\langle \mathcal{G}_{\mathcal{T}}^{\mathcal{A}}(\Omega), \sqsubseteq \rangle$ and $\langle T, \leq \rangle$, where $h_1 \sqsubseteq h_2$ means that $h_1(\omega) \leq h_2(\omega)$ holds for each $\omega \in \Omega$. Hence, $\Pi_\nu$ behaves like a $\mathcal{T}$-possibilistic measure which can be called a $\nu$-*extension* of $\Pi$ from $\mathcal{A}$ to $\mathcal{G}_{\mathcal{T}}^{\mathcal{A}}(\Omega)$.

# 10 Possibilistic Variables

Also in this chapter, when introducing the notion of possibilistic variable which should play, within the framework of possibility theory, the same role as random (or stochastic) variables in standard probability theory, we follow the pattern of explanation from [10]. In probability theory, and in measure theory in general, it is possible to transfer a (probability) measure from one universe to another, using a mapping between these universes [4]. It is not difficult to show that something similar can be done with possibilistic measures.

Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $Y$ be a nonempty set and let $f : \Omega \to Y$ be a mapping, where $< \Omega, \mathcal{A}, \Pi_1 >$ is a $\langle T, \leq \rangle$-possibilistic space. Let us define the system $\mathcal{A}_1^{(f)}$ of subsets of $Y$ and the mapping $\Pi_1^{(f)} : \mathcal{A}_1^{(f)} \to T$ as follows:

$$\mathcal{A}_1^{(f)} = \{B : B \subset Y,\ f^{-1}(B) \in \mathcal{A}\},\ f^{-1}(B) = \{\omega \in \Omega : f(\omega) \in B\}, \tag{10.1}$$

$$\Pi_1^{(f)}(B) = \Pi_1(f^{-1}(B)),\ B \in \mathcal{A}_1^{(f)}. \tag{10.2}$$

As can be easily seen, $\mathcal{A}_1^(f)$ is an ample field of subsets of $Y$ and $\Pi_1^{(f)}$ is a $\langle T, \leq \rangle$-possibilistic measure on $\langle Y, \mathcal{A}_1^{(f)} \rangle$ which can be called the *transformed* $\langle T, \leq \rangle$*-possibilistic measure on* $\langle Y, \mathcal{A}_1^{(f)} \rangle$ *induced by* $\Pi_1$ *and* $f$. Moreover, $\Pi_1$ is normalized iff $\Pi_1^{(f)}$ is. The $\mathcal{T}$-possibilistic distribution $\pi_1^{(f)}$ of $\Pi_1^{(f)}$ satisfies, for any $x \in Y$, the relation

$$\pi_1^{(f)}(x) = \bigvee\{\pi_1(x_1) : f(x_1) \in [x]_{\mathcal{A}_1^{(f)}}\}, \tag{10.3}$$

when $\pi_1$ is the $\mathcal{T}$-possibilistic distribution of $\Pi_1$.

In what follows, let us apply the very general definition of measurability of mappings by Jacobs [21].

**Definition 10.1** *Let* $Y_1, Y_2$ *be nonempty sets, let* $\mathcal{B}_1 \subset \mathcal{P}(Y_1),\ \mathcal{B}_2 \subset \mathcal{P}(Y_2)$ *be systems of subsets of the corresponding sets. A mapping* $f : Y_1 \to Y_2$ *is called* $\mathcal{B}_1 - \mathcal{B}_2$-measurable*, if the inclusion*

$$\{\{x \in Y_1 : f(x) \in B_2\} : B_2 \in \mathcal{B}_2\} \subset \mathcal{B}_1 \tag{10.4}$$

*holds, i.e., if the inverse image, w.r. to* $f$*, of each set from* $\mathcal{B}_2$ *belongs to* $\mathcal{B}_1$*.*

Applying this definition to the case above, when $Y_1 = \Omega,\ Y_2 = Y,\ \mathcal{B}_2 = \mathcal{A},\ \mathcal{B}_2 = \mathcal{A}_1^{(f)}$ and $f : \Omega \to Y$ is a mapping, we obtain easily that $f$ is $\mathcal{A} - \mathcal{A}_1^{(f)}$-measurable. Let $\mathcal{Y}$ be an ample fieled of subsets of $Y$. If $f$ is $\mathcal{A} - \mathcal{Y}$-measurable, it follows immediately that the inclusion $\mathcal{Y} \subset \mathcal{A}_1^{(f)}$ holds, in other terms, $\mathcal{A}_1^{(f)}$, is the greatest w.r. to inclusion ample field of subsets of $Y$ such that $f$ is $\mathcal{A}-\mathcal{Y}$-measurable.

Hence, let $\mathcal{Y}$ be an ample field of subsets of the set $Y$, let $f : \Omega \rightarrow Y$ be an $\mathcal{A}-\mathcal{Y}$-measurable mapping. As $\Pi_1^{(f)}$ is a $\langle T, \leq\rangle$-possibilistic measure on $\langle Y, \mathcal{A}_1^{(f)}\rangle$ and $\mathcal{Y} \subset \mathcal{A}_1^{(f)}$ holds, we may consider the restriction of $\Pi_1^{(f)}$ to $\mathcal{Y}$ ($\Pi_1^{(f)}) \upharpoonright \mathcal{Y}$, in symbols). It is a $\langle T, \leq\rangle$-possibilistic measure on $\langle Y, \mathcal{Y}\rangle$ which can be called the *transformed* $\langle T, \leq\rangle$*-possibilistic measure on* $\langle Y, \mathcal{Y}\rangle$ induced by $\Pi_1$ and $f$. For any $B \in \mathcal{Y}$ we obtain that

$$(\Pi_1^{(f)} \upharpoonright \mathcal{Y})(B) = \Pi_1(f^{-1}(B)) \tag{10.5}$$

and the $\mathcal{T}$-possibilistic distribution $\pi_2$ of $\Pi_1^{(f)} \upharpoonright \mathcal{Y}$ satisfies, for each $x \in Y$ the relation

$$\pi_2(x) = \bigvee\{\pi_1(x_1) : f(x_1) \in [x]_{\mathcal{Y}}\}. \tag{10.6}$$

Clearly, $\mathcal{A}_1^{(f)}$ is the greatest w.r. to inclusion ample field of subsets of $Y$ on which a transformed $\langle T, \leq\rangle$-possibilistic measure induced by $\Pi_1$ and $f$ can be defined.

Informally, a variable may be defined as an abstract object (entity) which can take values in a certain universe. Hence, let us consider a universe $Y$ and a variable $X$ in $Y$ of the type just informally described. In general, $X$ can take any value in $Y$, however, it is possible to restrict the values which $X$ may take in $Y$, e.g., saying that only values in a proper subset $D$ of $Y$ can be assumed by $X$. By imposing such a restriction the uncertainty concerning the value actually taken by $X$ is reduced, so that we get more information about the value taken by $X$ in $Y$.

In general, such information need not always be so strict as giving a subset $D$ of $Y$, e.g., it can be given in the form of a probability measure (a detailed discussion can be found in [13]). More explicitly, we consider a $\sigma$-field $\mathcal{S}$ of subsets of $Y$ and a probability measure $Pr$ on the measurable space $\langle Y, \varphi\rangle$, supposing that for any $A \in \mathcal{S}$, $Pr(A)$ is the probability with which the variable $X$ takes a value in $A$. When doing so, we also impose a restriction on the values which can $X$ take in $Y$, but this restriction is more flexible than that specified by a subset $D$ of $Y$ and it could be said that the information we have about the values taken by $X$ in $Y$ is probabilistic or stochastic.

In probability theory, the formalized mathematical model describing what we have just informally explained goes as follows. First of all, a basic space $\Omega$ is considered together with a nonempty $\sigma$-field of subsets $\mathcal{S}_\Omega$, the elements of $\Omega$ are called *elementary random events* and the subsets of $\Omega$ which are in $\mathcal{S}_\Omega$ are *random events*. Then a probability measure $Pr_\Omega$ on $\mathcal{S}$ is introduced, so arriving at the notion of *probability space* $\langle \Omega, \mathcal{S}_\Omega, Pr_\Omega\rangle$, the basic stone of the axiomatic mathematical probability theory. A mapping $X : \Omega \rightarrow Y$ is called *random or stochastic variable* defined on $\langle \Omega, Pr_\Omega\rangle$ and taking into values in $\langle Y, \mathcal{S}\rangle$, if it is $\mathcal{S}_\Omega - \mathcal{S}$-measurable, i.e., if the inclusion

$$\{\{\omega \in \Omega : X(\omega) \in B\} : B \in \mathcal{S}\} \subset \mathcal{S}_\Omega \tag{10.7}$$

holds. The universe $Y$ is then called the *sample space* and, for each $A \subset Y,\ A \in \mathcal{S}$, the probability with which the random (stochastic) variable $X$ takes its values in $A$ is given by

$$Pr_\Omega(X^{-1}(A)) = Pr_\Omega(\{\omega \in \Omega : X(\omega) \in A\}). \tag{10.8}$$

If the information concerning the values taken by a variable $X$ and being at our disposal is given by a possibilistic measure, the situation can be processed quite analogously as in the probabilistic case, on the intuitive as well as on the formalized level, let us limit ourselves to the explicit introduction of the resulting formalized model.

**Definition 10.2** *Let* $\mathcal{T} = \langle T, \leq\rangle$ *be a complete lattice, let* $\Omega$ *be a nonempty set, let* $\mathcal{A}$ *be an ample field of subsets of* $\Omega$*, let* $\Pi$ *be a* $\mathcal{T}$*-possibilistic measure on* $\mathcal{A}$*, let* $Y$ *be a nonempty set, let* $\mathcal{Y}$ *be an ample field of subsets of* $Y$*. A mapping* $X : \Omega \rightarrow Y$ *is called a* possibilistic variable defined on the $\mathcal{T}$-possibilistic space $\langle \Omega, \mathcal{A}, \Pi\rangle >$ and taking its values in the sample space $\langle Y, \mathcal{Y}\rangle$, *if it is* $\mathcal{A}-\mathcal{Y}$ *measurable, i.e., if (10.7) holds with* $\mathcal{S}$ *replaced by* $\mathcal{Y}$ *and* $\mathcal{S}_\Omega$ *replaced by* $\mathcal{A}$*.*

# 11 Partial Real-Valued Monotone and Possibilistic Measures and Their Extensions

It is not our aim, in this work, to repeat the ideas, constructions and results contained in [10] or elsewhere and dealing with the formalizations of lattice-valued possibilistic measures and their analogies, formal as well as perhaps semantic, with the axiomatic probability theory, even if we are ready to take profit of [10] or of other related works in any case when such a reference seems to be necessary or adequate. Hence, let us return, for a while, to Chapters 2 and 3 above when pursued our path leading from uncertainty as informal phenomenon from the surrounding us world to mathematical models for uncertainty quantification and processing.

Namely, let us focus our attention to the stage of our reasoning when we arrived at the idea of elementary random events and the space $\Omega$ of all such events and when we agreed to take random events as sets of elementary random events, i.e., as subsets of $\Omega$, and to take the degrees of uncertainties ascribed to random events as values of set functions ascribing to subsets of $\Omega$ real numbers from the unit interval [0,1]. These set functions should obey some conditions, different in the case of probabilities from those in the case of possibilities, but in both the cases meeting the most general properties behind the intuitive idea of size of sets; these common properties were formalized in the notion of monotone measures.

In the case of finite spaces $\Omega$ and with uncertainty taken in the sense of randomness, the obtained formalization completely and conservatively met and extended the common sense, ideas and results of the most elementary combinatoric probability theory describing our everyday's experience with coins and dices tossing or with various simple lotteries. The relation between a random events and subsets of $\Omega$ can be taken as one-to-one, i.e., *every* subset of $\Omega$ defines random event. Real numbers from [0,1] as degrees of probabilities (uncertainties in the sense of randomness) also seem to be quite natural and intuitive, as these degrees can be approximated by relative frequences of successes taken from sufficiently large collections of statistically independent random samples, and values of relative frequences obviously belong to the real interval [0,1].

As a matter of fact, still keeping in mind probability theory but admitting also infinite and, in particular, uncountable spaces $\Omega$ of elementary random events, the situation significantly changes. Namely, it is not reasonable to expect that every set function, perhaps satisfying the demands imposed on probability measure when considering only certain subsets of $\Omega$ can be conservatively and consistently extended to probability measure on the whole power-set $\mathcal{P}(\Omega)$ of all subsets of $\Omega$. E.g., the well-known and intuitive probability measure on the subsets of the unit interval [0,1] of real numbers, ascribing ins length $b - a$ to any interval $[a, b]$ of [0,1] can be extended in a uniquely defined, consistent and conservative way just to the $\sigma$-field of Lebesque subsets of [0,1], but there exist also subsets of [0,1] not measurable in this sense (cf. [20] or any elementary textbook on measure theory or mathematical analysis). The solution to this problem accepted by the axiomatic probability theory is well-known: probability measure is defined as *partial* function on $\mathcal{P}(\Omega)$ the values of which are defined only for *some* subsets $A \subset \Omega$. The system $\mathcal{A}$ of such subsets, i.e., the definition domain of probability measure $P$ in question, is supposed to define a $\sigma$-field of subsets of $\Omega$ because of obvious practical reasons and this $\sigma$-field occurs among the most fundamental stones defining the notion of probability space.

The real-valued possibilistic measures were also conceived as total set functions taking $\mathcal{P}(\Omega)$ into [0,1] and the cardinality of the space $\Omega$ did not play such an important role as in the case of probability measures. Indeed, the operation of supremum applies to any nonempty subset of [0,1] and for the empty subset of [0,1] the convention sup $(\emptyset) = 0$ can be accepted. Hence, the idea to define also possibilistic measures as *partial* functions on $\mathcal{P}(\Omega)$ reflected rather our limited abilities to obtain the detailed empirical values of possibilistic measures in various cases of their applications. In the case of possibilistic measures, however, another way of generalization was more inspirative and general: to abandon the idea that uncertainty degrees are real numbers from the unit interval in favor of taking them as elements of some more general, even if not so rich as the unit interval, structure, e.g., a partially ordered set or, in particular, a complete lattice. Some arguments supporting this choice were already briefly mentioned above referring to [10], [19] and to other relevant sources for more detail. We will also return our attention from time to time, in what follows, to this point.

Let us turn back to real-valued set functions taking their values in the unit interval [0,1]. Perhaps the most general and still nontrivial mappings of this kind, reflecting at least the weakest properties of numerically quantified sizes of sets are real-valued monotone measures in the sense of Definition 3.1 above. Denoting by $\mathcal{R}$ general subsystems of $\mathcal{P}(\Omega)$ with the aim to use $\mathcal{A}$ only for $\sigma$-fields or ample fields, the definition can be re-called as follows. Given a nonempty set $\Omega$ and a nonempty system $\mathcal{R}$ of subsets of $\Omega$, a mapping $\varphi : \mathcal{R} \to [0,1]$ is called a *real-valued partial normalized monotone measure* on $\mathcal{R}$, if $\varphi(\emptyset) = 0$ and/or $\varphi(\Omega) = 1$ supposing that $\emptyset \in \mathcal{R}$ and/or $\Omega \in \mathcal{R}$, and if the inequality $\varphi(A) \leq \varphi(B)$ holds for each $A \subset B, A, B \in \mathcal{R}$. The *inner (or lower) monotone measure $\varphi_\star$ induced on $\mathcal{P}(\Omega)$ by $\varphi$* is defined, for each $A \subset \Omega$ by

$$\varphi_\star(A) = \bigvee\{\varphi(B) : B \subset A,\ B \in \mathcal{R}\}. \tag{11.1}$$

*The outer (or upper) monotone measure $\varphi^\star$ induced on $\mathcal{P}(\Omega)$ by $\varphi$* is defined for each $A \subset \Omega$, by

$$\varphi_\star(A) = \bigwedge\{\varphi(B) : B \supset A,\ B \in \mathcal{R}\}. \tag{11.2}$$

Let us recall that $\bigwedge$ denotes the infimum and $\bigvee$ the supremum defined in [0,1] by its standard linear ordering and let us apply the conventions $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$ for the empty subset of [0,1], then $\varphi^\star$ and $\varphi_\star$ are defined on whole the system $\mathcal{P}(\Omega)$. A set $A \subset \Omega$ is called *$< \varphi, \mathcal{R} >$-measurable*, if $\varphi_\star(A) = \varphi^\star(A)$.

E.g., in the most simple case with $\mathcal{R} = \{\emptyset, \Omega\}$, the only monotone measure on $\mathcal{R}$ is such that $\varphi(\emptyset) = 0$ and $\varphi(\Omega) = 1$. Then $\varphi^\star(A) = \varphi(\emptyset) = 0$ for each $A \subset \Omega,\ A \neq \Omega$, and $\varphi^\star(A) = 1$ for each $\emptyset \neq A \subset \Omega$, so that the only $< \varphi, \mathcal{R} >$-measurable sets are $\emptyset$ and $\Omega$.

**Lemma 11.1** *Let $\{\emptyset, \Omega\} \subset \mathcal{R} \subset \mathcal{P}(\Omega)$, let $\varphi$ be a monotone measure on $\mathcal{R}$. Then both $\varphi_\star$ and $\varphi^\star$ are monotone measures on $\mathcal{P}(\Omega)$, conservatively extending $\varphi$ from $\mathcal{R}$ to $\mathcal{P}(\Omega)$.*

**Proof 11.1** *The assertions follow immediately from the definitions of $\varphi_\star$ and $\varphi^\star$ and from the elementary properties of the operation of supremum and infimum in [0,1].*

Let us focus our attention to a partial case of partial monotone measures, namely to normalized real-valued partial possibilistic measures. The formal definition will read as follows.

**Definition 11.1** *Let $\Omega$ be a nonempty set, let $\{\emptyset, \Omega\} \subset \mathcal{R} \subset \mathcal{P}(\Omega)$ be a system of subsets of $\Omega$. A mapping $\Pi : \mathcal{R} \to [0,1]$ is called a* normalized real-valued partial possibilistic measure *on $\mathcal{R}$, if $\Pi(\emptyset) = 0,\ \Pi(\Omega) = 1$ and $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ for each $A, B \in \mathcal{R}$ such that $A \cup B \in \mathcal{R}$ as well. The partial possibilistic measure $\Pi$ on $\mathcal{R}$ is* finitely complete*, if for each* finite *system $\mathcal{R}_0 \subset \mathcal{R}$ of subsets of $\Omega$ such that $\bigcup \mathcal{R}_0 = \bigcup_{A \in \mathcal{R}_0} A$ is in $\mathcal{R}$ the equality $\Pi(\bigcup \mathcal{R}_0) = \bigvee\{\Pi(A) : A \in \mathcal{R}_0\}$ holds. Supposing that this equality holds for* each *nonempty $\mathcal{R}_0 \subset \mathcal{R}$ such that $\bigcup \mathcal{R}_0 \in \mathcal{R}$, the partial possibilistic measure $\Pi$ on $\mathcal{R}$ is called* complete.

In the case of possibilistic measure defined on $\mathcal{P}(\Omega)$ or on a rich enough systems of subsets of $\Omega$ each possibilistic measure is obviously finitely complete (in the case of $\mathcal{P}(\Omega)$ even complete), but in general this need not be the case. Indeed, take $\mathcal{R} = \{\emptyset, A_1, A_2, A_3, \Omega\}$, where $A_1, A_2, A_3$ defines a disjoint covering of $\Omega$ by nonempty sets. Then each $\Pi : \mathcal{R} \to [0,1]$ such that $\Pi(\emptyset) = 0$ and $\Pi(\Omega) = 1$ holds defines a partial possibilistic measure on $\mathcal{R}$ no matter whether the relation $\Pi(A_1) \vee \Pi(A_2) \vee \Pi(A_3) = 1 = \Pi(\Omega)$ holds, hence, $\Pi$ need not be finitely complete.

It is perhaps worth stating explicitly that the definitions of $\varphi_\star$ and $\varphi^\star$, given a partial monotone measure on $\mathcal{R} \subset \mathcal{P}(\Omega)$, cannot be "refined" or "improved" when replacing the simple values $\varphi(B)$ for $B \subset A - (B \supset A$, resp.) and $B \in \mathcal{R}$ by the sumprema (infima, resp.) of values taken over systems $\mathcal{C} \subset \mathcal{R}$ such that $\bigcup \mathcal{C} (= \bigcup_{C \in \mathcal{C}} C) \subset A (\bigcap \mathcal{C} \supset \mathcal{A}$, resp.) but not necessarily $\bigcup \mathcal{C} \in \mathcal{R} (\bigcap \mathcal{C} \in \mathcal{R}$, resp.). In symbols, we can easily prove that

$$\varphi_\star(A) = \bigvee \left\{ \bigvee_{B \in \mathcal{C}} \varphi(B) : \mathcal{C} \subset \mathcal{R},\ \bigcup \mathcal{C} \subset A \right\} \tag{11.3}$$

and

$$\varphi^\star(A) = \bigwedge \left\{ \bigwedge_{B\in\mathcal{C}} \varphi(B) : \mathcal{C} \subset \mathcal{R},\ \bigcap \mathcal{C} \supset A \right\}. \tag{11.4}$$

**Lemma 11.2** *Let $\Pi$ be a possibilistic measure defined on an upper semilattice of subsets of $\Omega$ such that $\{\emptyset, \Omega\} \subset \mathcal{R}$. Then $\Pi^\star$ is a possibilistic measure on $\mathcal{P}(\Omega)$.*

**Proof 11.2** *Given $A, B \subset \Omega$, Lemma 11.1 yields that*

$$\Pi^\star(A) \vee \Pi^\star(B) \leq \Pi^\star(A \cup B) \tag{11.5}$$

*holds. For each $\varepsilon > 0$ there exist $A_\varepsilon,\ B_\varepsilon \in \mathcal{R}$ such that $A_\varepsilon \supset A, B_\varepsilon \supset B$ and inequalities $\Pi(A_\varepsilon) < \Pi^\star(A)+\varepsilon,\ \Pi(B_\varepsilon) < \Pi^\star(B)+\varepsilon$ are valid. As $\mathcal{R}$ is an upper semilattice, $A_\varepsilon \cup B_\varepsilon \in \mathcal{R},\ A_\varepsilon \cup B_\varepsilon \supset A \cup B$, so that an easy calculation yields that*

$$\Pi^\star(A \cup B) \leq \Pi(A_\varepsilon \cup B_\varepsilon) = \Pi(A_\varepsilon) \vee \Pi(B_\varepsilon) < (\Pi^\star(A) \vee \Pi^\star(B)) + \varepsilon. \tag{11.6}$$

*The assertion follows immediately from (11.5) and (11.6).*

As a matter of fact, an analogous assertion or $\Pi_\star$ does not hold. Indeed, take $\Omega \neq \emptyset,\ \emptyset \neq C \subset \Omega,\ C \neq \Omega,\ \emptyset \neq A, B \subset \Omega$ such that $A \cup B, C, A, B \neq C$ (hence, $A \neq B$), take $\mathcal{R} = \{\emptyset, C, \Omega\}$ and set $\Pi(\emptyset) = 0,\ \Pi(\Omega) = 1,\ \Pi(C) = \alpha,\ 0 < \alpha < 1$. Then $\mathcal{R}$ is evidently a lattice and $\Pi$ is a possibilistic measure on $\mathcal{R}$. Nevertheless, for both $A, B$,

$$\Pi_\star(A) = \bigvee \{\Pi(D) : D \subset A,\ D \in \mathcal{R}\} = \bigvee \{\Pi(D) : D \subset B,\ D \in \mathcal{R}\} = \Pi_\star(B) = 0, \tag{11.7}$$

so that $\Pi_\star(A) \vee \Pi_\star(B) = 0$. However, $\Pi_\star(A \cup B) = \Pi_\star(C) = \alpha > 0$. Let us recall that the roles of set-theoretical operations of the union and intersection are not dual in the context of possibilistic measures, namely, for each $A, B \in \mathcal{R}$ such that $A \cap B \in \mathcal{R}$ just the inequality $\Pi(A \cap B) \leq \Pi(A) \wedge \Pi(B)$ can be proved with the equality holding only under strong supplementary conditions.

A system $\mathcal{R}$ of subsets of a nonempty set $\Omega$ is called *nested* (or *hereditary*, cf. [20]), if for all $A \in \mathcal{R}$ and all $B \subset A$ also $B \subset \mathcal{R}$ holds. Consequently, in this case $\mathcal{R} = \bigcup_{A\in\mathcal{R}} \mathcal{P}(A)$.

**Lemma 11.3** *Let $\Pi$ be a possibilistic measure defined on a nonempty nested system $\mathcal{R} \subset \mathcal{P}(\Omega)$ such that $\bigvee_{C\in\mathcal{R}} \Pi(C) = 1$ holds. Then $\Pi_\star$ is a possibilistic measure on $\mathcal{P}(\Omega)$.*

**Proof 11.3** *There exists $A \in \mathcal{R}$, so that $\emptyset \subset A$, $\emptyset \in \mathcal{R}$ and $\Pi_\star(\emptyset) = 0$ follows. The condition $\bigvee_{C\in\mathcal{R}} \Pi(C) = 1$ immediately yields that $\Pi_\star(\Omega) = 1$. Given $A, B \subset \Omega$ we obtain that*

$$\Pi_\star(A \cup B) = \bigvee_{C\subset A\cup B,\ C\in\mathcal{R}} \Pi(C) = \bigvee_{C\subset A\cup B,\ C\in\mathcal{R}} \Pi(C \cap (A \cup B)) = \bigvee_{C\subset A\cup B,\ C\in\mathcal{R}} \Pi((C \cap A) \cup (C \cap B)). \tag{11.8}$$

*As $C$ is in $\mathcal{R}$, also $C \cap A$ and $C \cap B$ are in $\mathcal{R}$, so that*

$$\Pi_\star(A\cup B) = \bigvee_{C\subset A\cup B,\ C\in\mathcal{R}} (\Pi(C\cap A) \vee \Pi(C\cap B)) = \left[ \bigvee_{C\subset A\cup B,\ C\in\mathcal{R}} \Pi((C \cap A) \right] \vee \left[ \bigvee_{C\subset A\cup B,\ C\in\mathcal{R}} \Pi((C \cap B) \right]. \tag{11.9}$$

*If $C \subset A \cup B,\ C \in \mathcal{R}$ holds, $C \cap A \subset C,\ C \cap A \in \mathcal{R}$ holds as well, hence,*

$$\bigvee_{C\subset A\cup B,\ C\in\mathcal{R}} \Pi(C \cap A) \leq \bigvee_{D\subset A,\ D\in\mathcal{R}} \Pi(D) = \Pi_\star(A). \tag{11.10}$$

*Inversely, for each $D \subset A,\ D \in \mathcal{R}$, we obtain that $D \subset A \cup B,\ D \in \mathcal{R}$ and $D \cap A = D$, so that*

$$\Pi_\star(A) = \bigvee_{D\subset A,\ D\in\mathcal{R}} \Pi(D) \leq \bigvee_{C\subset A\cup B,\ C\in\mathcal{R}} \Pi(C) \tag{11.11}$$

*holds as well. Consequently,*

$$\Pi_\star(A) = \bigvee_{C\subset A\cup B,\ C\in\mathcal{R}} \Pi(C\cap A), \Pi_\star(B) = \bigvee_{C\subset A\cup B,\ C\in\mathcal{R}} \Pi(C\cap B), \tag{11.12}$$

*so that*

$$\Pi_\star(A\cup B) = \Pi_\star(A)\vee\Pi_\star(B) \tag{11.13}$$

*and the proof is completed.*

**Theorem 11.1** *Let $\{\emptyset,\Omega\}\subset\mathcal{R}\subset\mathcal{P}(\Omega)$ be a complete upper semilattice, let $\Pi$ be a complete possibilistic measure on $\mathcal{R}$, let $\mathcal{M}(\Pi,\mathcal{R})$ be the system of all $(\Pi,\mathcal{R})$-measurable subsets of $\Omega$. Then*

$$\Pi_\star\left(\bigcup\mathcal{A}\right) = \bigvee_{A\in\mathcal{A}}\Pi_\star(A),\ \Pi^\star\left(\bigcup\mathcal{A}\right) = \bigvee_{A\in\mathcal{A}}\Pi^\star(A) \tag{11.14}$$

*hold for each $\emptyset\neq\mathcal{A}\subset\mathcal{M}(\Pi,\mathcal{R})$.*

**Proof 11.4** *As $\Pi_\star$ is a monotone measure on $\mathcal{P}(\Omega)$, the inequalities $\Pi_\star(A)\leq\Pi_\star(\bigcup\mathcal{A}),\ A\in\mathcal{A}$ and $\bigvee_{A\in\mathcal{A}}\Pi_\star(A)\leq\Pi_\star(\bigcup\mathcal{A})$ easily follow. In order to arrive at a contradiction suppose that*

$$\bigvee_{A\in\mathcal{A}}\Pi^\star(A) = \bigvee_{A\in\mathcal{A}}\Pi_\star(A) < \Pi_\star\left(\bigcup\mathcal{A}\right) \tag{11.15}$$

*holds (the equality in (11.15) follows from the fact that $\mathcal{A}\subset\mathcal{M}(\Pi,\mathcal{R})$). So, there exists $C\in\mathcal{R}$, $C\subset\bigcup\mathcal{A}$, such that $\Pi(C)>\bigvee_{A\in\mathcal{A}}\Pi^\star(A)$ holds, consequently, there exists $\varepsilon>0$ such that the inequality $\Pi(C)>\Pi^\star(A)+\varepsilon$ holds for all $A\in\mathcal{A}$. Hence, there exists $\varepsilon_1>0$ and, for every $A\in\mathcal{A}$, a set $A_0(A)\subset\Omega$ such that $A_0(A)\supset A,\ A_0(A)\in\mathcal{R}$ and $\Pi(C)>\Pi(A_0(A))+\varepsilon_1$ hold for each $A\in\mathcal{A}$. However, we obtain that $\bigcup_{A\in\mathcal{A}}A_0(A)\supset\bigcup\mathcal{A}$ and, due to the condition imposed on $\mathcal{R}$ and $\Pi, \bigcup_{A\in\mathcal{A}}A_0(A)\in\mathcal{R}$ and the relation*

$$\Pi(C) > \bigvee_{A\in\mathcal{A}}\Pi(A_0(A))+\varepsilon_1 = \mathcal{P}\left(\bigcup_{A\in\mathcal{A}}A_0(A)\right)+\varepsilon_1 > \Pi^\star\left(\bigcup_{A\in\mathcal{A}}A\right)+\varepsilon_1 \tag{11.16}$$

*follows. This contradicts the fact that $C\subset\bigcup\mathcal{A}$, hence, $C\subset D$ for each $D\in\mathcal{R}$ such that $D\supset\bigcup\mathcal{A}$, so that the strict inequality in (11.15) cannot hold.*

*The inequality $\Pi^\star(\bigcup\mathcal{A})\geq\bigvee_{A\in\mathcal{A}}\Pi^\star(A)$ is obvious, so let its strict version hold. Then there exists $\varepsilon>0$ such that $\Pi^\star(\bigcup\mathcal{A})>\Pi^\star(A)+\varepsilon$ is valid for every $A\in\mathcal{A}$, consequently, there exists $\varepsilon_0>0$ and, for each $A\in\mathcal{A},\ A_0(A)\subset\Omega$ such that $A_0(A)\supset A,\ A_0(A)\in\mathcal{R}$, and $\Pi^\star(\bigcup\mathcal{A})>\Pi(A_0(A))+\varepsilon_0$ for every $A\in\mathcal{A}$. Using the supposed properties of $\mathcal{R}$ and $\Pi$ we arrive at the inequality*

$$\Pi^\star\left(\bigcup\mathcal{A}\right) > \bigvee_{A\in\mathcal{A}}\Pi(A_0(A))+\varepsilon_0 = \Pi\left(\bigcup_{A\in\mathcal{A}}A_0(A)\right)+\varepsilon_1. \tag{11.17}$$

*However, as $\bigcup_{A\in\mathcal{A}}A_0(A)\supset\bigcup\mathcal{A}$ holds, this relation contradicts the definition of $\Pi^\star(\bigcup\mathcal{A})$. So, the equality $\Pi^\star(\bigcup\mathcal{A})=\bigvee_{A\in\mathcal{A}}\Pi^\star(A)$ must hold and Theorem 11.1 is proved. In other terms, the system $\mathcal{M}(\Pi,\mathcal{R})$ of $(\Pi,\mathcal{R})$-measurable subsets of $\Omega$ is a complete lattice, like $\mathcal{R}$ itself.*

**Corollary 11.1** *Omitting the adjective "complete" from the conditions imposed on $\mathcal{R}$ and $\Pi$ in Theorem 11.1, (11.14) can be proved for any* finite *$\mathcal{A},\emptyset\neq\mathcal{A}\subset\mathcal{M}(\Pi,\mathcal{R})$.*

The idea behind the notions of inner and outer measures seems to be reasonable and fruithful either when the original set function cannot be, in principle, defined for all subsets of the universe of discourse $\Omega$ (as it is, e.g., the case of Borel measure), or the actual values, ascribed in the Platonist sense to some (or even to most of the) subsets of $\Omega$ cannot be effectively obtained or processed due to some theoretical and/or practical restrictions. This problem occurs, e.g., when some elements of the universe $\Omega$ are indistinquishable elements are just the equivalence classes induced in $\Omega$ by an equivalence relation, we arrive at the notion of *rough sets* introduced by Z. Pawlak in [41].

Hence, from a formalized point of view, a rough set is a pair $\langle \Omega, \approx \rangle$, where $\Omega$ is a nonempty set and $\approx$ is an equivalence relation on $\Omega$, i.e., $\approx$ is a reflexive, symmetric and transitive binary relation on $\Omega$. Denote by $\mathcal{P}_0 \subset \mathcal{P}(\Omega)$ the factor-space $\emptyset / \approx$, so that, for every $\omega \in \Omega$, $[\omega] = \{\omega_1 \in \Omega : \omega_1 \approx \omega\}$ denotes the equivalence class (i.e., the subset of $\Omega$) of all elements of $\Omega$ equivalent to (or: indistinguishable from) the element $\omega \in \Omega$. Denoting by $\mathcal{P}_1$, two mappings $\dot{[]}, \langle\rangle : \mathcal{P}(\Omega) \to \mathcal{P}_1$ can be defined. Given $A \subset \Omega$,

$$[A] = \bigcup \{[\omega] : [\omega] \subset A\}, \langle A \rangle = \bigcup \{[\omega] : [\omega] \cap A \neq \emptyset\}, \tag{11.18}$$

applying the conventions that $[\emptyset] = \langle \emptyset \rangle = \emptyset$. The similarity of the symbols $\dot{[]}$ and $\langle\rangle$ to those of necessity and possibility functors in modal logics is justified by a common philosophical idea behind.

The inclusion $[A] \subset \langle A \rangle$ holds for each $A \subset \Omega$ with the equality being valid iff $[A] = \langle A \rangle = A \in \mathcal{P}_1$. Consequently, for each $A \subset \Omega, \langle\langle [A] \rangle = [A]$ and $[\langle A \rangle] = \langle A \rangle$. For every $\emptyset \neq \mathcal{A} \subset \mathcal{P}(\Omega)$, in particular, for each $A, B \subset \Omega$, the following inclusions and identities hold:

$$\begin{aligned}
\bigcup_{A \in \mathcal{A}} [A] &\subset \left[\bigcup \mathcal{A}\right], \ [A] \cup [B] \subset [A \cup B], \\
\bigcup_{A \in \mathcal{A}} \langle A \rangle &= \langle \bigcup \mathcal{A} \rangle, \ \langle A \rangle \cup \langle B \rangle = \langle A \cup B \rangle, \\
\bigcap_{A \in \mathcal{A}} [A] &= \left[\bigcap \mathcal{A}\right], \ [A] \cap [B] = [A \cap B], \\
\bigcap_{A \in \mathcal{A}} \langle A \rangle &\supset \langle \bigcap \mathcal{A} \rangle, \ \langle A \rangle \cap \langle B \rangle \supset \langle A \cap B \rangle.
\end{aligned} \tag{11.19}$$

Easy to obtain counter-examples demonstrate that none of the inclusions above can be, in general, replaced by the equality relation.

Let $\pi$ be a possibilistic distribution on $\mathcal{P}_0 = \Omega | \approx$, so that $\pi : \mathcal{P}_0 \to [0,1], \bigvee_{\alpha \in \mathcal{P}_0} \pi(\alpha) = 1$. $\pi$ is called *compact*, if there exists $\omega_0 \in \Omega$ such that $\pi([\omega_0]) = 1$. Setting for every $R \in \mathcal{P}_1$

$$\Pi(R) = \bigvee_{\alpha \in R} \pi(\alpha), \tag{11.20}$$

we can easily verify that $\Pi$ is a partially defined possibilistic measure on $\mathcal{P}(\Omega)$ with $Dom\,(\Pi) = \mathcal{P}_1$. For the inner and outer measures induced by $\Pi$ on $\mathcal{P}(\Omega)$ we obtain easily that

$$\Pi_\star(A) = \bigvee \{\Pi(B) : B \in \mathcal{P}_1, \ B \subset A\} = \Pi([A]), \tag{11.21}$$

$$\Pi^\star(A) = \bigwedge \{\Pi(B) : B \in \mathcal{P}_1, \ B \supset A\} = \Pi(\langle A \rangle) \tag{11.22}$$

for each $A \subset \Omega$, as could be expected. The following facts are almost self-evident:

- [i] Both $\Pi_\star$ and $\Pi^\star$ are monotone measures on $\mathcal{P}(\Omega)$, conservatively extending $\Pi$ from $\mathcal{P}_1$ to $\mathcal{P}(\Omega)$. Hence, for each $A \subset B \subset \Omega$ and each $C \in \mathcal{P}_1$, $\Pi_\star(A) \leq \Pi_\star(B), \Pi^\star(A) \leq \Pi^\star(B)$, and $\Pi_\star(C) = \Pi^\star(C)$ hold.
- [ii] $\Pi^\star$ is a complete possibilistic measure on $\mathcal{P}(\Omega)$.

Let us close this chapter by focusing our attention to the particular case of possibilistic measures taking only the extremal values 0 and 1 as their possible degrees. Consequently, everything that follows can be expressed, stated and proved within the standard language of (crisp) set theory. Let $\Omega$ be a nonempty set, let $A_0$ be a fixed (nonempty and proper, to avoid the most trivial cases) subset of

$\Omega$, let $\pi_{A_0}$ be the characteristic function (identifier) of $A_0$, so that $\pi_{A_0}(\omega) = 1$, if $\omega \in A_0$, $\pi_{A_0}(\omega) = 0$, if $\omega \in \Omega - A_0$. Given $A \subset \Omega$, setting $\Pi_{A_0}(A) = 1$, if $A \cap A_0 \neq \emptyset$, $\Pi_{A_0}(A) = 0$ otherwise, we can easily check that $\Pi_{A_0}$ is a complete (two-valued or binary) possibilistic measure on $\mathcal{P}(\Omega)$ defined by the possibilistic distribution $\pi_{A_0}$ (the identity $\Pi_{A_0}(\bigcup \mathcal{A}) = \bigvee_{A \in \mathcal{A}} \Pi_{A_0}(A)$ evidently holds for every $\emptyset \neq \mathcal{A} \subset \mathcal{P}(\Omega)$). Any binary complete possibilistic measure defined on the whole power-set $\mathcal{P}(\Omega)$ can be expressed as $\Pi_{A_0}$, where $A_0 = \{\omega \in \Omega : \Pi(\{\omega\}) = 1\}$.

As $\Pi_{A_0}$ is defined for every $A \subset \Omega$, it is beyond any sense to define the inner and the outer measures induced by $\Pi_{A_0}$. The situation becomes less trivial when setting $\Pi$ as a *partial* mapping on $\mathcal{P}(\Omega)$. To limit ourselves to a very simple case, we will keep continuity with the explanation above and we will suppose that the definition domain of $\Pi$ consists of equivalence classes generated by an equivalence relation on $\Omega$. So, let $\Omega, \approx, \mathcal{P}_0$ and $\mathcal{P}_1$ keep their former meaning, let $\pi_0 : \mathcal{P}_0 \to \{0,1\}$ be a binary possibilistic distribution on $\mathcal{P}_0$, let $\Pi$ be the binary possibilistic measure defined by $\pi_0$ on $\mathcal{P}_1$. For the corresponding inner and outer measures $\Pi_{A_0,\star}$ and $\Pi^{\star}_{A_0}$ we obtain that for each $A \subset \Omega$

$$\Pi_{A_0,\star}(A) = \Pi_{A_0}([A]), \ \Pi^{\star}_{A_0}(A) = \Pi_{A_0}(\langle A \rangle). \tag{11.23}$$

In other terms, $\Pi_{A_0,\star}(A) = 1$ iff $[A] \cap A_0 \neq \emptyset$, $\Pi^{\star}_{A_0}(A) = 1$ iff $\langle A \rangle \cap A_0 \neq \emptyset$. The following simple assertion defines some conditions under which the class of $(\Pi_{A_0}, \mathcal{P}_1)$-measurable sets is larger than $\mathcal{P}_1$.

**Lemma 11.4** *Let $\langle \Omega, \approx \rangle$ be a rough set, let $\Pi$ be a binary complete possibilistic measure on $\mathcal{P}_1$, let $A_0 \subset \Omega$ be such that there exists $\omega_0 \in A_0$ possessing the property that $\approx$ is not the identity on $\Omega - [\omega_0]$, hence, that there exist $\omega_1, \omega_2 \in \Omega - [\omega_0]$ such that $\omega_1 \neq \omega_2$, but $\omega_1 \approx \omega_2$. Then there exists a $(\Pi_{A_0}, \mathcal{P}_1)$-measurable set which is not in $\mathcal{P}_1$.*

**Proof 11.5** *Let $A \subset \Omega$ be such that $[\omega_0] \subset A$ and just one of the elements $\omega_1, \omega_2$ is in $A$. Hence, $A$ is not in $\mathcal{P}_1$, but $\Pi_{A_0,\star}(A) = \Pi_{A_0}([A]) = 1$, as $[\omega_0] \subset A$ yields that $[\omega_0] \subset [A_0]$, so that $A_0 \cap [A] \neq \emptyset$. The inequality $\Pi_{A_0,\star}(A) \leq \Pi^{\star}_{A_0}(A)$ holds in general, so that $\Pi_{A_0,\star}(A) = \Pi^{\star}_{A_0}(A) = 1$ and the set $A$ is $(\Pi_{A_0}, \mathcal{P}_1)$-measurable.*

# 12 Almost-Measurability Induced by Real-Valued Monotone and Possibilistic Measures

The notion of almost-measurability in a natural way weakens and generalizes that of $\langle \Pi, \mathcal{R} \rangle$-measurability introduced and briefly analyzed in the foregoing chapter. In order to arrive at this generalization in an intuitive way, let us recall the well-known standard construction applied in measure theory in order to obtain the system of subsets of the real line measurable in the Lebesgue sense, cf. [20] or other textbook on classical measure and integration theory. In order to simplify the construction let us limit ourselves to the unit interval [0,1] of real numbers.

We will start from the system $\mathcal{R}$ of all semi-open intervals $[a, b) \subset [0, 1]$, ascribing to each such interval the non-negative real number $\mu([a, b)) = b - a$. i.e., its length. As a matter of fact, we could begin with the open or closed subintervals of [0,1] as well. This mapping can be uniquely extended to the $\sigma$-additive (probability) measure $\mu$ defined on the minimal $\sigma$-field $\mathcal{B}$ of subsets of [0,1] such that $\mathcal{B}$ contains $\mathcal{R}$. The sets in $\mathcal{B}$ are just the so called *Borel subsets of* [0,1] or *subsets of* [0,1] *measurable in the Borel sense*. It is a well known fact that far not any subset of [0,1] is measurable in the Borel sense. Hence, we define the inner and the outer measures $\mu_{\star}$ and $\mu^{\star}$ induced by $\mu$ and we denote by $\mathcal{L}$ the system of all subsets of [0,1] for which the equality $\mu_{\star}(A) = \mu^{\star}(A)$ holds. The sets from $\mathcal{L}$ are called *measurable in the Lebesgue sense* and, as a matter of fact, cf. again [20], [38] or elsewhere, $A$ is measurable in the Lebesgue sense iff $A$ differs from a Borel set on a subset of a Borel set of zero measure. In symbols

$$\mathcal{L} = \{A \subset [0, 1] : \text{ there exist } A_0, B_0 \in \mathcal{B} \text{ such that } \mu(B_0) = 0 \text{ and } A \div A_0 \subset B_0\}, \tag{12.1}$$

where $\div$ denotes the set-theoretic operation of symmetric difference. Obviously, $\mathcal{B} \in \mathcal{L}$ holds and, as can be proved, $\mathcal{B} \neq \mathcal{L}$, but still for not any subset of [0,1] is measurable in the Lebesgue sense.

Moreover, what is important in our context, there exist subsets $A \subset [0,1]$ such that $\mu_\star(A) = 0$ and $\mu^\star(A) = 1$ (cf. [20], e.g). However, this is the price we have to pay for the intuitive semantic behind the measure $\mu$ enabling to reduce it to the length of intervals. In the foregoing chapter we applied this way of reasoning when defining $(\varphi, \mathcal{R})$-measurable sets with respect to a real-valued monotone (or, in particular, possibilistic) measure defined on a system $\mathcal{R}$ of subsets of the universe of discourse $\Omega$.

The condition $\mu_\star(A) = \mu^\star(A)$ which defines the system $\mathcal{L}$ of subsets of [0,1] which are measurable in the Lebesgue sense can be obviously written as $|\mu_\star(A) - \mu^\star(A)| = 0$. The straighforward idea now reads as follows: fiven a real number $\varepsilon \geq 0$, a subset $A$ of [0.1] could be called *$\varepsilon$-almost-measurable*, if the inequality $|\mu_\star(A) - \mu^\star(A)| \leq \varepsilon$ holds. This notion generalizes (in the sense of weakening) the notion of measurability in the Lebesgue sense which is identical with the 0-almost-measurability. On the other side, this extension is nontrivial in the sense that for each $\varepsilon < 1$ there exist subsets of [0.1] which are not $\varepsilon$-almost-measurable, as there exist $A \subset [0,1]$ such that $\mu_\star(A) = 0$ and $\mu^\star(A) = 1$.

In the case of a $\sigma$-additive measure $\mu$ (probabilistic, say), defined on the system of sets measurable in the Lebesgue sense and generated by a $\sigma$-field of subsets of a universe $\Omega$, the notion of almost measurability is of a rather limited use, because in this case neither the values $\mu_\star(A)$ nor $\mu^\star(A)$ can be used as reasonable extensions of the set function $\mu$ from $\mathcal{L}$ to $\mathcal{P}(\Omega)$. Indeed, given an infinite sequence $A_1, A_2, \ldots$ of mutually disjoint subsets of $\Omega$, we would like to obtain that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \tag{12.2}$$

holds, as it is the case with subsets $A_1, A_2, \ldots$ measurable in the Lebesgue sense. But, given a sequence $\alpha_1, \alpha_2, \ldots$ of reals from [0,1] such that $|\mu(A_i) - \alpha_i| \leq \varepsilon$ holds for a fixed $\varepsilon > 0$ and for each $i = 1, 2, \ldots$, the series $\sum_{i=1}^{\infty} \alpha_i$ still may vary along all the extended line $[-\infty, \infty]$. Hence, neither the uniform $\varepsilon$-approximation of measure values ascribed to particular subsets of $\Omega$ offer any reasonable approximation of the measure value ascribed to their union.

However, due to the qualitative differences between the properties of summation (series taking) and those of standard supremum in [0,1] the notion of almost-measurability induced by possibilistic measures can be taken as quite reasonable and useful. Let us begin with the following definition.

**Definition 12.1** *Let $\varphi$ be a real-valued normalized monotone measure defined on a nonempty system $\mathcal{R}$ of subsets of a universe $\Omega \neq \emptyset$, let $0 \leq \varepsilon \leq 1$ be a real number. A set $A \subset \Omega$ is called $(\varphi, \mathcal{R}, \varepsilon)$-*almost measurable, *if the inequality $\varphi^\star(A) - \varphi_\star(A) \leq \varepsilon$ holds, and $A$ is called $(\varphi, \mathcal{R}, \varepsilon)$-*strictly almost measurable, *if $\varphi^\star(A) - \varphi_\star(A) < \varepsilon$ is the case. The system of all $(\varphi, \mathcal{R}, \varepsilon)$-almost measurable subsets of $\Omega$ will be denoted by $\mathcal{M}(\varphi, \mathcal{R}, \varepsilon^+)$ and that of all $(\varphi, \mathcal{R}, \varepsilon)$-strictly almost measurable subsets of $\Omega$ by $\mathcal{M}(\varphi, \mathcal{R}, \varepsilon^-)$.*

As the following assertion demonstrates, in the case of possibilistic measures and under some more conditions imposed on the definition domain $\mathcal{R}$, the system of all $\varepsilon$-almost measurable sets is closed with respect to unions.

**Lemma 12.1** *Let $\Pi$ be a complete real-valued possibilistic measure defined on a nonempty system $\mathcal{R}$ of subsets of $\Omega$ which is closed w.r. to unions, i.e., for each $\emptyset \neq \mathcal{A} \subset \mathcal{R}$, $\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A \in \mathcal{R}$ holds. Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ be a system of subsets of $\Omega$ such that, for a given $\varepsilon \geq 0$, the relation $\Pi^\star(A) - \Pi_\star(A) \leq \varepsilon$ holds for every $A \in \mathcal{A}$. Then also $\Pi^\star(\bigcup \mathcal{A}) - \Pi_\star(\bigcup \mathcal{A}) \leq \varepsilon$ holds.*

**Proof 12.1** *Take a $\delta > 0$ and combine the inequality $\Pi^\star(A) - \Pi_\star(A) \leq \varepsilon$, supposed to hold for each $A \in \mathcal{A}$, with the definition of $\Pi^\star(A)$ and $\Pi_\star(A)$. So there exist, for each $A \in \mathcal{A}$, sets $A_\delta^+$ and $A_\delta^-$ from $\mathcal{R}$ such that $A_\delta^- \subset A \subset A_\delta^+$ and*

$$\Pi(A_\delta^+) - \Pi(A_\delta^-) < \varepsilon + \delta \tag{12.3}$$

*is valid for each $A \in \mathcal{A}$. It follows easily that*

$$\bigvee_{A \in \mathcal{A}} \Pi(A_\delta^+) - \bigvee_{A \in \mathcal{A}} \Pi(A_\delta^-) \leq \varepsilon + \delta \tag{12.4}$$

*holds. Indeed, supposing that this inequality does not hold, i.e., that the difference in (12.4.) is greater that $\mathcal{E}+\delta$, it follows that there exists $A_0 \in \mathcal{A}$ such that the inequality*

$$\Pi(A^+_{0,\delta}) - \bigvee_{A\in\mathcal{A}} \Pi(A^-_\delta) > \varepsilon + \delta \tag{12.5}$$

*is valid. However, in this case*

$$\Pi(A^+_{0,\delta}) - \Pi(A^-_\delta) > \varepsilon + \delta \tag{12.6}$$

*holds for all $A \in \mathcal{A}$ including $A_0$, but this contradicts (12.3). The inclusions*

$$\bigcup_{A\in\mathcal{A}} A^-_\delta \subset \bigcup \mathcal{A} \subset \bigcup_{A\in\mathcal{A}} A^+_\delta \tag{12.7}$$

*are valid and both the sets $\bigcup_{A\in\mathcal{A}} A^-_\delta$ and $\bigcup_{A\in\mathcal{A}} A^+_\delta$ are in $\mathcal{R}$ due to the assumptions imposed on $\mathcal{R}$. Applying the assumption of completeness of the possibilistic measure $\Pi$ on $\mathcal{R}$, we obtain that*

$$\bigvee_{A\in\mathcal{A}} \Pi(A^+_\delta) \geq \Pi^\star\left(\bigcup \mathcal{A}\right) \geq \Pi_\star\left(\bigcup \mathcal{A}\right) \geq \bigvee_{A\in\mathcal{A}} \Pi(A^-_\delta) \tag{12.8}$$

*holds, so that the inequality*

$$\Pi^\star\left(\bigcup \mathcal{A}\right) - \Pi_\star\left(\bigcup \mathcal{A}\right) \leq \varepsilon \tag{12.9}$$

*easily follows from (12.4) and from the fact that $\delta > 0$ was arbitrary. The assertion is proved.*

**Definition 12.2** *A nonempty system $\mathcal{R}$ of subsets of a nonempty space $\Omega$ is called an* upper *(a* lower*, resp.)* semilattice*, if for each nonempty finite system $\mathcal{A} \subset \mathcal{R}$ its union $\bigcup \mathcal{A} = \bigcup_{A\in\mathcal{A}} A$ (its intersection $\bigcap \mathcal{A} = \bigcap_{A\in\mathcal{A}} A$, resp.) also belongs to $\mathcal{R}$. If this relation holds for* every *$\mathcal{A} \subset \mathcal{R}$, the upper (lower, resp.) semilattice $\mathcal{A}$ is called* complete.

**Theorem 12.1** *Let $\Pi$ be a possibilistic measure defined on a nonempty upper semilattice $\mathcal{R}$ of subsets of $\Omega$. Then, for each $\varepsilon > 0$ the system $\mathcal{M}(\Pi, \mathcal{R}, \varepsilon^-)$ of all $\Pi, \mathcal{R}, \varepsilon)$-strictly almost measurable subsets of $\Omega$ is also an upper semilattice.*

**Proof 12.2** *Let $\emptyset \neq \mathcal{A} \subset \mathcal{M}(\Pi, \mathcal{R}, \varepsilon^-)$ be a finite system of subsets of $\Omega$, so that $\Pi^\star(A) - \Pi_\star(A) < \varepsilon$ holds for each $A \in \mathcal{A}$. Due to the definitions of $\Pi^\star(A)$ and $\Pi_\star(A)$ there exist, for each $A \in \mathcal{A}$, subsets $A^+$ and $A^-$ in $\mathcal{R}$ such that $A^- \subset A \subset A^+$ and $\Pi(A^+) - \Pi(A^-) < \varepsilon$ holds. Hence,*

$$\bigcup_{A\in\mathcal{A}} A^+ \supset \bigcup \mathcal{A} \supset \bigcup_{A\in\mathcal{A}} A^- \tag{12.10}$$

*immediately implies that*

$$\Pi(\bigcup_{A\in\mathcal{A}} A^+) = \bigvee_{A\in\mathcal{A}} \Pi(A^+) \geq \Pi^\star\left(\bigcup \mathcal{A}\right) \geq \Pi_\star\left(\bigcup \mathcal{A}\right) \geq \Pi\left(\bigcup_{A\in\mathcal{A}} A^-\right) = \bigvee_{A\in\mathcal{A}} \Pi(A^-). \tag{12.11}$$

*Suppose, in order to arrive at a contradiction, that the relation*

$$\bigvee_{A\in\mathcal{A}} \Pi(A^+) - \bigvee_{A\in\mathcal{A}} \Pi(A^-) \geq \varepsilon \tag{12.12}$$

*is valid. Then*

$$\Pi(A_0^+) - \bigvee_{A\in\mathcal{A}} \Pi(A^-) \geq \varepsilon \tag{12.13}$$

*also holds, taking $A_0 \in \mathcal{A}$ such that $\Pi(A_0^+) = \bigvee_{A\in\mathcal{A}} \Pi(A^+)$. As the system $\mathcal{A}$ is finite, such $A$ always exists. In particular, (12.12) yields that*

*Pi*$(A_0^+) - \Pi(A_0^-) \geq \varepsilon$ *holds, but this contradicts the way in which* $A^+$ *and* $A^-$ *have been chosen. Consequently, the inequality*

$$\Pi^\star\left(\bigcup\right) - \Pi_\star\left(\bigcup\mathcal{A}\right) \leq \bigvee_{A\in\mathcal{A}} \Pi(A^+) - \bigvee_{A\in\mathcal{A}} \Pi(A^-) < \varepsilon \tag{12.14}$$

*follows, so that* $\bigcup\mathcal{A}$ *also belongs to* $\mathcal{M}(\Pi,\mathcal{R},\varepsilon^-)$*, hence,* $\mathcal{M}(\Pi,\mathcal{R},\varepsilon^-)$ *is an upper semilattice.*

A slight and easy to prove modification of the assertion just achieved reads as follows.

**Theorem 12.2** *Let* $\Pi$ *be a complete possibilistic measure defined on a nonempty complete upper semilattice* $\mathcal{R}$ *of subsets of* $\Omega$*. Then, for each* $\varepsilon \geq 0$*, the system* $\mathcal{M}(\Pi,\mathcal{R},\varepsilon^+)$ *of* $(\Pi,\mathcal{R},\varepsilon)$*-almost measurable subsets of* $\Omega$ *is also a complete upper semilattice.*

**Proof 12.3** *Let* $\emptyset \neq \mathcal{A} \subset \mathcal{M}(\Pi,\mathcal{R},\varepsilon^+)$*, so that* $\Pi^\star(A) - \Pi_\star(A) \leq \varepsilon$ *holds for each* $A \in \mathcal{A}$*. Given* $\delta_1 > 0$ *there exist, for each* $A \in \mathcal{A}$*, two subsets* $A^+, A^- \in \mathcal{R}$ *such that* $A^+ \supset A \supset A^-$ *and* $\Pi(A^+) - \Pi(A^-) < \varepsilon + \delta_1$ *holds. Consequently, (12.10) holds and also (12.11) can be deduced due to assumption of completeness imposed on* $\mathcal{R}$ *and* $\Pi$*. In order to arrive at a contradiction, suppose that the inequality*

$$\bigvee_{A\in\mathcal{A}} \Pi(A^+) - \bigvee_{A\in\mathcal{A}} \Pi(A^-) > \varepsilon + \delta_1 \tag{12.15}$$

*is valid. Given* $\delta_2 > 0$*, there exists* $A_0 \in \mathcal{A}$ *such that*

$$\left(\bigvee_{A\in\mathcal{A}} \Pi(A^+)\right) - \Pi(A_0^+) < \delta_2 \tag{12.16}$$

*holds, hence, also*

$$\Pi(A_0^+) - \Pi(A_0^-) > \varepsilon + \delta_2 \tag{12.17}$$

*holds. This is compatible with the inequality* $\Pi(A_0^+) - \Pi(A_0^-) < \varepsilon + \delta_1$ *only when* $\delta_1 = \delta_2 = 0$*, so that the inequality*

$$\bigvee_{A\in\mathcal{A}} \Pi(A^+) - \bigvee_{a\in\mathcal{A}} \Pi(A^-) \leq \varepsilon \tag{12.18}$$

*is valid and the same way of reasoning as in the end of the proof of Theorem 12.1 yields that* $\bigcup\mathcal{A} \in \mathcal{M}(\Pi,\mathcal{R},\varepsilon^\star)$ *holds. The assertion is proved.*

Assertions dual to Theorems 12.1 and 12.2 for lower semilattices and complete lower semilattices cannot be achieved by a simple dualization of the proofs above. It is caused by the fact that for a possibilistic measure $\Pi$ just the inequalities $\Pi(A \cap B) \leq \Pi(A) \wedge \Pi(B)$ (or $\Pi(\bigcap_{A\in\mathcal{A}} A) \leq \bigwedge_{A\in\mathcal{A}} \Pi(A)$) can be proved, but not the equalities, as it is, by definition, the case for unions and suprema. Consequently, under the notation used above, the value $\bigwedge_{A\in\mathcal{A}} \Pi(A^-)$ cannot be used as a lower bound for $\Pi_\star(\bigcap_{A\in\mathcal{A}} A)$ and the dual analogies of the proofs presented above fail.

Given a monotone measure $\Pi$ defined on a nonempty system $\mathcal{R}$ of a basic space $\Omega$ and given a real number $\varepsilon \geq 0$, a subset $A \subset \Omega$ is called to be of $\varepsilon$-*differential* $\Pi$-*measure*, if there exist, for every $\delta > \varepsilon$, sets $A^-.A^+ \in \mathcal{R}$ such that $A^- \subset A^+$, $A \subset A^+ - A^-$, and $\Pi(A^+) - \Pi(A^-) < \delta$ holds. It follows easily that a subset $B \subset \Omega$ is $(\Pi,\mathcal{R},\varepsilon^+)$-almost measurable, i.e., $B \in \mathcal{M}(\Pi,\mathcal{R},\varepsilon^+)$, if there exists a subset $A \subset \Omega$ which is of $\varepsilon$-differential $\Pi$-measure and sets $B^-.B^+ \in \mathcal{R}$, $B^- \subset B \subset B^+$, such that $B$ differs from $B^-$ as well as from $B^+$ on subsets of the set $A$. In particular, for $\varepsilon = 0$ we obtain that $B$ is in $\mathcal{M}(\Pi,\mathcal{R},0^+)$, iff there are, for every $\delta > 0$, sets $B^-, B^+ \in \mathcal{R}$ such that $B^- \subset B \subset B^+$ and $\Pi(B^+) - \Pi(B^-) < \delta$ holds, consequently, iff $\Pi^\star(B) = \Pi_\star(B)$. Hence, using the terms introduced in the foregoing chapter, $(\Pi,\mathcal{R})$-measurable sets are those which differ from a set in $\mathcal{R}$ on a subset of a set of zero-differential $\Pi$-measure. This formulation reads very closely to that from the classical measure theory according to which sets measurable in the Lebesgue sense are those differing from Borel sets on subsets of sets of zero Borel measure (cf. [20], e.g., for more detail).

# 13 Pseudo-Complement Operation and Lattice-Valued Metric Spaces over Complete Lattices

In several following chapters, our aim will be to look for appropriate analogies of the notions of inner and outer measures, measurability in the Lebesgue sense and almost-measurability in the case of lattice-valued possibilistic measures and to analyze them in more detail. Doing so in the case of real-valued possibilistic measures, which take their values in the complete lattice $\langle [0,1], \leq \rangle$, we have taken profit of the operation of complement 1-. in [0,1] and of the metric $|x-y|$ describing to each pair $x, y$ of reals from [0,1] the absolute value of their difference as their distance from each other. In this chapter we will try to introduce some more or less appropriate analogies of these two notions in the case of a general complete lattice $\mathcal{T} = \langle T, \leq \rangle$.

The notion of complete lattice is perhaps the most specific particular case of partially ordered sets still covering both the best-known and most often used structures in which various quantification functions take their values – the unit interval of reals together with their standard linear ordering and the power-set of all subsets of a nonempty universe of discourse partially ordered by the relation of set-theoretic inclusion. However, the structure of a complete lattice does not contain an analogy of the operation of complement compatible with the operation of substraction $1-x$ in the case of $\langle [0,1], \leq \rangle$, $x \in [0,1]$, and with the operation of set-theoretic complement $X - A$ in the case of $\langle \mathcal{P}(X), \subset \rangle$, $A \subset X$. One remedy would consist in enriching the partially ordered set (or complete lattice, in particular), by definition, by a new primary unary operation of complement, binding its properties by appropriate axiomatic demands. E.g., when taking inspiration from the properties of supremum and infimum operations in the power-set over a nonempty set, we arrive at the well-known idea of complete Boolean algebra. However, taking inspiration from the notion of the so called residuated lattices, a way how to define at least a weakened notion of complement in every complete lattice may read as follows.

Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice. Elements $t_1, t_2 \in T$ are called *mutually orthogonal*, if $t_1 \wedge t_2 = \mathbf{0}_{\mathcal{T}}$ $(= \bigwedge T$, by definition). Given $t \in T$, let $C_t$ denote the set of all elements of $T$ which are mutually orthogonal with $t$, hence, $C_t = \{s \in T : s \wedge t = \mathbf{0}_{\mathcal{T}}\}$. $C_t$ is nonempty for each $t \in T$, as $\mathbf{0}_{\mathcal{T}} \in C_t$ obviously holds. Consequently, $\bigvee C_t$ is defined (even without the conventions concerning the empty set), will be denoted by $t^C$ and called the *(pseudo-)complement* of $t$ in $\mathcal{T}$ (cf. Definition (7.4) and (7.7)). The possible dual version of definition of pseudo-complement according to which $t^d = \bigwedge \{s \in T : s \vee t = \mathbf{1}_{\mathcal{T}}\}$ is also introduced in Chapter 7 (cf. (7.8)), but will not be applied in what follows.

In the particular case when $\mathcal{T} = \mathcal{P}(X), \subset>$ for some nonempty set $X$, the notion of the pseudo-complement obviously coincides with the standard set-theoretic definition of complement, so that for each $A \subset X$ the identity

$$A^C = \bigcup \{B \subset X : B \cap A = \emptyset\} = X - A \tag{13.1}$$

easily follows. Consequently, $(A^C)^C = A$ holds.

Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice. For each $s \in T$, $s \wedge \mathbf{0}_{\mathcal{T}} = \mathbf{0}_{\mathcal{T}}$ holds, on the other side, $s = \mathbf{0}_{\mathcal{T}}$ is the only $s \in T$ such that $s \wedge \mathbf{1}_{\mathcal{T}} = \mathbf{0}_{\mathcal{T}}$. So

$$\mathbf{0}_{\mathcal{T}}^{\mathbf{C}} = \bigvee \{s \in T : s \wedge \mathbf{0}_{\mathcal{T}} = \mathbf{0}_{\mathcal{T}}\} = \bigvee T = \mathbf{1}_{\mathcal{T}}, \tag{13.2}$$

$$\mathbf{1}_{\mathcal{T}}^{\mathbf{C}} = \bigvee \{s \in T : s \wedge \mathbf{1}_{\mathcal{T}} = \mathbf{0}_{\mathcal{T}}\} = \bigvee \{\mathbf{0}_{\mathcal{T}}\} = \mathbf{0}_{\mathcal{T}}, \tag{13.3}$$

and the relations $\mathbf{0}_{\mathcal{T}}^{\mathbf{C}} \vee \mathbf{0}_{\mathcal{T}} = \mathbf{1}_{\mathcal{T}}$, $\mathbf{0}_{\mathcal{T}}^{\mathbf{C}} \wedge \mathbf{0}_{\mathcal{T}} = \mathbf{0}_{\mathcal{T}}$ (and similarly for $\mathbf{1}_{\mathcal{T}}$ and $\mathbf{1}_{\mathcal{T}}^{\mathbf{C}}$) follow. However, neither $t \wedge t^C = \mathbf{0}_{\mathcal{T}}$ nor $t \vee t^C = \mathbf{1}_{\mathcal{T}}$ need hold for each $t \in T$. Indeed, consider the complete lattice $\langle \{0, t_1, t_2, t_3, 1\}, \leq \rangle$ introduced in Chapter 7. Then

$$t_1^C = \bigvee \{s \in T : s \wedge t_1 = \mathbf{0}_{\mathcal{T}}\} = \bigvee C_1 = 1, \tag{13.4}$$

so that $t_1 \wedge t_1^C = t_1 \neq \mathbf{0}_{\mathcal{T}}$ (and similarly for $t_2$ and $t_3$). Now, take the complete lattice $\langle [0,1], \leq \rangle$. Then, for each $0 < t < 1$,

$$t^C = \bigvee\{s \in [0,1] : s \wedge t = 0\} = 0, \tag{13.5}$$

so that $t \vee t^C = t \neq 1$. If $\mathcal{T} = \langle T, \leq\rangle$ is a completely distributive complete lattice, then $t \wedge t^C = \mathbf{0}_{\mathcal{T}}$ holds for each $t \in T$. Indeed,

$$t \wedge t^C = t \wedge \left(\bigvee\{s \in T : s \wedge t = \mathbf{0}_{\mathcal{T}}\}\right) = \bigvee\{s \wedge t : s \in T, s \wedge t = \mathbf{0}_{\mathcal{T}}\} = \bigvee\{\mathbf{0}_{\mathcal{T}}\} = \mathbf{0}_{\mathcal{T}}. \tag{13.6}$$

However, as can be easily proved, the complete lattice $\langle [0,1], \leq\rangle$ is completely distributive, but still in this case $t \vee t^C < 1$ holds for every $0 < t < 1$.

Applying the operation of the pseudo-complement twice, step by step, neither $t^C \leq (t^C)^C$ nor $t \geq (t^C)^C$ hold in general. Considering, again, the complete lattice $\langle \{0, t_1, t_2, t_3, 1\}, \leq\rangle$, we obtain that $(t_i^C)^C = 1^C = 0 < t_i$ holds for each $i = 1, 2, 3$. On the other hand, for $\mathcal{T} = \langle [0,1], \leq\rangle$ and $0 < t < 1$ we have that $(t^C)^C = 0^C = 1 > t$ holds. If $\mathcal{T} = \langle T, \leq\rangle$ is a completely distributive complete lattice, then $t \wedge t^C = \mathbf{0}_{\mathcal{T}}$ holds, so that $t \in \{s \in T : s \wedge t^C = \mathbf{0}_{\mathcal{T}}\}$ holds, hence, the inequality

$$t \leq \bigvee\{s \in T : s \wedge t^C = \mathbf{0}_{\mathcal{T}}\} = (t^C)^C \tag{13.7}$$

follows. However, neither in this case the inverse inequality $t \geq (t^C)^C$ can be proved in general, as the example with $\mathcal{T} = \langle [0,1], \leq\rangle$ demonstrates.

**Lemma 13.1** *Let $\mathcal{T} = \langle T, \leq\rangle$ be a completely distributive complete lattice. Then for every $t_1, t_2 \in T$ the identity*

$$(t_1 \vee t_2)^C = t_1^C \wedge t_2^C \tag{13.8}$$

*holds.*

**Proof 13.1** *For each $s \in T$, if $s \wedge (t_1 \vee t_2) = \mathbf{0}_{\mathcal{T}}$, then $s \wedge t_i = \mathbf{0}_{\mathcal{T}}$ for both $i = 1, 2$. Hence, for both $i = 1, 2$,*

$$(t_1 \vee t_2)^C = \bigvee\{s \in T : s \wedge (t_1 \vee t_2) = \mathbf{0}_{\mathcal{T}}\} \leq \bigvee\{s \in T : s \wedge t_i = \mathbf{0}_{\mathcal{T}}\} = t_i^C \tag{13.9}$$

*holds, so that the inequality*

$$(t_1 \vee t_2)^C \leq t_1^C \wedge t_2^C \tag{13.10}$$

*immediately follows. On the other hand,*

$$(t_1 \vee t_2) \wedge (t_1^C \wedge t_2^C) = [t_1 \wedge (t_1^C \wedge t_2^C)] \vee [t_2 \wedge (t_1^C \wedge t_2^C)] \leq (t_1 \wedge t_1^C) \vee (t_2 \wedge t_2^C) = \mathbf{0}_{\mathcal{T}} \vee \mathbf{0}_{\mathcal{T}} = \mathbf{0}_{\mathcal{T}}, \tag{13.11}$$

*as $t \wedge t^C = \mathbf{0}_{\mathcal{T}}$ holds for each $t \in T$ in each completely distributive complete lattice, as proved above. So,*

$$t_1^C \wedge t_2^C \in \{s \in T : s \wedge (t_1 \vee t_2) = \mathbf{0}_{\mathcal{T}}\}, \tag{13.12}$$

*so that*

$$t_1^C \wedge t_2^C \leq \bigvee\{s \in T : s \wedge (t_1 \vee t_2) = \mathbf{0}_{\mathcal{T}}\} = (t_1 \vee t_2)^C \tag{13.13}$$

*and (13.8) immediately follows.*

If $\mathcal{T}$ is not completely distributive, (13.8) does not hold in general. Indeed, taking again the complete lattice $\langle \{0, t_1, t_2, t_3, 1\}, \leq\rangle$ from Chapter 7 we obtain that

$$(t_i \vee t_j)^C = 1^C = 0 \neq t_i^C \wedge t_j^C = 1 \wedge 1 = 1 \tag{13.14}$$

holds for each $i \neq j, i, j = 1, 2, 3$. Let us note that the identity

$$(t_i \wedge t_j)^C = t_i^C \vee t_j^C, \tag{13.15}$$

dual to (13.8), is valid in $\langle \{0, t_1, t_2, t_3, 1\}, \leq \rangle$, as in this case the identity

$$(t_i \wedge t_j)^C = 0^C = 1 = t_i^C \vee t_j^C = 1 \vee 1 = 1 \tag{13.16}$$

holds for each $i \neq j, i, j = 1, 2, 3$. Leaving aside the question whether (13.6) holds in general for each completely distributive complete lattice, let us remark that this is the case when $(t^C)^C = t$ holds for each $t \in T$, e.g., when $\langle, \vee, \wedge, ^C \rangle$ is a Boolean algebra. Indeed, in this case (13.8) immediately yields that

$$(t_1 \wedge t_2)^C = (t_1^C)^C \wedge (t_2^C)^C = ((t_1^C \vee t_2^C)^C)^C = t_1^C \vee t_2^C. \tag{13.17}$$

Approaching the task how to define a metric or a relation possessing the properties of numerical metrics in an appropriate degree, we take, again, an inspiration from the naive and intuitive set theory. Within the framework of this theory, it is the operation of symmetric difference which proves some properties which a distance between two subsets of a set $\Omega$ should satisfy. Indeed, setting $A \div B = (A - B) \cup (B - A)$, the first two axioms of metrics hold trivially, as $A \div A = \emptyset$ and $A \div B = B \div A$ are valid for every $A, B \subset \Omega$. Given $A, B, C \subset \Omega$, we obtain easily that

$$\begin{aligned} A - C &= ((A - C) \cap B) \cup ((A - C) \cap B^C) = (A \cap B - C \cap B) \cup \\ &\cup (A \cap B^C - C \cap B^C) \subset (B - C) \cup (A \cap B^C \cap C^C) \subset \\ &\subset (B - C) \cup (A - B). \end{aligned} \tag{13.18}$$

Analogously, $C - A \subset (C - B) \cup (B - A)$ holds, so that

$$A \div C = (A - C) \cup (C - A) \subset (A - B) \cup (B - C) \cup (C - B) \cup (B - A) = (A \div B) \cup (B \div C) \tag{13.19}$$

follows. However, just this relation can be taken as the set-valued equivalent of the triangular inequality for real-valued measures according to which $\rho(x, z) \leq \rho(x, y) + \rho|y, z)$ holds for each $x, y$, and $z$.

As the following assertion demonstrates, this idea can be generalized to distributive complete lattices, i.e., to complete lattices in which the identities $s \wedge \bigvee S = \bigvee_{t \in S}(s \wedge t)$ and $s \vee (\bigwedge S) = \bigwedge_{t \in S}(s \vee t)$ are valid for each $s \in T$ and each *finite* $S \subset T$.

**Theorem 13.1** *Let $\mathcal{T} = \langle T, \leq \rangle$ be a distributive complete lattice such that the pseudo-complement operation $(\cdot)^C$, defined by (7.7), is Boolean-like, i.e., $t \wedge t^C = \mathbf{0}_{\mathcal{T}}$ and $t \vee t^C = \mathbf{1}_{\mathcal{T}}$ hold for every $t \in T$. Let $\rho : T \times T \to T$ be the mapping defined by*

$$\rho(t_1, t_2) = (t_1 \wedge t_2^C) \vee (t_2 \wedge t_1^C) \tag{13.20}$$

*for each $t_1, t_2 \in T$. Then the mapping $\rho$ is a $\mathcal{T}$-valued metric on $T$ in the sense that it is*

- *[i]* reflexive*, i.e., $\rho(t_1, t_1) = \mathbf{0}_{\mathcal{T}}$ for each $t_1 \in T$,*
- *[ii]* symmetric*, i.e., $\rho(t_1, t_2) = \rho(t_2, t_1)$ for each $t_1, t_2 \in T$,*
- *[iii]* triangular inequality *holds, i.e., $\rho(t_1, t_3) \leq \rho(t_1, t_2) \vee \rho(t_2, t_3)$ for each $t_1, t_2, t_3 \in T$*

**Proof 13.2** *For each $t_1 \in T$,*

$$\rho(t_1, t_1) = (t_1 \wedge t_1^C) \vee (t_1^C \wedge t_1) = \mathbf{0}_{\mathcal{T}}, \tag{13.21}$$

*as $(\cdot)^C$ is supposed to be Boolean-like, so that $t_1 \wedge t_1^C = \mathbf{0}_{\mathcal{T}}$ holds. The symmetry of $\rho$ is obvious. Let $t_1, t_2, t_3 \in T$. Then*

$$
\begin{aligned}
& [(t_1 \wedge t_2^C) \vee (t_2 \wedge t_3^C)] \wedge (t_1 \wedge t_3^C) = (t_1 \wedge t_2^C \wedge t_1 \wedge t_3^C) \vee (t_2 \wedge t_3^C \wedge t_1 \wedge t_3^C) = \\
= \;& (t_1 \wedge t_2^C \wedge t_3^C) \vee (t_1 \wedge t_2 \wedge t_3^C) = (t_1 \wedge t_3^C) \wedge (t_2 \vee t_2^C) = t_1 \wedge t_3^C, \qquad (13.22)
\end{aligned}
$$

*applying the distributivity and the Boolean property of* $(\cdot)^C$*, according to which* $t_2 \vee t_2^C = \mathbf{1}_{\mathcal{T}}$*. Hence,*

$$
[(t_1 \wedge t_2^C) \vee (t_2 \wedge t_3^C)] \geq t_1 \wedge t_3^C \qquad (13.23)
$$

*holds. Interchanging mutually the role of* $t_1$ *and* $t_3$*, we obtain that the inequality*

$$
[(t_1^C \wedge t_2) \vee (t_2^C \wedge t_3)] \geq t_1^C \wedge t_3 \qquad (13.24)
$$

*is also valid. Combining (13.23) and (13.24), the inequality*

$$
\begin{aligned}
& \rho(t_1, t_2) \vee \rho(t_2, t_3) = \\
= \;& [(t_1 \wedge t_2^C) \vee (t_2 \wedge t_1^C)] \vee [(t_2 \wedge t_3^C) \vee (t_3 \wedge t_2^C)] \geq \\
\geq \;& (t_1 \wedge t_3^C) \vee (t_3 \wedge t_1^C) = \rho(t_1, t_3) \qquad (13.25)
\end{aligned}
$$

*easily follows and the proof is completed.*

# 14 Dual Axiomatic Approach to Partial Lattice-Valued Possibility and Necessity Measures

Let us begin our reasoning with almost the most general definitions of partial lattice-valued possibility (or possibilistic) and necessity measures, re-phrasing more or less routinely their definitions applied to the case of real-valued set functions. Step by step we will impose, below, more conditions on these measures approaching some more specific but perhaps also more interesting examples of lattice-valued possibility and necessity measures. Some notions have been already introduced, but we take their explicit re-calling in the context of the corresponding dual notions as useful and perhaps convenient for the reader.

**Definition 14.1** *Let* $\Omega$ *be a nonempty set, let* $\mathcal{R}$ *be a nonempty system of subsets of* $\Omega$*, let* $\mathcal{T} = \langle T, \leq \rangle$ *be a complete lattice. A mapping* $\Pi$ *which takes* $\mathcal{R}$ *into* $T(\Pi : \mathcal{R} \to T$*, in symbols) is called a* partial $\mathcal{T}$-(valued) monotone measure on $\mathcal{R}$*, if*

- *[i]* $\Pi(\emptyset) = \mathbf{0}_{\mathcal{T}} (= \bigwedge T)$ *supposing that* $\emptyset \in \mathcal{R}$*,*
- *[ii]* $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}} (= \bigvee T)$ *supposing that* $\Omega \in \mathcal{R}$
- *[iii]* $\Pi(A) \leq \Pi(B)$ *holds for each* $A, B \in \mathcal{R}$ *such that* $A \subset B$ *holds.*

*A partial* $\mathcal{T}$*-monotone measure* $\Pi$ *on* $\mathcal{R}$ *is called a* partial $\mathcal{T}$-(valued) possibilistic measure *on* $\mathcal{R}$*, if* $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ *for each* $A, B, A \cup B \in \mathcal{R}$*. A partial* $\mathcal{T}$*-possibilistic measure* $\Pi$ *on* $\mathcal{R}$ *is called* complete*, if* $\Pi(\bigcup \mathcal{A}) = \bigvee_{A \in \mathcal{A}} \Pi(A)$ *holds for every* $\emptyset \neq \mathcal{A} \subset \mathcal{R}$ *such that* $\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A \in \mathcal{R}$ *holds.*

*A mapping* $\Pi : \mathcal{R} \to T$ *is called a* dual partial $\mathcal{T}$-(valued) possibilistic measure *on* $\mathcal{R}$*, if (i) and (ii) hold and if* $\Pi(A \cap B) = \Pi(A) \wedge \Pi(B)$ *for each* $A, B, A \cap B \in \mathcal{R}$*. A dual partial* $\mathcal{T}$*-possibilistic measure* $\Pi$ *on* $\mathcal{R}$ *is called* complete*, if* $\Pi(\bigcap \mathcal{A}) = \bigwedge_{A \in \mathcal{A}} \Pi(A)$ *holds for each* $\emptyset \neq \mathcal{A} \subset \mathcal{R}$ *such that* $\bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} A \in \mathcal{R}$ *also holds.*

Each dual partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$ is obviously also a $\mathcal{T}$-monotone measure on $\mathcal{R}$. Indeed, given $A, B \in \mathcal{R}$ such that $A \subset B$, we obtain that

$$\Pi(A) = \Pi(A \cap B) = \Pi(A) \wedge \Pi(B) \leq \Pi(B). \tag{14.1}$$

For each partial $\mathcal{T}$-possibilistic measure $\Pi$ on $\mathcal{R}$, and for each $A, B \in \mathcal{R}$ such that $A \cap B \in \mathcal{R}$, the relation $\Pi(A \cap B) \leq \Pi(A) \wedge \Pi(B)$ holds. Dually, for each dual partial $\mathcal{T}$-possibilistic measure $\Pi$ on $\mathcal{R}$, and for each $A, B \in \mathcal{R}$ such that $A \cup B \in \mathcal{R}$, the relation $\Pi(A) \vee \Pi(B) \leq \Pi(A \cup B)$ holds; in no of the two cases the equality holds in general. Indeed, take $\Pi(\emptyset) = \mathbf{0}_{\mathcal{T}},\ \Pi(A) = \mathbf{1}_{\mathcal{T}}$ for all $\emptyset \neq A \subset \Omega$, card $\Omega \geq 2$, in the first case, and $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}},\ \Pi(A) = \mathbf{0}_{\mathcal{T}}$ for each $A \subset \Omega,\ A \neq \Omega$, card $\Omega \geq 2$, in the other case, as the most trivial counter-examples.

In the way analogous to that applied when investigating the real-valued possibilistic measures, we can introduce the notion of $\mathcal{T}$-(valued) partial necessity measure induced by a given partial $\mathcal{T}$-possiblistic measure, and to investigate the relation between dual partial $\mathcal{T}$-possibilistic measures and the partial $\mathcal{T}$-necessity measures.

**Definition 14.2** *Let $\mathcal{T}$ and $\mathcal{R}$ be as in Definition 14.1, let $\Pi : \mathcal{R} \to T$ be a partial $\mathcal{T}$-monotone measure on $\mathcal{R}$, let $\mathcal{R}^- = \{A \subset \Omega : \Omega - A \in \mathcal{R}\}$ be the system of all complements of the subsets from $\mathcal{R}$. The mapping $N_\Pi : \mathcal{R}^- \to T$, defined by*

$$N_\Pi(A) = (\Pi(\Omega - A))^C \tag{14.2}$$

*for every $A \in \mathcal{R}^-$ is called the* partial $\mathcal{T}$-(valued) necessity measure *induced by $\Pi$ on $\mathcal{R}^-$; here $(\cdot)^C$ is the (pseudo-)complement operation defined in Chapter 13.*

It seems to be worth stating explicitly that Definitions 14.1 and 14.2 have been purposely conceived in a very general form. E.g., if $\mathcal{R}$ is a decomposition of $\Omega$ into disjoint nonempty subsets, then any mapping $\Pi : \mathcal{R} \to T$ defines a complete partial $\mathcal{T}$-possibilistic measure, but also a complete dual partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$, simply due to the fact that all the conditions imposed on $\Pi$ expire tautologically.

**Lemma 14.1** *Under the notations and conditions of Lemma 13.1 and Definition 14.2, and supposing that $\Pi$ is a partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$, the mapping $N_\Pi$ is a dual partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}^-$.*

**Proof 14.1** *If $\emptyset \in \mathcal{R}^-$, then $\Omega \in \mathcal{R}$ and $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$, so that $N_\Pi(\emptyset) = (\Pi(\Omega - \emptyset))^C = \mathbf{1}_{\mathcal{T}}^{\mathbf{C}} = \mathbf{0}_{\mathcal{T}}$. Dually, if $\Omega \in \mathcal{R}^-$, then $\emptyset \in \mathcal{R}$ and $\Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}$, so that $N_\Pi(\Omega) = (\Pi(\Omega - \Omega))^C = \mathbf{0}_{\mathcal{T}}{}^C = \mathbf{1}_{\mathcal{T}}$. If $A, B \in \mathcal{R}^-$ are such that $A \cap B \in \mathcal{R}^-$, then $\Omega - A,\ \Omega - B$, and $\Omega - (A \cap B) = (\Omega - A) \cup (\Omega - B)$ are in $\mathcal{R}$, so that*

$$\begin{aligned}
& N_\Pi(A \cap B) = (\Pi(\Omega - (A \cap B)))^C = \\
= \;& (\Pi((\Omega - A) \cup (\Omega - B)))^C = (\Pi(\Omega - A) \vee \Pi(\Omega - B))^C = \\
= \;& (\Pi(\Omega - A))^C \wedge (\Pi(\Omega - B))^C = N_\Pi(A) \wedge N_\Pi(B)
\end{aligned} \tag{14.3}$$

*due to Lemma 13.1. The assertion is proved.*

**Theorem 14.1** *Let the notations and conditions of Definition 14.2 hold, let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice such that $(t^C)^C = t$ for each $t \in T$. Then there exists, for each dual partial $\mathcal{T}$-possibilistic measure $\Sigma$ defined on a nonempty system $\mathcal{R}$ of subsets of $\Omega$, a partial $\mathcal{T}$-possibilistic measure $\Pi$ on $\mathcal{R}^-$ such that $\Sigma$ is identical with $N_\Pi$ on $\mathcal{R}$.*

**Proof 14.2** *Given a dual partial $\mathcal{T}$-possibilistic measure $\Sigma$ on $\emptyset \neq \mathcal{R} \subset \mathcal{P}(\Omega)$, apply the same construction as in Lemma 14.1, setting*

$$\Pi_\Sigma(A) = (\Sigma(\Omega - A))^C \tag{14.4}$$

*for every $A \in \mathcal{R}^- = \{\Omega - A : A \in \mathcal{R}\}$. In the same way as in the proof of Lemma 14.1 we obtain that $\Pi_\Sigma(\emptyset) = \mathbf{0}_\mathcal{T}$ and $\Pi_\Sigma(\Omega) = \mathbf{1}_\mathcal{T}$. Let $A, B \in \mathcal{R}$ be such that $A \cap B \in \mathcal{R}$. Then $\Omega - A \in \mathcal{R}^-$, $\Omega - B \in \mathcal{R}^-$ and $\Omega - (A \cap B) = (\Omega - A) \cup (\Omega - B) \in \mathcal{R}^-$ hold, so that*

$$\begin{aligned}
\Pi_\Sigma(A \cup B) &= (\Sigma(\Omega - (A \cup B)))^C = \\
&= (\Sigma((\Omega - A) \cap (\Omega - B)))^C = \\
&= (\Sigma(\Omega - A) \wedge \Sigma(\Omega - B))^C = \\
&= (\Sigma(\Omega - A))^C \vee (\Sigma(\Omega - B))^C = \Pi_\Sigma(A) \vee \Pi_\Sigma(B)
\end{aligned} \tag{14.5}$$

*due to (13.8) (Lemma 13.1) which holds supposing that $(t^C)^C = t$ for each $t \in T$. Hence, $\Pi_\Sigma$ is a partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}^-$. Defining the corresponding partial $\mathcal{T}$-necessity measure on $(\mathcal{R}^-)^- = \mathcal{R}$ we obtain that*

$$N_{\Pi_\Sigma}(A) = (\Pi_\Sigma(\Omega - A))^C = ((\Sigma(\Omega - (\Omega - A)))^C)^C = \Sigma(A) \tag{14.6}$$

*for every $A \in \mathcal{R}$. The assertion is proved.*

It is almost self-evident that if $\Pi$ is a $\mathcal{T}$-monotone measure on $\mathcal{R}$, then $N_\Pi$ is a $\mathcal{T}$-monotone measure on $\mathcal{R}^-$. The relations $N_\Pi(\emptyset) = \mathbf{0}_\mathcal{T}$ and $N_\Pi(\Omega) = \mathbf{1}_\mathcal{T}$ are obvious. If $A, B \in \mathcal{R}^-$ are such that $A \subset B$ holds, then $\Omega - A \in \mathcal{R}$, $\Omega - B \in \mathcal{R}$, and $\Omega - A \supset \Omega - B$, hence, $\Pi(\Omega - A) \geq \Pi(\Omega - B)$ holds and the inequality

$$N_\Pi(A) = (\Pi(\Omega - A))^C \leq (\Pi(\Omega - B))^C = N_\Pi(B) \tag{14.7}$$

follows.

Let $\pi : \Omega \to T$ be a $\mathcal{T}$-possibilistic distribution on $\Omega$. The *dual $\mathcal{T}$-possibilistic distribution* $\nu = \nu_\pi : \Omega \to T$ will be defined as follows:

$$\nu(\omega) = (\pi(\omega))^C = \bigvee \{s \in T : s \wedge \pi(\omega) = \mathbf{0}_\mathcal{T}\}. \tag{14.8}$$

**Theorem 14.2** *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\Omega$ be a nonempty set, let $\pi : \Omega \to T$ be a $\mathcal{T}$-possibilistic distribution on $\Omega$ which defines the $\mathcal{T}$-possibilistic measure $\Pi$ on $\mathcal{P}(\Omega)$. Then for every $A \subset \Omega$ the inequality*

$$N_\Pi(A) \leq \bigwedge_{\omega \in \Omega - A} \nu(\omega) \tag{14.9}$$

*holds. Moreover, if $T$ is completely distributive in the sense that the identity*

$$s \wedge \bigvee S = \bigvee_{t \in S} (s \wedge t) \tag{14.10}$$

*is valid for each $s \in T$ and each $S \subset T$, the the equality holds in (14.9).*

**Proof 14.3** *By definition,*

$$N_\Pi(A) = (\Pi(\Omega - A))^C = \left( \bigvee_{\omega \in \Omega - A} \pi(\omega) \right)^C = \bigvee \left\{ s \in T : s \wedge \bigvee_{\omega \in \Omega - A} \pi(\omega) = \mathbf{0}_\mathcal{T} \right\}. \tag{14.11}$$

*For each $s \in T$ and each $\omega \in \Omega - A$, if $s \wedge \bigvee_{\omega \in \Omega - A} \pi(\omega) = \mathbf{0}_\mathcal{T}$, then $s \wedge \pi(\omega) = \mathbf{0}_\mathcal{T}$ follows, hence, the inequalities*

$$\bigvee \left\{ s \in T : s \wedge \bigvee_{\omega \in \Omega - A} \pi(\omega) = \mathbf{0}_\mathcal{T} \right\} \leq \bigvee \{s \in T : s \wedge \pi(\omega) = \mathbf{0}_\mathcal{T}\} = (\pi(\omega))^C \tag{14.12}$$

*and*

$$N_\Pi(A) = \bigvee \left\{ s \in T : s \wedge \bigvee_{\omega \in \Omega - A} \pi(\omega) = \mathbf{0}_\mathcal{T} \right\} \leq \bigwedge_{\omega \in \Omega - A} (\pi(\omega))^C = \bigwedge_{\omega \in \Omega - A} \nu(\omega) \tag{14.13}$$

*can be obtained immediately.*
*If $\mathcal{T}$ is completely distributive, then for each $s \in T$ and $A \subset \Omega$ the identity*

$$s \wedge \left( \bigvee_{\omega \in \Omega - A} \pi(\omega) \right) = \bigvee_{\omega \in \Omega - A} (s \wedge \pi(\omega)) \tag{14.14}$$

*is valid. In order to prove the equality in (14.9) we have to prove the inverse inequality*

$$\bigwedge_{\omega \in \Omega - A} (\pi(\omega))^C \leq \left( \bigvee_{\omega \in \Omega - A} \pi(\omega) \right)^C . \tag{14.15}$$

*Due to the supposed complete distributively we obtain that*

$$\begin{aligned} & \left( \bigwedge_{\omega \in \Omega - A} (\pi(\omega))^C \right) \wedge \left( \bigvee_{\omega \in \Omega - A} (\pi(\omega)) \right) = \\ = \quad & \bigvee_{\omega \in \Omega - A} \left[ \left( \bigwedge_{\omega \in \Omega - A} (\pi(\omega))^C \right) \wedge \pi(\omega) \right] \leq \\ \leq \quad & \bigvee_{\omega \in \Omega - A} ((\pi(\omega))^C \wedge \pi(\omega)) = \bigvee_{\omega \in \Omega - A} \mathbf{0}_\mathcal{T} = \mathbf{0}_\mathcal{T}, \end{aligned} \tag{14.16}$$

*and (14.15) immediately follows, so that the equality in (14.9) is proved.*

# 15 Inner and Outer $\mathcal{T}$-Monotone and $\mathcal{T}$-Possibilistic Measures

The pattern applied below in order to extend a partial set function from its domain $\mathcal{R} \subset \mathcal{P}(\Omega)$ to the whole power-set $\mathcal{P}(\Omega)$ is the same as that one used in standard measure theory dealing with real-valued measures (cf. 20] or [38], e.g.), and its most general form reads as follows.

**Definition 15.1** *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\Omega$ be a nonempty set, let $\mathcal{R}$ be a system of subsets of $\Omega$, let $\Pi : \mathcal{R} \to T$ be a partial $\mathcal{T}$-valued set function on $\mathcal{R}$. The* inner (lower) measure $\Pi_\star$ *and the* outer (upper) measure $\Pi^\star$*, induced by $\Pi$ on $\mathcal{P}(\Omega)$, are defined for each $A \subset \Omega$ by*

$$\Pi_\star(A) = \bigvee \{\Pi(B) : B \subset A,\ B \in \mathcal{R}\}, \tag{15.1}$$

$$\Pi^\star(A) = \bigwedge \{\Pi(B) : B \supset A,\ B \in \mathcal{R}\}. \tag{15.2}$$

Both the values $\Pi_\star(A)$ and $\Pi^\star(A)$ are uniquely defined when applying the standard conventions according to which $\bigvee \emptyset = \mathbf{0}_\mathcal{T} = \bigwedge T$ and $\bigwedge \emptyset = \mathbf{1}_\mathcal{T} = \bigvee T$ holds for the empty subset of $T$. Our definition applies also in the most trivial (or the most pathological) case when $\mathcal{R} = \emptyset$, i.e., when $\Pi$ is nowhere defined $\mathcal{T}$-valued function. In this case the conventions apply to each $A \subset \Omega$ and we obtain that $\Pi_\star(A) = \mathbf{0}_\mathcal{T}$ and $\Pi^\star(A) = \mathbf{1}_\mathcal{T}$ for each $A \subset \Omega$. If $\mathcal{R} = \{\emptyset, \Omega\}$ and $\Pi(\emptyset) = \mathbf{0}_\mathcal{T},\ \Pi(\Omega) = \mathbf{1}_\mathcal{T}$, the only difference will be that $\Pi_\star(\Omega) = \mathbf{1}_\mathcal{T}$ and $\Pi^\star(\emptyset) = \mathbf{0}_\mathcal{T}$ follows.

**Lemma 15.1** *Let $\mathcal{T} = \langle T, \leq \rangle, \Omega, \mathcal{R},$ and $\Pi$ be as in Definition 15.1, let $\{\emptyset, \Omega\} \subset \mathcal{R}$, let $\Pi(\emptyset) = \mathbf{0}_\mathcal{T},\ \Pi(\Omega) = \mathbf{1}_\mathcal{T}$. Then both the mappings $\Pi_\star$ and $\Pi^\star$ are $\mathcal{T}$-monotone measures on $\mathcal{P}(\Omega)$.*

**Proof 15.1** *The relations* $\Pi_\star(\emptyset) = \Pi^\star(\emptyset) = \mathbf{0}_\mathcal{T}$ *and* $\Pi_\star(\Omega) = \Pi^\star(\Omega) = \mathbf{1}_\mathcal{T}$ *are obvious. Let* $A \subset B \subset \Omega$ *hold. Then, for each* $C \in \mathcal{R}$, $C \subset A$, *also* $C \subset B$ *holds, and for each* $C \in \mathcal{R}$, $C \supset B$, *also* $C \supset A$ *holds. Hence, the inequalities*

$$\Pi_\star(A) = \bigvee\{\Pi(C) : C \subset A,\ C \in \mathcal{R}\} \leq \bigvee\{\Pi(C) : C \subset B,\ C \in \mathcal{R}\} = \Pi_\star(B), \tag{15.3}$$

$$\Pi^\star(A) = \bigwedge\{\Pi(C) : C \supset A,\ C \in \mathcal{R}\} \leq \bigwedge\{\Pi(C) : C \supset B,\ C \in \mathcal{R}\} = \Pi^\star(B). \tag{15.4}$$

*easily follow and the assertion is proved.*

Lemma 15.1 perhaps justifies the substantive "measure" used for $\Pi_\star$ and $\Pi^\star$.

A weakened form of Lemma 15.1 can be found in [31], Theorem 3.1: If $\emptyset \in \mathcal{R}$ and $\mathcal{T}$ is a complete upper semilattice, then $\Pi_\star$ is a $\mathcal{T}$-monotone measure on $\mathcal{P}(\Omega)$ and, dually, if $\Omega \in \mathcal{R}$ and $\mathcal{T}$ is a complete lower semilattice, then $\Pi^\star$ is a $\mathcal{T}$-monotonous measure on $\mathcal{P}(\Omega)$ (in the weakened sense of $\mathcal{T}$-monotone measure as introduced in [31], when conditions $\Pi(\emptyset) = \mathbf{0}_\mathcal{T}$ and $\Pi(\Omega) = \mathbf{1}_\mathcal{T}$ are omitted). Moreover, if $\Pi$ is a $\mathcal{T}$-monotone measure on $\mathcal{R}$, then both $\Pi_\star$ and $\Pi^\star$ extend $\Pi$ from $\mathcal{R}$ to $\mathcal{P}(\Omega)$ conservatively. Indeed, for each $A \in \mathcal{R}$,

$$\Pi_\star(A) = \bigvee\{\Pi(B) : B \subset A,\ B \in \mathcal{R}\} = \Pi(A) \tag{15.5}$$

and, dually, $\Pi^\star(A) = \Pi(A)$.

As a matter of fact, if $\Pi$ is a $\mathcal{T}$-possibilistic measure on $\mathcal{R}$,. then neither $\Pi_\star$ nor $\Pi^\star$ need be a $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$. Indeed, take $R_1 = \{\emptyset, \Omega\}$ and set $\Pi_1(\emptyset) = \mathbf{0}_\mathcal{T}$, $\Pi_1(\Omega) = \mathbf{1}_\mathcal{T}$, so that $\Pi_1$ is the most trivial partial $\mathcal{T}$-possiblistic measure. Given a nonempty proper subset $A$ of $\Omega$, we obtain that $\Pi_{1\star}(A) = \Pi_{1\star}(\Omega - A) = \mathbf{0}_\mathcal{T}$, so that $\Pi_{1\star}(A \cup (\Omega - A)) = \Pi_{1\star}(\Omega) = \mathbf{1}_\mathcal{T} > \mathbf{0}_\mathcal{T} = \Pi_{1\star}(A) \vee \Pi_{1\star}(\Omega - A)$. Now, take $\mathcal{R}_2 = \{\emptyset, A, B, \Omega\}$, where $A$ and $B$ are mutually disjoint nonempty subsets of $\Omega$ such that $A \cup B \neq \Omega$, hence, $A \cup B$ is not in $\mathcal{R}$, and set $\Pi_2(\emptyset) = \Pi_2(A) = \Pi_2(B) = \mathbf{0}_\mathcal{T}$, $\Pi_2(\Omega) = \mathbf{1}_\mathcal{T}$. So, $\Pi_2$ is a partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}_2$ and we obtain that $\Pi_2^\star(\Omega) = \Pi_2^\star(A \cup B) = \mathbf{1}_\mathcal{T} > \mathbf{0}_\mathcal{T} = \Pi_2^\star(A) \vee \Pi_2^\star(B)$. Consequently, neither $\Pi_{1\star}$ nor $\Pi_2^\star$ are $\mathcal{T}$-possibilistic measures on $\mathcal{P}(\Omega)$.

The next statement introduces some sufficient conditions under which the mapping $\Pi_\star$ induced by a partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$ is also a $\mathcal{T}$-possibilistic measure.

**Theorem 15.1** *Let* $\mathcal{T}, \Omega$, *and* $\mathcal{R}$ *be as in Definition 15.1, let* $\Pi$ *be a partial* $\mathcal{T}$*-possibilistic measure on* $\mathcal{R}$. *Let*

$$\mathcal{R}_0 = \{A \subset \Omega : A \cap C \in \mathcal{R} \text{ for each } C \in \mathcal{R}\}. \tag{15.6}$$

*Then* $\Pi_\star$ *is a partial* $\mathcal{T}$*-possibilistic measure on* $\mathcal{R}_0$.

**Proof 15.2** *For every* $A, B \in \mathcal{R}_0$,

$$\begin{aligned} \Pi_\star(A \cup B) = & \bigvee_{C \in \mathcal{R}, C \subset A \cup B} \Pi(C) = \\ = & \bigvee_{C \in \mathcal{R}, C \subset A \cup B} \Pi(C \cap (A \cup B)) = \\ = & \bigvee_{C \in \mathcal{R}, C \subset A \cup B} \Pi((C \cap A) \cup (C \cap B)). \end{aligned} \tag{15.7}$$

*As* $C \in \mathcal{R}$ *and* $A, B \in \mathcal{R}_0$, *we obtain that* $C \cap A \in \mathcal{R}$ *and* $C \cap B \in \mathcal{R}$, *so that*

$$\Pi((C \cap A) \cup (C \cap B)) = \Pi(C \cap A) \vee \Pi(C \cap B), \tag{15.8}$$

*as* $\Pi$ *is a* $\mathcal{T}$*-possibilistic measure on* $\mathcal{R}$. *Hence,*

$$\Pi_\star(A\cup B) = \bigvee_{C\in\mathcal{R},C\subset A\cup B} (\Pi(C\cap A)\vee\Pi(C\cap B)) = \left(\bigvee_{C\in\mathcal{R},C\subset A\cup B} \Pi(C\cap A)\right)\vee\left(\bigvee_{C\in\mathcal{R},C\subset A\cup B} \Pi(C\cap B)\right). \tag{15.9}$$

*Consider the value* $\bigvee_{C\in\mathcal{R},C\subset A\cup B}\Pi(C\cap A)$*. If* $C\in\mathcal{R},\ A\in\mathcal{R}_0$ *holds, then* $C\cap A\subset A,\ C\cap A\in\mathcal{R}_0$ *follows, so that* $\Pi(C\cap A)$ *is among the values the supremum of which defines* $\Pi_\star(A)$ *and the inequality*

$$\bigvee_{C\in\mathcal{R},C\subset A\cup B} (\Pi(C\cap A) \leq \bigvee_{D\in\mathcal{R},D\subset A} \Pi(D) = \Pi_\star(A) \tag{15.10}$$

*follows. On the other side, if* $D\in\mathcal{R},\ D\subset A$ *holds, then* $D\subset A\cup B,\ D=D\cap A$ *holds so well, so that* $\Pi(D)$ *is among the values the supremum of which stands on the left-hand side of (15.10). Consequently, the inequality inverse to (15.10) is also proved so that the equality follows. In the same way we obtain that*

$$\bigvee_{C\in\mathcal{R},C\subset A\cup B} (\Pi(C\cap B) = \Pi_\star(B), \tag{15.11}$$

*and we arrive at the conclusion that* $\Pi_\star(A\cup B)=\Pi_\star(A)\vee\Pi_\star(B)$*, hence,* $\Pi_\star$ *is a* $\mathcal{T}$*-possibilistic measure on* $\mathcal{R}_0$*.*

The three following remarks are worth being stated explicitly

- [i] If $\mathcal{R}$ is closed with respect to intersection, i.e., if $A\cap B\in\mathcal{R}$ for each $A,B\in\mathcal{R}$, then the inclusion $\mathcal{R}\subset\mathcal{R}_0$ easily follows, so that $\Pi_\star$ extends $\Pi$ conservatively from $\mathcal{R}$ to $\mathcal{R}_0$.
- [ii] If $\mathcal{R}$ is closed w.r. to the subsets, i.e., if $C\in\mathcal{R}$ and $D\subset C$ implies that $D\in\mathcal{R}$, and if $\Omega\in\mathcal{R}$, then $\mathcal{R}_0=\mathcal{P}(\Omega)$ (the system closed w.r. to subsets are called *hereditary* in [20]).
- [iii] If $\Pi$ is a *complete* partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$ and the other conditions of Theorem 15.1 are fulfilled, then $\Pi_\star$ is a *complete* $\mathcal{T}$-possibilistic measure on $\mathcal{R}_0$, as an easy modification of the proof of Theorem 15.1 demonstrates.

A partially ordered set $\mathcal{T}=\langle T,\leq\rangle$ is called *continuous from above ($\downarrow$-continuous)*, if for each $x\in T$ the infimum of all elements greater than $x$ is defined and equals to $x$, in symbols, if $x=\bigwedge\{t\in T : t>x\}$ holds for each $x\in T$.

**Theorem 15.2** *Let* $\mathcal{T},\Omega,$ *and* $\mathcal{R}$ *be as in Definition 15.1, let* $\mathcal{R}$ *be closed with respect to union, i.e.,* $A\cup B\in\mathcal{R}$ *for each* $A,B\in\mathcal{R}$*, let* $\mathcal{T}$ *be a* $\downarrow$*-continuous complete lower semilattice and (not necessarily complete) upper semilattice, let* $\Pi$ *be a partial* $\mathcal{T}$*-possibilistic measure on* $\mathcal{R}$*. Then* $\Pi^\star$ *is a* $\mathcal{T}$*-possibilistic measure on* $\mathcal{P}(\Omega)$*.*

**Proof 15.3** *For each* $A,B\subset\Omega$ *the value*

$$\Pi^\star(A\cup B) = \bigwedge_{C\in\mathcal{R},C\supset A\cup B} \Pi(C) \tag{15.12}$$

*is defined, as* $\mathcal{T}$ *is supposed to be a complete lower semilattice. For each* $C\in\mathcal{R}$*, if* $C\supset A\cup B$*, then* $C\supset A$ *and* $C\supset B$*, hence, the relations*

$$\Pi^\star(A\cup B) \geq \bigvee_{C\in\mathcal{R},C\supset A} \Pi(C) = \Pi^\star(A), \tag{15.13}$$

$$\Pi^\star(A\cup B) \geq \bigvee_{C\in\mathcal{R},C\supset B} \Pi(C) = \Pi^\star(B), \tag{15.14}$$

*and*

$$\Pi^\star(A \cup B) \geq \Pi^\star(A) \vee \Pi^\star(B) \tag{15.15}$$

*follow, the last supremum being defined due to the assumption that $\mathcal{T}$ is an upper semilattice. Due to the definition of infimum in $\mathcal{T}$, for each $t_1 \in T,\ t_1 > \Pi^\star(A)$, there exists $C_1 \in \mathcal{R}$ such that $C_1 \supset A$ and $\Pi(C_1) < t_1$ holds, analogously, for each $t_2 \in T,\ t_2 > \Pi^\star(B)$, there exists $C_2 \in \mathcal{R}$ such that $C_2 \supset B$ and $\Pi(C_2) < t_2$ hold. As $\mathcal{R}$ is closed with respect to unions, $C_1 \cup C_2 \in \mathcal{R},\ C_1 \cup C_2 \supset A \cup B$, hence, $\Pi^\star(A \cup B) \leq \Pi(C_1 \cup C_2) = \Pi(C_1) \vee \Pi(C_2) \leq t_1 \vee t_2$ follows, as $\Pi$ is a partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$. In particular, for every $t > \Pi^\star(A) \vee \Pi^\star(B)$ we obtain that $t > \Pi^\star(A),\ t > \Pi^\star(B)$, hence, $\Pi^\star(A \cup B) \leq t$. So,*

$$\Pi^\star(A \cup B) = \bigwedge\{t \in T : t > \Pi^\star(A) \vee \Pi^\star(B)\} = \Pi^\star(A) \vee \Pi^\star(B), \tag{15.16}$$

*as $\mathcal{T}$ is $\downarrow$-continuous. So, $\Pi^\star$ is a $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$.*

Given a partial $\mathcal{T}$-monotone measure $\Pi$ on a system $\mathcal{R}$ of subsets of a nonempty set $\Omega$ and supposing, in order to simplify further considerations and reasoning, that $\{\emptyset, \Omega\} \subset \mathcal{R}$, we may either extend it to $\mathcal{P}(\Omega)$, introducing $\Pi_\star$ and/or $\Pi^\star$ as above, or we may define the necessity measure $N_\Pi$ on $\mathcal{R}^- = \{A \subset \Omega : \Omega - A \in \mathcal{R}\}$, setting $N_\Pi(A) = (\Pi(\Omega - A))^C$ for each $A \in \mathcal{R}^-$. These operations can be applied also sequentially, step by step, and in different order. So, we arrive at $N_{(\Pi_\star)}$, if $\Pi_\star$ is defined first, or to $(N_\Pi)_\star$, if $N_\Pi$ on $\mathcal{R}^-$ is defined and then extended to the whole power-set $\mathcal{P}(\Omega)$. The dual case with $N_{(\Pi^\star)}$ and $(N_\Pi)^\star$ will be investigated later.

**Lemma 15.2** *Under the notations just introduced, the inequality*

$$(N_\Pi)_\star(A) \leq N_{(\Pi_\star)}(A) \tag{15.17}$$

*holds foe each $A \subset \Omega$.*

**Proof 15.4** *For each $A \subset \Omega$ we obtain that*

$$\begin{aligned}
(N_\Pi)_\star(A) &= \bigvee\{N_\Pi(B) : B \subset A, B \in \mathcal{R}^-\} = \\
&= \bigvee\{(\Pi(\Omega - B))^C : \Omega - B \supset \Omega - A,\ \Omega - B \in \mathcal{R}\} = \\
&= \bigvee\left\{\bigvee\{s \in T : s \wedge \Pi(\Omega - B) = \mathbf{0}_\mathcal{T}\} : \Omega - B \supset \Omega - A,\ \Omega - B \in R\right\}.
\end{aligned} \tag{15.18}$$

*For each $B \subset \Omega$, if $\Omega - B \in \mathcal{R}$ and $\Omega - B \supset \Omega - A$ holds, then the inequality*

$$\Pi(\Omega - B) \geq \Pi^\star(\Omega - A) \geq \Pi_\star(\Omega - A) \tag{15.19}$$

*follows. Consequently, for each $B \subset \Omega,\ \Omega - B \in \mathcal{R}$, and each $s \in T$, if $s \wedge \Pi(\Omega - B) = \mathbf{0}_\mathcal{T}$, then also $s \wedge \Pi_\star(\Omega - B) = \mathbf{0}_\mathcal{T}$ holds. Hence, the inclusion*

$$\begin{aligned}
\bigcup\ & \{\{s \in T : s \wedge \Pi(\Omega - B) = \mathbf{0}_\mathcal{T}\} : B \subset \Omega,\ \Omega - B \in \mathcal{R},\ \Omega - B \supset \Omega - A\} \subset \\
\subset\ & \{s \in T : s \wedge \Pi_\star(\Omega - A) = \mathbf{0}_\mathcal{T}\}
\end{aligned} \tag{15.20}$$

*is valid. Taking the suprema of both the sets in (15.20) and applying (15.18) we obtain that the relation*

$$(N_\Pi)_\star(A) \leq \bigvee\{s \in T : s \wedge \Pi_\star(\Omega - A) = \mathbf{0}_\mathcal{T}\} = (\Pi_\star(\Omega - A))^C = N_{(\Pi_\star)}(A) \tag{15.21}$$

*holds. The assertion is proved.*

The most trivial example illustrates that equality in (15.17) need not hold in general. Take $\mathcal{R} = \{\emptyset, \Omega\},\ \Pi(\emptyset) = \mathbf{0}_{\mathcal{T}},\ \Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$. Then

$$N_{\Pi}(\emptyset) = (\Pi(\Omega - \emptyset))^C = (\Pi(\Omega))^C = \mathbf{1}_{\mathcal{T}}{}^C = \mathbf{0}_{\mathcal{T}}, \tag{15.22}$$

$$N_{\Pi}(\Omega) = (\Pi(\Omega - \Omega))^C = (\Pi(\emptyset))^C = \mathbf{0}_{\mathcal{T}}{}^C = \mathbf{1}_{\mathcal{T}}. \tag{15.23}$$

Hence, for each $A \subset \Omega,\ \emptyset \neq A \neq \Omega$, we obtain that

$$(N_{\Pi})_{\star}(A) = \bigvee\{N_{\Pi}(B) : B \subset A,\ B \in \mathcal{R}^{-}\} = N_{\Pi}(\emptyset) = \mathbf{0}_{\mathcal{T}}, \tag{15.24}$$

as $\emptyset$ is the only subset of $A$ which is in $\mathcal{R}^{-}$. However,

$$N_{(\Pi_{\star})}(A) = (\Pi_{\star}(\Omega - A))^C = \left(\bigvee\{\Pi(B) : B \subset A,\ B \in \mathcal{R}\}\right)^C = (\Pi(\emptyset))^C = \mathbf{1}_{\mathcal{T}} \tag{15.25}$$

as $\emptyset$ is the only subset of $\Omega - A$ which is in $\mathcal{R}$, so that the strict inequality $(N_{\Pi})_{\star}(A) < N_{(\Pi_{\star})}(A)$ follows.

The assertion dual to Lemma 15.2 reads as follows.

**Lemma 15.3** *Under the notations and conditions introduced, the inequality*

$$(N_{\Pi})^{\star}(A) \geq N_{(\Pi^{\star})}(A) \tag{15.26}$$

*holds for each $A \subset \Omega$.*

**Proof 15.5** *Analyzing the definitions of $(N_{\Pi})^{\star}(A)$ and $N_{(\Pi^{\star})}(A)$, we obtain that*

$$\begin{aligned}
(N_{\Pi})^{\star}(A) &= \bigwedge\{N_{\Pi}(B) : B \supset A,\ B \in \mathcal{R}^{-}\} = \\
&= \bigwedge\{(\Pi(\Omega - B))^C : B \supset A,\ B \in \mathcal{R}^{-}\} = \\
&= \bigwedge\left\{\left[\bigvee\{s \in T : s \wedge \Pi(\Omega - B) = \mathbf{0}_{\mathcal{T}}\right\}\right] : B \supset A,\ B \in R^{-}\}.
\end{aligned} \tag{15.27}$$

*and*

$$N_{(\Pi^{\star})}(A) \leq (\Pi^{\star}(\Omega - A))^C = \bigvee\{s \in T : s \wedge \Pi^{\star}(\Omega - A) = \mathbf{0}_{\mathcal{T}}\}. \tag{15.28}$$

*Take $A \subset \Omega$, take $B \in \mathcal{R}^{-}$ such that $B \supset A$ holds, then $\Omega - B \in \mathcal{R},\ \Omega - B \subset \Omega - A$ and, consequently, also*

$$\Pi(\Omega - B) \leq \Pi_{\star}(\Omega - A) = \bigvee\{\Pi(C) : C \subset \Omega - A,\ C \in \mathcal{R}\} \leq \Pi^{\star}(\Omega - A) \tag{15.29}$$

*follows. Hence, for each $s \in T$ such that $s \wedge \Pi^{\star}(\Omega - A) = \mathbf{0}_{\mathcal{T}}$ also $s \wedge \Pi(\Omega - B) = \mathbf{0}_{\mathcal{T}}$ holds, so that the set inclusion*

$$\{s \in T : s \wedge \Pi^{\star}(\Omega - A) = \mathbf{0}_{\mathcal{T}}\} \subset \{s \in T : s \wedge \Pi(\Omega - B) = \mathbf{0}_{\mathcal{T}}\} \tag{15.30}$$

*is valid for each $B \in \mathcal{R}^{-},\ B \supset A$. Consequently, for each such $B$ we obtain that*

$$\bigvee\{s \in T : s \wedge \Pi^{\star}(\Omega - A) = \mathbf{0}_{\mathcal{T}}\} \leq \bigvee\{s \in T : s \wedge \Pi(\Omega - B) = \mathbf{0}_{\mathcal{T}}\}, \tag{15.31}$$

*so that the inequality*

$$\bigvee\{s \in T : s \wedge \Pi^{\star}(\Omega - A) = \mathbf{0}_{\mathcal{T}}\} \leq \bigwedge\left\{\left[\bigvee\{s \in T : s \wedge \Pi(\Omega - B) = \mathbf{0}_{\mathcal{T}}\right\}\right] : B \in R^{-},\ B \supset A\} \tag{15.32}$$

*follows. Due to (15.27) and (15.28) this inequality immediately applies (15.26), so that the assertion is proved.*

The example above with $\mathcal{R} = \{\emptyset, \Omega\}$, $\Pi(\emptyset) = \mathbf{0}_{\mathcal{T}}$ and $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$ demonstrates that also in (15.26) the equality does not hold in general. Again, taking $A \subset \Omega$, $\emptyset \neq A \neq \Omega$, we obtain that

$$(N_\Pi)^\star(A) = \bigwedge\{N_\Pi(B) : B \in \mathcal{R}^-,\ B \supset A\} = N_\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}, \tag{15.33}$$

but

$$N_{(\Pi^\star)}(A) = (\Pi^\star(\Omega - A))^C = \left(\bigwedge\{\Pi(C) : C \supset A,\ C \in \mathcal{R}\}\right)^C = (\Pi(\Omega))^C = \mathbf{0}_{\mathcal{T}}, \tag{15.34}$$

so that the strict inequality $(N_\Pi)^\star(A) > N_{(\Pi^\star)}(A)$ follows.

# 16 Possibilistic Completion of Partial Lattice-Valued Possibilistic Measures

Besides the inner and outer measures, another way how to extend a partial $\mathcal{T}$-possiblistic measure $\Pi$ from its domain $\mathcal{R} \subset \mathcal{P}(\Omega)$ to the whole power-set $\mathcal{P}(\Omega)$ can be inspired by the following property of the total (i.e., defined on $\mathcal{P}(\Omega)$) $\mathcal{T}$-possibilistic measures. Let $\mathcal{T}$ be a complete lattice, let $\Pi$ be a $\mathcal{T}$-monotone measure on $\mathcal{P}(\Omega)$. Hence, $\{\omega\} \subset R$ and $\Pi(\{\omega\}) \leq \Pi(R)$ holds for each $R, \omega \in R \subset \Omega$. As $\{\omega\}$ itself is one of such $R$'s, the identity

$$\Pi(\{\omega\}) = \bigwedge\{\Pi(R) : \omega \in R \subset \Omega\} \tag{16.1}$$

follows. If $\Pi$ is a $\mathcal{T}$-possibilistic measure on $\mathcal{P}(\Omega)$, defined by a $\mathcal{T}$-possibilistic distribution $\pi : \Omega \to T$, then $\pi(\omega) = \Pi(\{\omega\})$ for every $\omega \in \Omega$, and for each $A \subset \Omega$ we obtain that

$$\Pi(A) = \bigvee_{\omega \in A} \left(\bigwedge\{\Pi(R) : R \in \mathcal{R}_\omega\}\right), \tag{16.2}$$

where $\mathcal{R}_\omega = \{R \in \mathcal{P}(\Omega) : \omega \in R\}$. Each $\mathcal{R}_\omega$ is nonempty, as $\Omega \in \mathcal{R}_\omega$ for every $\omega \in \Omega$.

If $\Omega$ is finite, then $\{\Pi(\{\omega\}) : \omega \in \Omega\}$ is the $\mathcal{T}$-possibilistic distribution which defines $\Pi$, as in this case the identities

$$\mathbf{1}_{\mathcal{T}} = \Pi(\Omega) = \Pi\left(\bigcup_{\omega \in \Omega} \{\omega\}\right) = \bigvee_{\omega \in \Omega} \Pi(\{\omega\}) \tag{16.3}$$

and, for each $A \subset \Omega$,

$$\Pi(A) = \Pi\left(\bigcup_{\omega \in A} \{\omega\}\right) = \bigvee_{\omega \in A} \Pi(\{\omega\}) \tag{16.4}$$

are obvious. However, if $\Omega$ is infinite, this need not be the case. Indeed, let $\Pi : \mathcal{P}(\Omega) \to T$ be such that $\Pi(A) = \mathbf{0}_{\mathcal{T}}$, if $A$ is empty or finite, and $\Pi(A) = \mathbf{1}_{\mathcal{T}}$ for infinite $A$'s. Then $\Pi(\{\omega\}) = \bigwedge\{\Pi(R) : \omega \in R \subset \Omega\} = \mathbf{0}_{\mathcal{T}}$ for each $\omega \in \Omega$, but this mapping does not define a $\mathcal{T}$-possibilistic distribution on $\Omega$.

Let $\mathcal{R}$ be a system of subsets of $\Omega$ on which a partial $\mathcal{T}$-possibilistic measure $\Pi$ is defined. As above, let us set

$$\pi_0(\omega) = \bigwedge\{\Pi(R) : \omega \in R \in \mathcal{R}\} = \bigwedge\{\Pi(R) : R \in \mathcal{R}_\omega\} \tag{16.5}$$

for every $\omega \in \Omega$, here $\mathcal{R}_\omega = \{R \in \mathcal{R} : \omega \in R\}$. Either we may suppose that $\{\emptyset, \Omega\} \subset \mathcal{R}$, so that $\Omega \in \mathcal{R}_\omega$, so that $\Omega \in \mathcal{R}_\omega$ for every $\omega \in \Omega$, or we may apply the convention according to which $\bigwedge \emptyset = \mathbf{1}_{\mathcal{T}}$ for the empty subset of $T$, so that $\pi_0(\omega) = \mathbf{1}_{\mathcal{T}}$ for every $\omega \in \Omega - \bigcup \mathcal{R} (\bigcup \mathcal{R} = \bigcup_{R \in \mathcal{R}} R)$. Obviously, the resulting mapping $\pi_0 : \Omega \to T$ will be the same as if joining $\Omega$ with $\mathcal{R}$ and setting $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$.

Below, we will investigate whether, and under which conditions, $\pi_0$ defines a $\mathcal{T}$-possibilistic distribution on $\Omega$. This being the case, we set $\Pi_0(A) = \bigvee_{\omega \in A} \pi_0(\omega)$ for each $A \subset \Omega$ and we will analyze the properties of the possibilistic measure $\Pi_0$ and its relations to $\Pi, \Pi_\star$ and $\Pi^\star$.

**Lemma 16.1** *Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\Omega$ be a finite nonempty set, let $\mathcal{R} \subset \mathcal{P}(\Omega)$ be such that $\{\emptyset, \Omega\} \subset \mathcal{R}$, let $\Pi$ be a partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$. Let, for each $\omega \in \Omega$,*

$$\pi_0(\omega) = \bigwedge \{\Pi(R) : \omega \in R \in \mathcal{R}\}. \tag{16.6}$$

*Then $\pi_0 : \Omega \to T$ is a $\mathcal{T}$-possibilistic distribution on $\Omega$, i.e. $\bigvee_{\omega \in \Omega} \pi_0(\omega) = \mathbf{1}_\mathcal{T}$.*

**Proof 16.1** *We have to prove that*

$$\bigvee_{\omega \in \Omega} \pi_0(\omega) = \bigvee_{\omega \in \Omega} \left( \bigwedge \{\Pi(R) : \omega \in R \in \mathcal{R}\} \right) = \mathbf{1}_\mathcal{T}, \tag{16.7}$$

*where $\mathcal{R}_\omega = \{R \in \mathcal{R} : \omega \in \mathcal{R}\}$ as above. First, suppose that $\mathcal{T} = \langle T, \leq \rangle$ is such that $\bigvee \{t \in T : t < \mathbf{1}_\mathcal{T}\} = t_0 < \mathbf{1}_\mathcal{T}$ holds, so that $t \leq t_0 < \mathbf{1}_\mathcal{T}$ is valid for each $t < \mathbf{1}_\mathcal{T}$, and suppose that (16.7) does not hold. Hence,*

$$\bigvee_{\omega \in \Omega} \left( \bigwedge \{\Pi(R) : \omega \in R \in \mathcal{R}\} \right) \leq t_0 \tag{16.8}$$

*and*

$$\bigwedge \{\Pi(R) : \omega \in R \in \mathcal{R}\} \leq t_0 \tag{16.9}$$

*for each $\omega \in \Omega$ follows. If $\Pi(R) > t_0$ holds for each $R \in \mathcal{R}_\omega$, the only case may be that $\Pi(R) = \mathbf{1}_\mathcal{T}$ for each $R \in \mathcal{R}_\omega$, but in this case $\bigwedge \{\Pi(R) : \omega \in R \in \mathcal{R}\} = \mathbf{1}_\mathcal{T} > t_0$. Consequently, for each $\omega \in \Omega$ there exists $R_\omega \in \mathcal{R}_\omega$ such that $\Pi(R_\omega) \leq t_0$ holds. However, $\omega \in R_\omega$, so that $\bigcup_{\omega \in \Omega} R_\omega = \Omega$ and $\Pi(\bigcup_{\omega \in \Omega} R_\omega) = \bigvee_{\omega \in \Omega} \Pi(R_\omega) = \mathbf{1}_\mathcal{T}$ follows. On the other hand, $\Pi(R_\omega) \leq t_0$ for each $\omega \in \Omega$ yields that $\bigvee_{\omega \in \Omega} \Pi(R_\omega) \leq t_0 < \mathbf{1}_\mathcal{T}$ should be valid. This contradiction implies that under the condition that $\bigvee \{t \in T : t < \mathbf{1}_\mathcal{T}\} < \mathbf{1}_\mathcal{T}$ holds, (16.7) follows.*

*Hence, suppose that $\bigvee \{t \in T : t < \mathbf{1}_\mathcal{T}\} = \mathbf{1}_\mathcal{T}$ holds and suppose, moreover, that there exists $t \in T,\ t < \mathbf{1}_\mathcal{T}$ such that, for each $\omega \in \Omega$, there exists $R_\omega \in \mathcal{R}_\omega$ such that $\Pi(R_\omega) \leq t$ holds, in symbols, suppose that*

$$(\exists t \in T,\ t < \mathbf{1}_\mathcal{T})(\forall \omega \in \Omega)(\exists R_\omega \in \mathcal{R}_\omega)(\Pi(R_\omega) \leq t). \tag{16.10}$$

*Using the same way of reasoning as above, we arrive again at the contradiction that $\bigcup_{\omega \in \Omega} R_\omega = \Omega$, but $\bigvee_{\omega \in \Omega} \Pi(R_\omega) \leq t < \mathbf{1}_\mathcal{T}$ should be valid simultaneously. Hence, the only what remains is that $\bigvee \{t \in T : t < \mathbf{1}_\mathcal{T}\} = \mathbf{1}_\mathcal{T}$ and the negation of (16.10) holds simultaneously. However, the negation of (16.10) means that*

$$(\forall t \in T,\ t < \mathbf{1}_\mathcal{T})(\exists \omega \in \Omega)(\forall R_\omega \in \mathcal{R}_\omega)(\Pi(R_\omega) > t). \tag{16.11}$$

*This implies, using (16.6), that*

$$(\forall t \in T,\ t < \mathbf{1}_\mathcal{T})(\exists \omega \in \Omega)(\pi_0(\omega) \geq t), \tag{16.12}$$

*consequently,*

$$\bigvee_{\omega \in \Omega} \pi_0(\omega) = \bigvee \{t \in T : t < \mathbf{1}_\mathcal{T}\} = \mathbf{1}_\mathcal{T}. \tag{16.13}$$

*The assertion is proved.*

The idea immediately arising is to define the possibilistic measure $\Pi_0$ on $\mathcal{P}(\Omega)$, setting

$$\Pi_0(A) = \bigvee_{\omega\in\Omega} \pi_0(\omega) \tag{16.14}$$

for every $A \subset \Omega$. For each $A \in \mathcal{R}$ and each $\omega \in A$ the relation $A \in \mathcal{R}_\omega = \{R \in \mathcal{R} : \omega \in R\}$ is obvious, so that the relations

$$\pi_0(\omega) = \bigwedge\{\Pi(R) : R \in \mathcal{R}_\omega\} \leq \Pi(A) \tag{16.15}$$

and

$$\Pi_0(A) = \bigvee_{\omega\in A} \pi_0(\omega) \leq \Pi(A) \tag{16.16}$$

easily follow. In general, the inequality in (16.16) cannot be replaced by equality, as the following simple example demonstrates.

Let $\Omega = \{1,2,3,4,5\}$, let $A_1 = \{1,2\}$, $A_2 = \{3,4\}$, $A_3 = \{2,3\}$, let $\mathcal{R} = \{\emptyset, A_1, A_2, A_3, \Omega\}$, let $\Pi : \mathcal{R} \to T$ be such that $\Pi(\emptyset) = \Pi(A_1) = \Pi(A_2) = \mathbf{0}_\mathcal{T}$, $\Pi(A_3) = \Pi(\Omega) = \mathbf{1}_\mathcal{T}$. Up to trivial cases $\emptyset \cup \Omega$, $\emptyset \cup A_i$ and $A_i \cup \Omega$, $i = 1,2,3$, no unions created by sets $A_1, A_2, A_3$ are in $\mathcal{R}$, so that $\Pi$ is a partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$.

For each $i = 1,2,3,4$,

$$\pi_0(i) = \bigwedge\{\Pi(R) : i \in R \in \mathcal{R}\} = \mathbf{0}_\mathcal{T}, \tag{16.17}$$

as there always exists $R \in \mathcal{R}$ such that $i \in R$ and $\Pi(R) = \mathbf{0}_\mathcal{T}$ (namely, $R = A_1$, for 1,2, $R = A_2$ for 3,4). For 5 we obtain that $\pi_0(5) = \mathbf{1}_\mathcal{T} = \Pi(\Omega)$, as $\Omega$ is the only set in $\mathcal{R}$ which contains 5, consequently, $\{\pi_0(i) : i = 1,2,\ldots,5\}$ defines, indeed, a $\mathcal{T}$-possibilistic distribution on $\Omega$. So

$$\Pi_0(A_3) = \Pi_0(\{2,3\}) = \pi_0(2) \vee \pi_0(3) = \mathbf{0}_\mathcal{T} < \mathbf{1}_\mathcal{T} = \Pi(A_3). \tag{16.18}$$

Let us recall that for singletons the equality $\Pi_0(\{\omega\}) = \Pi(\{\omega\})$ holds for each $\omega$ such that $\{\omega\} \in \mathcal{R}$. Indeed, $\Pi_0(\{\omega\}) \leq \Pi(\{\omega\})$ follows from (16.16) as a particular case. If $\{\omega\}, A \in \mathcal{R}$ are such that $\omega \in A$, then $\{\omega\} \subset A$ and $\Pi(\{\omega\}) \leq \Pi(A)$ follows. Hence, the relation

$$\Pi(\{\omega\}) \leq \bigwedge\{\Pi(A) : \omega \in A \in \mathcal{R}\} = \pi_0(\omega) = \Pi_0(\{\omega\}) \tag{16.19}$$

is valid due to the definition of $\pi_0(\omega)$. Consequently, if $\Omega$ is finite and all singletons are in $\mathcal{R}$, then the only $\Pi_1$ which extends conservatively $\Pi$ from $\mathcal{R}$ to $\mathcal{P}(\Omega)$ is such that, for every $A \subset \Omega$,

$$\Pi_1(A) = \Pi\left(\bigcup_{\omega\in A}\{\omega\}\right) = \bigvee_{\omega\in A} \Pi(\{\omega\}) = \bigvee_{\omega\in A} \pi_0(\omega) = \Pi_0(A). \tag{16.20}$$

The same result, i.e., the identity of the only conservative extension of $\Pi$ with $\Pi_0$ on the whole power-set $P(\Omega)$, follows also when $\mathcal{R}$ is closed with respect to intersections, i.e., when $A \cap B \in \mathcal{R}$ holds for each $A, B \in \mathcal{R}$. Indeed, given $A \in \mathcal{R}$, consider the system $\mathcal{R}_A = \{R \cap A : R \in \mathcal{R}\}$ of subsets of $A$; due to the assumption, $\mathcal{R}_A \subset \mathcal{R}$ holds, so that $\Pi(B) \leq \Pi(A)$ is defined for each $B \in \mathcal{R}_A$. Each covering $\mathcal{A} \subset \mathcal{R}$ such that $\bigcup \mathcal{A} \supset A$ can be replaced by covering $\mathcal{A}_A = \{B \cap A : B \in \mathcal{A}\}$ such that $\bigcup \mathcal{A}_A = A$, $\bigcup \mathcal{A}_1 \in \mathcal{R}$, and $\Pi(\bigcup \mathcal{A}_A) \leq \bigvee\{\Pi(C) : C \in \mathcal{A}\}$ holds, so that the main idea of the counter-example above with $\Omega = \{1,2,\ldots,5\}$ fails. Consequently, the equality $\bigvee_{\omega\in A} \pi_0(\omega) = \Pi(A)$ can be proved in the same way as the relation $\bigvee_{\omega\in\Omega} \pi_0(\omega) = \mathbf{1}_\mathcal{T} = \Pi(\Omega)$ in Lemma 16.1, just with $\Omega$ replaced by $A$, $\mathbf{1}_\mathcal{T}$ replaced by $\Pi(A)$, and $\mathcal{R} \subset \mathcal{P}(\Omega)$ replaced by $\mathcal{R}_A \subset \mathcal{P}(A)$.

**Lemma 16.2** *Let $\Pi$ be a partial $\mathcal{T}$-possibilistic measure defined on a system $\mathcal{R} \subset \mathcal{P}(\Omega)$ such that $\{\emptyset, \Omega\} \subset \mathcal{R}$. Then, for each $A \in \mathcal{R}^- = \{\Omega - B : B \in \mathcal{R}\}$, the inequality*

$$(N_\Pi)_0(A) \leq N_\Pi(A) \leq N_{(\Pi_0)}(A) \tag{16.21}$$

*holds.*

**Proof 16.2** *For each $A \in \mathcal{R}^-$,*

$$(N_\Pi)_0(A) = \bigvee_{\omega \in A} \lambda^0(\omega), \tag{16.22}$$

*where*

$$\lambda^0(\omega) = \bigwedge \{N_\Pi(R) : \omega \in R \in \mathcal{R}\}. \tag{16.23}$$

*As $A \in \mathcal{R}^-$, the inequality $\lambda^0(\omega) \leq N_\Pi(A)$ easily follows for each $\omega \in A$, so that the inequality*

$$(N_\Pi)_0(A) = \bigvee_{\omega \in A} \lambda^0(\omega) \leq N_\Pi(A) \tag{16.24}$$

*is obvious.*

*For $N_{(\Pi_0)}(A)$ we compute easily that*

$$N_{(\Pi_0)}(A) = (\Pi_0(\Omega - A))^C = \left( \bigvee_{\omega \in \Omega - A} \pi_0(\omega) \right)^C. \tag{16.25}$$

*If $A \in R^-$, then $\Omega - A \in \mathcal{R}$, so that $\Pi(\Omega - A)$ is defined and the inequality*

$$\Pi_0(\Omega - A) \leq \Pi(\Omega - A) \tag{16.26}$$

*follows from (16.16). Consequently, the inequality*

$$N_{(\Pi_0)}(A) = (\Pi_0(\Omega - A))^C \geq (\Pi(\Omega - A))^C = N_\Pi(A) \tag{16.27}$$

*also holds and the assertion is proved.*

No inequality relation valid in general binds the values of the mappings $\Pi_0$ and $\Pi_\star$. Indeed, take $\mathcal{R} = \{\emptyset, \Omega\}$ and $\Pi(\emptyset) = \mathbf{0}_\mathcal{T}$, $\Pi(\Omega) = \mathbf{1}_\mathcal{T}$. Then, foe each $\omega \in \Omega$

$$\pi_0(\omega) = \bigwedge \{\Pi(R) : \omega \in R \in \mathcal{R}\} = \Pi(\Omega) = \mathbf{1}_\mathcal{T}, \tag{16.28}$$

so that

$$(\Pi_0(A) = \bigvee_{\omega \in A} \pi_0(\omega) = \mathbf{1}_\mathcal{T} \tag{16.29}$$

for every $\emptyset \neq A \subset \Omega$. On the other side,

$$\Pi_\star(A) = \bigvee \{\Pi(B) : B \in \mathcal{R},\ B \subset A\} = \Pi(\emptyset) = \mathbf{0}_\mathcal{T} \tag{16.30}$$

for every $A \subset \Omega,\ A \neq \Omega$. Hence, the inequality $\Pi_\star(A) < \Pi_0(A)$ holds foe each $\emptyset \neq A \neq \Omega,\ A \subset \Omega$. However, considering example with $\Omega = \{1, 2, \ldots, 5\}$ described above, we remember that $\Pi_o(A_3) = \mathbf{0}_\mathcal{T}$, but

$$\Pi_\star(A_3) = \bigvee \{\Pi(B) : B \in \mathcal{R},\ B \subset A_3\} = \Pi(A_3) = \mathbf{1}_\mathcal{T}, \tag{16.31}$$

so that the inequality $\Pi_0(A_3) < \Pi_\star(A_3)$ follows.

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