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Hájek, Petr
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Institute of Computer Science
Academy of Sciences of the Czech Republic

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Petr Hájek

Technical report No. 937

May 2005

Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:ics@cs.cas.cz

Institute of Computer Science
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Abstract:

General semantics of the logic whose axioms are those axioms of MTL (monoidal t-norm based fuzzy logic) using only the connectives $\&$ and $\rightarrow$ (i.e. not mentioning $\wedge, \bar{0}$) is given by the variety of algebras called basic quasihoops. Completeness theorem is proved for propositional calculus and a corresponding predicate calculus (complete with respect to linearly ordered quasihoops) is presented.

Keywords:
fuzzy logic, continuous t-norms, quasihoops

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[2]Institute of Computer Science, Academy of Sciences of the Czech Republic, 182 07 Prague, Czech Republic

1 Introduction

Standard semantics of fuzzy (propositional) logic is given on the unit real interval $[0,1]$ by a continuous or at least left-continuous t-norm $*$ and its residuum $\Rightarrow$. If $*$ is continuous then the lattice operations $\wedge, \vee$ are definable from $\Rightarrow, *$; thus the t-norm algebra $([0,1], *, \Rightarrow)$ determines a residuated lattice with divisibility and prelinearity. The variety generated by all t-norm algebras in the language $(*, \Rightarrow, \wedge, \vee, 1)$ consists of all basic hoops and in the language $(*, \Rightarrow, \wedge, \vee, 1, 0)$ of all BL-algebras. The lattice operations can be omitted as definable.

In the case of left continuous t-norms one has to consider $\wedge$ as primitive; $\vee$ remains definable (by Dummett's formula) and the variety generated by all left continuous t-norm algebras in the language $(*, \Rightarrow, \wedge, \vee, 1)$ is that of all basic semihoops and in $(*, \Rightarrow, \wedge, \vee, 1, 0)$ that of all MTL-algebras. (See [4] for definitions.) Not that $\vee$ can be omitted as definable.

Our first question reads: What is the general semantics of the logic whose axioms are those axioms of MTL using only the connectives & and $\to$ (i.e. not mentioning $\wedge, \bar{0}$)? This means: what is the class of algebras of truth functions making those axioms valid?

Our answer is: the variety of basic quasihoops as defined below. These are some partially quasiordered algebras (not necessarily lattices) but admitting factorization we get a partial order; ordered basic quasihoops are subalgebras of reducts of MTL-algebras to the language $*, \Rightarrow, 1$; the quasihoop logic is complete w.r.t. basic quasihoops, as well as to (reducts of) MTL-algebras, MTL-chains, and to left continuous t-norm algebras with $(*, \Rightarrow, 1)$. We easily get also results on subdirect representability and conservation extensions. Note that our quasihoops are equivalent to (reversed left) BCK(RP)-algebras of Iorgulescu, see [5]. Our logic fits well to the general hierarchy of fuzzy logics proposed by Cintula [1].

Our second question reads: what is the variety generated by all left continuous t-norm algebras just in the language $*, \Rightarrow$? It is *not* the variety of basic quasihoops but some smaller variety. (See Remark 1 point (2) below.) It would be nice to describe it by an explicit system of equations; we only show that the *quasivariety* generated by those algebras is strictly smaller (and consists of *ordered* basic quasihoops).

Our third question reads: Can we build a quasihoop predicate logic complete w.r.t. models over linearly ordered quasihoops? The last section contains a positive answer.

2 The quasihoop logic

Take the axioms (A1), (A2), (A3), (A5) of BL, i.e.

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $(\varphi \& \psi) \to (\psi \& \varphi)$
(A3) $(\varphi \& \psi) \to \varphi$
(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A5b) $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

and add, for each positive natural n, the axiom

(A6)$_\mathrm{n}$ $((\varphi \to \psi)^n \to \chi) \to (((\psi \to \varphi)^n \to \chi) \to \chi)$.

Deduction rule is modus ponens; α^n is $\alpha \& \ldots \& \alpha$ (n factors). Clearly, all these axioms are $*$-tautologies for each left-continuous t-norm $*$. Call this logic quHL – the logic of (basic) quasihoops. An easy checking gets the following:

Theorem 1 (1) The formulas (1) – (8), (19) – (20), (24) – (28) from [3] 2.2.7 – 2.2.16 (for $\bar{1}$ being $p \to p$ for any fixed p, and $\varphi \equiv \psi$ being $(\varphi \to \psi) \& (\psi \to \varphi)$) are provable using only axioms (A1) – (A3), (A5ab).

(2) For the obvious notion of a theory, the deduction theorem [3] 2.2.18 holds over quHL (by the same proof).

Call a theory T *prime* (or *complete* as in [3]) if for each φ, ψ $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$.

Theorem 2 Let $T \not\vdash \alpha$. Then there is a prime theory $\hat{T} \supseteq T$ such that $\hat{T} \not\vdash \alpha$.

Modify the proof from [3] using $(\mathrm{A6})_\mathrm{n}$.

3 Quasihoops

Definition 1 A *quasihoop* is an algebra $\mathbf{L} = (L, *, \Rightarrow, 1)$ satisfying the following, for each $x, y, z \in L$:

$$(x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)) = 1,$$
$$(x * y) = (y * x), \quad (x * y) \Rightarrow x = 1,$$
$$((x * y) \Rightarrow z) = (x \Rightarrow (y \Rightarrow z)),$$
$$1 * x = x, \quad x \Rightarrow x = 1, \quad 1 \Rightarrow x = x.$$

Define further $x \leq y$ iff $(x \Rightarrow y) = 1$ and $x \equiv y$ iff $x \leq y$ and $y \leq x$.

Theorem 3 (1) $\leq$ is a quasiorder (reflexive and transitive).

(2) 1 is greatest, $x \leq 1$ for each x.

(3) $x \leq y \Rightarrow z$ iff $x * y \leq z$ (residuum)

(4) $x \equiv 1$ iff $x = 1$.

(5) If $x = 1$ and $x \Rightarrow y = 1$ then $y = 1$ (soundness of modus ponens).

Proof: (1) Reflexivity is evident. Assume $x \Rightarrow y = y \Rightarrow z = 1$. Then $1 = ((x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)) = (1 \Rightarrow (1 \Rightarrow (x \Rightarrow z)) = (1 \Rightarrow (x \Rightarrow z)) = x = z$. This proves transitivity of $\leq$.

(2) $x \Rightarrow 1 = (x \Rightarrow (x \Rightarrow x)) = ((x * x) \Rightarrow x) = 1$.

(3) $x \leq y \Rightarrow z$ iff $x \Rightarrow (y \Rightarrow z) = 1$ iff $((x * y) \Rightarrow z) = 1$ iff $x * y \leq z$.

(4) $x \equiv 1$ implies $1 \leq x$, thus $(1 \Rightarrow x) = 1$ hence $x = 1$.

(5) $x \Rightarrow y \leq x \Rightarrow y$, hence $x * (x \Rightarrow y) \leq y$. If $x = (x \Rightarrow y) = 1$ then $1 = 1 * 1 = x * (x \Rightarrow y) \leq y$, hence $y = 1$ by (4).

□

Definition 2 A quasihoop $\mathbf{L}$ is *basic* if for each $x, y, z \in L$ and each positive natural n,

$$[((x \Rightarrow y)^n \Rightarrow z) * ((y \Rightarrow x)^n \Rightarrow z)] \leq z$$

(analogue of $(\mathrm{A6})_\mathrm{n}$.)

Remark 1 (1) Clearly, quasihoops form a variety and so do basic quasihoops. Note that the quasiorder $\leq$ of a (basic) quasihoop need not be an order. A trivial counterexample is a four-element algebra $0 < a \equiv b < 1$ such that $\{0, a, 1\}$ and $\{0, b, 1\}$ are copies of the three-valued MV-algebra and $a * b = b * a = 0, a \Rightarrow b = b \Rightarrow a = 1$.

(2) A quasihoop is a hoop iff it satisfies $x * (x \Rightarrow y) = y * (y \Rightarrow x)$. In each hoop $\leq$ is an order which is an inf-semilattice with $x \cap y = x * (x \Rightarrow y)$. A quasihoop is a semihoop iff it is ordered and the order $\leq$ is an inf-semilattice (every pair of elements has an infimum).

(3) The proof of associativity of $*$ in hoops (see [4]) gives here $x * (y * z) \equiv (x * y) * z$ (not necessarily $=$). A counterexample: let $1 > a > b_1 \equiv b_2 > c_1 \equiv c_2$; let $x \Rightarrow y = 1$ if $x \leq y$ and $x \Rightarrow y = y$ if $x > y$. Let $1 * x = x, x * y = y * x, x * x = x$ and further $b_1 * b_2 = b_1, c_1 * c_2 = c_1$, $a * b_1 = a * b_2, a * c_i = c_i, b_i * c_j = c_i(!)$.

Check that this is a linearly quasiordered quasihoop (thus it is basic) and $(a * b_1) * c_1 = b_2 * c_1 = c_2, a * (b_1 * c_1) = a * c_1 = c_1$.

(4) The logic quHL is obviously sound for interpretations in basic quasihoops; it follows among other things that the relation $\equiv$ on a quasihoop is a congruence.

Define $F \subseteq L$ to be a *filter* on $\mathbf{L} = (L, *, \Rightarrow, 1)$ if, for each $x, y \in L$, $x \in F$ and $x \leq y$ implies $y \in F$, and $x, y \in F$ implies $x * y \in F$. The *factor algebra* $\mathbf{L}/F$ is defined as usual: $[x]_F = \{y | x \equiv_F y\}$ (where $x \equiv_F y$ stands for $(x \Rightarrow y) \in F$ and $(y \Rightarrow x) \in F$); $[x]_F * [y]_F = [x * y]_F$ etc. The smallest filter is $\{1\}$; observe that $\mathbf{L}/\{1\}$ is an *ordered* quasihoop.

Each *linearly* ordered quasihoop is a lattice w.r.t. $\leq$ and hence it is a semihoop. By usual way (cf. the last theorem of the previous section) one proves the following:

Theorem 4 Ordered basic quasihoops have subdirect representation property: each ordered basic quasihoop is a subalgebra of (the $(*, \Rightarrow, 1)$-reduct of) a direct product of semihoop chains (or, if you want, of a direct product of MTL-chains, since each semihoop chain is a subalgebra of (the reduct of) a MTL-chain.

Corollary 1 Let φ be a formula of quHL. The following are mutually equivalent:

(i) quHL proves φ

(ii) MTLH proves φ

(iii) MTL proves φ

(iv) φ is a tautology over each basic quasihoop,

(v) ...over each ordered basic quasihoop,

(vi) ...over each linearly ordered quasihoop

(vii) ...over each (linearly ordered) basic semihoop

(viii) ...over each (linearly ordered) MTL-algebra.

(ix) ...over each algebra given by a left continuous t-norm on $[0, 1]$.

Thus MTL is a conservative extension of quHL.

Now let us turn to our second question. As promised, we claim the following:

Theorem 5 The quasivariety generated by algebras of left-continuous t-norms and their residua (in the language $*, \Rightarrow$) is the class of all ordered basic quasihoops.

Proof: Recall that a Horn formula is a formula of the form $\alpha_1 \& \ldots \& \alpha_n \to \beta$ where α_i and β are identities (n may be $0, 1, 2, \ldots$; for $n = 0$ it is just β). A quasivariety is the class of all algebras (of a given type) in which a given set of Horn formulas is valid. Clearly, a basic quasihoop is ordered iff the Horn sentence $((x \Rightarrow y = 1) \& (y \Rightarrow x = 1)) \to (x = y)$ is valid; thus it is a quasivariety. Each Horn formula valid in all ordered basic quasihoops is trivially valid in each left continuous t-norm algebra. Conversely, if a Horn formula is not valid in an ordered basic quasihoop then it is not valid in a linearly ordered quasihoop thanks to the subdirect representation property. Due to the fact that the quasihoop is ordered, each identity $\tau = \sigma$ is equivalent to $\tau \leq \sigma \,\&\, \sigma \leq \tau$, hence to $(\tau \Rightarrow \sigma) * (\sigma \Rightarrow \tau) = 1$; thus the Horn formula in question can be represented in the form

$$(\varphi_1 = 1 \,\&\ldots\&\, \varphi_n = 1) \to (\psi = 1)$$

which can be interpreted as saying that the (corresponding) formulas $\varphi_1, \ldots, \varphi_n$ of the logic quH are 1-true and the formula ψ is not 1-true (in the evaluation of variables making the Horn formula in question not satisfied). We may assume, as stated above, that in fact our linearly ordered quasihoop is a MTL-chain (since it is a semihoop and if it does not have a least element, you can add one). In other words, there is a model of the finite (propositional) theory $\{\varphi_1, \ldots, \varphi_n\}$ over a MTL-chain in which ψ is not 1-true. MTL has standard completeness and the proof of this in [6] immediately gets so-called strong finite standard completeness: there is a left-continuous t-norm $*$ and a $[0,1]$-evaluation of propositional variables making $\varphi_1, \ldots, \varphi_n$ 1-true and ψ not 1-true. Thus switching back to the algebraic language, the standard MTL-algebra given by $*$ satisfies, for the given evaluation of variables, the identities $\varphi_1 = 1, \ldots, \varphi_n = 1$ but does not satisfy $\psi = 1$, hence the Horn formula in question is not valid. This completes the proof. □

Example 1 (due to F. Esteva). The following is an ordered quasihoop whose order is not a lattice order:

$$A = \{1, a, b, c, d, e, 0\}, 1 > a > b > d > 0, 1 > a > c > e > 0, b > e, c > d;$$

(b, c) incomparable, (d, e) incomparable; $1 * x = x$, $x * y = 0$ for $x, y \neq 1$. (Residuum: $x \Rightarrow y = 1$ for $x \leq y$, $1 \Rightarrow y = y$, otherwise $x \Rightarrow y = a$.) Here $\inf(b, c)$ does not exist. The quasihoop is not basic:

$$((d \Rightarrow e) \Rightarrow a) \Rightarrow (((e \Rightarrow d) \Rightarrow a) \Rightarrow a = (1 \Rightarrow (1 \Rightarrow a)) = a \neq 1.$$

4 Quasihoop predicate logic

We define an **L**-*interpretation* **M** of a predicate logic over a basic quasihoop in the same way as usual and define the value $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ (v being an evaluation of variables). For quantified formulas we use inf and sup of instances and call **M** **L**-safe also as usual. In the analogy with BL∀, MTL∀ etc. we pay main attention to linearly ordered quasihoops (quasihoop chains). The problem is how to formulate axioms to prove (strong) completeness. Axioms $(\forall 1), (\forall 2), (\exists 1), (\exists 2)$ make no problems – they are expressed in the language of the logic of quasihoops. But the axiom $(\forall 3)$ is not since it contains disjunction: $(\forall x)(\varphi \vee \nu) \rightarrow ((\forall x)\varphi \vee \nu), x$ not free in ν. We first show that over MTL∀, the axiom $(\forall 3)$ can be equivalently replaced by four axioms formulated in the language of quasihoops (which seems to be of independent interest) and than we show that two axioms implied by our four ones can be added to $(\forall 1) - (\exists 2)$ to get (together with propositional axioms of quasihoops logic) an axiom system strongly complete for safe interpretations over quasihoop chains.

Definition 3 $\varphi \prec \psi$ is the formula $(\varphi \rightarrow \psi) \rightarrow \psi$. (Cf. [2]; in [7] the notation $\varphi \uparrow \psi$ is used.)

Recall that in MTL the disjunction $\alpha \vee \beta$ is defined as $(\alpha \prec \beta) \wedge (\beta \prec \alpha)$ (in our notation); thus $(\forall 3)$ can be written as

$$(\forall x)((\varphi \prec \nu) \wedge (\nu \prec \varphi)) \rightarrow [((\forall x)\varphi \prec \nu) \wedge (\nu \prec (\forall x)\varphi)]$$

By [3] 5.1.21 (15) (provable in MTL not using $(\forall 3)$), this is equivalent to

$$[(\forall x)(\varphi \prec \nu) \wedge (\forall x)(\nu \prec \varphi)] \rightarrow [((\forall x)\varphi \prec \nu) \wedge (\nu \prec (\forall x)\varphi)].$$

Write it as $(A1 \wedge A2) \rightarrow (S1 \wedge S2)$ ($A1$ being $(\forall x)(\varphi \prec \nu)$ etc.). Observe that for any α, β, the formula

$$(\alpha \wedge \beta) \equiv [(\alpha \& (\alpha \rightarrow \beta)) \vee (\beta \& (\beta \rightarrow \alpha))]$$

is a MTL-tautology (verify for any MTL-chain), thus $(A1 \wedge A2) \rightarrow (S1 \wedge S2)$ is equivalent to $[(A1 \& (A1 \rightarrow A2)) \vee (A2 \& (A2 \rightarrow A1))] \rightarrow (S1 \wedge S2)$. Over MTL, this formula as an axiom is equivalent to the following quadruple of axioms $(i = 1, 2)$:

$(\forall 31)$ $[A1 \& (A1 \rightarrow A2)] \rightarrow S_i$
$(A32)$ $[A2 \& (A2 \rightarrow A1)] \rightarrow S_i$

We have proved:

Theorem 6 In MTL∀, the axiom (∀3) can be replaced by the following four axioms (x not free in ν) :

$$[(\forall x)(\varphi \prec \nu)\&((\forall x)(\varphi \prec \nu) \to (\forall x)(\nu \prec \varphi))] \to [(\forall x)\varphi \prec \nu],$$
$$[(\forall x)(\varphi \prec \nu)\&((\forall x)(\varphi \prec \nu) \to (\forall x)(\nu \prec \varphi))] \to [\nu \prec (\forall x)\varphi],$$
$$[(\forall x)(\nu \prec \varphi)\&((\forall x)(\nu \prec \varphi) \to (\forall x)(\varphi \prec \nu))] \to [(\forall x)\varphi \prec \nu],$$
$$[(\forall x)(\nu \prec \varphi)\&((\forall x)(\nu \prec \varphi) \to (\forall x)(\varphi \prec \nu))] \to [\nu \prec (\forall x)\varphi].$$

(Remember them as $(\forall 31), (\forall 32)$ with our A_i, S_i).

Corollary 2 MTL∀ proves, for our A_i, S_i,

(∀33) $(A\&A2) \to S_i$,

i.e.

$$[(\forall x)(\varphi \prec \nu)\&(\forall x)(\nu \prec \varphi)] \to [(\forall x)\varphi \prec \nu],$$
$$[(\forall x)(\varphi \prec \nu)\&(\forall x)(\nu \prec \varphi)] \to [\nu \prec (\forall x)\varphi].$$

Remark 2 Observe that since $(A1\&A2) \to S_i$ is equvalent (in the quasihoop logic) to $A1 \to (A2 \to S_i)$ we can express (∀33) using implication as the only connective.

Definition 4 quHL∀ (the quasihoop predicate logic) is the fuzzy predicate logic over the quasihoop propositional logic with the axioms $(\forall 1), (\forall 2), (\exists 1), (\exists 2), (\forall 33)$ for quantifiers.

Theorem 7 *(Completeness theorem.)* Let T be a theory over quHL∀ and let φ be a formula. T proves φ (over quHL∀) iff φ is true in all models of T (i.e. all safe interpretations over quasihoop chains making all axioms of T true).

Proof: We just inspect the proof of completeness of BL∀ [3] 5.2.7 – 5.2.9 and construct theory $\hat{T} \supseteq T$ which is complete, Henkin and $\hat{T} \not\vdash \varphi$, which gives a model of T over the Lindenbaum algebra of $\hat{T}$, which is a quasihoop chain. The only place to be changed is Case 2 of the proof of 5.2.7 (handling $(\forall x)\chi(x)$). We proceed as follows: given T_n, α_n and $\chi(x)$, distinguish three subcases (c being a new constant):

(i) $T_n \not\vdash \alpha_n \prec \chi(c)$, thus $T_n \not\vdash \chi(c)$ and $T_n \not\vdash (\forall x)\chi(x)$. Let $T_{n+1} = T_n$, $\alpha_{n+1} = \alpha_n \prec \chi(c)$. We claim: For any theory $S \supseteq T_{n+1}$, if $S \not\vdash \alpha_n \prec \chi(c)$ then $S \not\vdash \chi(c)$ (evident by the definition of $\prec$) and $S \not\vdash \alpha_n$. Indeed, if $S \vdash \alpha_n$ then $S \vdash \chi(c) \equiv (\alpha_n \to \chi(c))$, hence $S \vdash \alpha_n \prec \chi(c)$, contradiction.

(ii) $T_n \not\vdash \chi(c) \prec \alpha_n$. Then analogously as above we show that $S \supseteq T_n$ and $S \not\vdash \alpha_n$ (evident) and $S \not\vdash \chi(c)$. We let $T_{n+1} = T_n$ and $\alpha_{n+1} = \chi(c) \prec \alpha_n$.

(iii) T_n proves both $\chi(c) \prec \alpha_n$ and $\alpha_n \prec \chi(c)$, hence T_n proves $(\forall x)(\chi(x) \prec \alpha_n)$ and $(\forall x)(\alpha_n \prec \chi(c))$ (since T_n assumes nothing on c). By ∀33), $T_n \vdash (\forall x)\chi(x) \prec \alpha_n$. This implies that $T_n + ((\forall x)\chi(x) \to \alpha_n \vdash \alpha_n$ (recall that $(\forall x)\chi(x) \prec \alpha_n$ is $((\forall x)\chi(x) \to \alpha_n) \to \alpha_n$) and therefore

$$T_n + (\alpha_n \to (\forall x)\chi(x)) \not\vdash \alpha_n$$

(since $T_n \not\vdash \alpha_n$, recall (A6)!). Thus we let $T_{n+1} = T_n + (\alpha_n \to (\forall x)\chi(x))$, $\alpha_{n+1} = \alpha_n$. This completes the proof.

□

This is a pleasing result: the fuzzy logic over (linearly ordered) quasihoops is a well-behaving t-norm based logic, both as propositional and as predicate logic.

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