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Abstract:

In the report, we present the full proof of the representation theorem for continuous Archimedean radial fuzzy systems. Radial fuzzy systems have antecedents of their rules represented by radial basis functions. The shape of such a function is considered to be the same as the shapes (membership functions) of individual fuzzy sets which form individual IF-THEN rules. Thus, a shape preservation property holds in radial fuzzy systems. Continuous Archimedean radial fuzzy systems are radial fuzzy systems employing a continuous Archimedean t-norms for and connective representation. As the main result we show, that a fuzzy system based on a continuous Archimedean t-norm t is radial if and only if the shapes of employed fuzzy sets are given by the composition of the pseudo-inverse of the additive generator of t with the polynomial of certain form.

Keywords: Radial fuzzy systems, Additive generators, Functional equations

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1 Introduction

Radial fuzzy systems are fuzzy systems which employ radial fuzzy sets in their rules and exhibit a shape preservation property in rules' antecedents. The property, which is called *the radial property*, simplifies the computational model of radial systems. Especially, a substantial simplification is reached for implicative (gradual) fuzzy systems [3]. Moreover, due to the property, an investigation of such the properties as coherence and redundancy of rules is effective with reasonable results.

Continuous Archimedean radial fuzzy systems are radial fuzzy systems based on continuous Archimedean t-norms. More specifically, they are the systems employing a continuous Archimedean t-norm for representation of *and* linguistic connective. Before we give an exact definition of a radial fuzzy system let us specify the notation.

In the report we will recognize the following subsets of set $\mathcal{R} = (-\infty, +\infty)$ of real numbers: the set of positive real numbers - $\mathcal{R}_+ = (0, +\infty)$, the set of non-negative real numbers - $\mathcal{R}_{0+} = [0, \infty)$. Cartesian products of these sets will be denoted as follows: $\mathcal{R}^n$, $\mathcal{R}^n_+$, $\mathcal{R}^n_{0+}$, where n is an integer from $\mathcal{N} = \{1, 2, \dots\}$.

Further, we will work with the extended real line $\mathcal{R}^* = [-\infty, +\infty] = \mathcal{R} \cup \{-\infty, +\infty\}$ and its parts: $\mathcal{R}^*_+ = [0, +\infty]$, $\mathcal{R}^*_{0+} = (0, \infty]$. Non-proper reals $-\infty, +\infty$ will be considered as elements of domains and ranges of continuous functions in such a way that we say that $f(a) = b$, $a, b \in \mathcal{R}^*$ if $\lim_{x \to a} f(x) = b$.

2 Radial fuzzy systems

Definition 1 *A fuzzy system is radial if*:

(i) *There exists a continuous function* $act : [0, +\infty] \to [0, 1]$*,* $act(0) = 1$*,* $act(+\infty) = 0$*, such that:* (a) *either there exists* $z_0 \in (0, +\infty)$ *such that act is strictly decreasing on* $[0, z_0]$ *and* $act(z) = 0$ *for* $z \in [z_0, +\infty]$ *or* (b) *act is strictly decreasing on* $[0, +\infty]$*. In this case we set* $z_0 = +\infty$*.*

(ii) *Fuzzy sets in antecedent and consequent of the* j*-th rule are specified as*

$$A_{ji}(x_i) = act\left(\left|\frac{x_i - a_{ji}}{b_{ji}}\right|\right), \tag{2.1}$$

$$B_j(y) = act\left(\frac{\max\{0, |y - c_j| - s_j\}}{d_j}\right), \tag{2.2}$$

where $n, m \in \mathcal{N}$; $i = 1, \dots, n$; $j = 1, \dots, m$; $\boldsymbol{x} \in \mathcal{R}^n$, $\boldsymbol{x} = (x_1, \dots, x_n)$; $\boldsymbol{a}_j \in \mathcal{R}^n$, $\boldsymbol{a}_j = (a_{j1}, \dots, a_{jn})$; $\boldsymbol{b}_j \in \mathcal{R}^n_+$, $\boldsymbol{b}_j = (b_{j1}, \dots, b_{jn})$, *(i.e.,* $b_{ji} > 0$*);* $c_j \in \mathcal{R}$; $d_j \in \mathcal{R}_+$, *(i.e.,* $d_j > 0$*);* $s_j \in \mathcal{R}_{0+}$, *(i.e.,* $s_j \geq 0$*).*

(iii) *For each* $\boldsymbol{x} \in \mathcal{R}^n$ *the radial property holds, i.e.,*

$$A_j(\boldsymbol{x}) = A_{j1}(x_1) \star \dots \star A_{jn}(x_n) = act(\,||\boldsymbol{x} - \boldsymbol{a}_j||_{\boldsymbol{b}_j}), \tag{2.3}$$

where $||\cdot||_{\boldsymbol{b}_j}$ *is a scaled version of some norm* $||\cdot||$ *in* $\mathcal{R}^n$*. This norm is common to all rules of the fuzzy system.*

Let us comment on the definition. The specification of a radial fuzzy system consists of three steps.

In the first step, point (i) of the definition, an activation function *act* is specified. The domain of *act* function corresponds to the set of non-negative extended reals $\mathcal{R}^*_{0+} = [0, +\infty]$ and the range to the unit interval $[0, 1]$. Values of the function at the limit points of the domain are required to be specified as follows: $act(0) = 1$ and $act(+\infty) = 0$, i.e., $\lim_{z \to +\infty} act(z) = 0$.

An *act* function can be of two types (ia) or (ib). The difference between two types is best presented graphically, see Fig. 2.1.

In the second step, point (ii) of the definition, the membership functions of fuzzy sets in antecedents and consequents are determined. It can be easily observed that the specification corresponds to a specification of one-dimensional radial functions.

Radial functions are generally defined by formula $f(\boldsymbol{x}) = \Phi(||\boldsymbol{x} - \boldsymbol{a}||)$, where Φ is a function from $\mathcal{R}_{0+}$ (or $\mathcal{R}^*_{0+}$) to $\mathcal{R}$, $||\cdot||$ is a norm in $\mathcal{R}^n$ and $\boldsymbol{a} \in \mathcal{R}^n$ is a central point of the function. Concerning

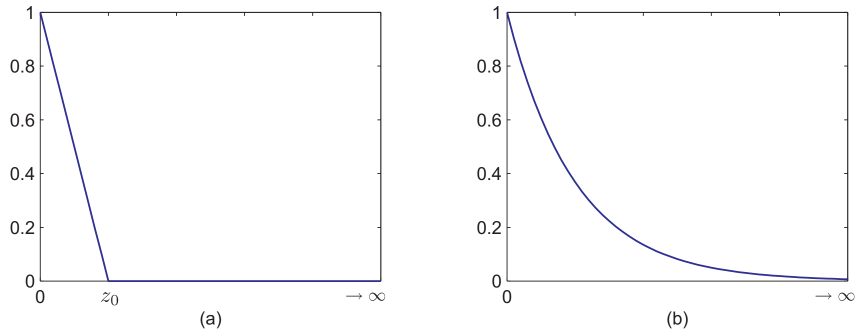

Figure 2.1: Examples of *act* function: (a) type (ia); (b) type (ib).

radial fuzzy systems, the class of so-called ℓ_p norms in $\mathcal{R}^n$ is important [4]. The definition formula of ℓ_p norms depends on parameter $p \in [1, +\infty]$ and reads as follows:

$$\begin{aligned} ||\boldsymbol{u}||_p &= (|u_1|^p + \cdots + |u_n|^p)^{1/p} \quad \text{for} \quad p \in [1, +\infty), \\ ||\boldsymbol{u}||_\infty &= \lim_{p\to+\infty} ||\boldsymbol{u}||_p = \max_i\{|u_i|\}. \end{aligned} \tag{2.4}$$

Scaled ℓ_p norms, denoted by $||\cdot||_{p_{\boldsymbol{b}}}$, are derived from corresponding ℓ_p norms by incorporating a vector $\boldsymbol{b} \in \mathcal{R}^n_+$ of scaling parameters, $\boldsymbol{b} = (b_1, \ldots, b_n)$, $b_i > 0$, into the above formulas. That is,

$$\begin{aligned} ||\boldsymbol{u}||_{p_{\boldsymbol{b}}} &= (|u_1/b_1|^p + \cdots + |u_n/b_n|^p)^{1/p}\ ;\ p \in [1, +\infty), \\ ||\boldsymbol{u}||_{\infty_{\boldsymbol{b}}} &= \lim_{p\to+\infty} ||\boldsymbol{u}||_{p_{\boldsymbol{b}}} = \max_i\{|u_i/b_i|\}. \end{aligned} \tag{2.5}$$

Clearly, original unscaled ℓ_p narm are obtained from scaled ones by choosing $\boldsymbol{b} = \boldsymbol{1} = (1, \ldots, 1)$. The most prominent examples of scaled ℓ_p norms are scaled ℓ_1 or octaedric, scaled ℓ_2 or Euclidean and scaled ℓ_∞ or cubic norms.

Antecedent fuzzy sets differ from consequent ones in specification of their cores (kernels) [6, 7]. An antecedent fuzzy set A_{ji} with membership function (2.1), has its core given by point a_{ji}. In the case of a consequent fuzzy set B_j, the core corresponds to closed interval $[c_j - s_j, c_j + s_j]$. That is, the central point of the core is determined by point c_j and core's length is driven by parameter $s_j \geq 0$. As both antecedent and consequent fuzzy sets are determined on the basis of common *act* function, consequent fuzzy sets B_js can be seen as enhanced or trapezoid-like versions of antecedent fuzzy sets, see Figs. 4.1, 4.2, 4.3 for examples. The introduction of trapezoid-like consequent fuzzy sets enhances computational capabilities of radial fuzzy systems [2].

Point (iii) of the definition is crucial, as it presents the requirement for the validity of the radial property in radial fuzzy systems. The property, formula (2.3 can be interpreted as the requirement for the preservation of radial shape of one-dimensional fuzzy sets A_{ji}s after their combination by a given t-norm. Thus, the radial property can be seen as a shape preservation property.

The property is not trivial. Having specified membership functions of A_{ji} sets and a t-norm $\star$, the representation of an antecedent is already determined: $A_j(\boldsymbol{x}) = A_{j1} \star \cdots \star A_{jn}$. The radial property requires $A_j(\boldsymbol{x})$ to be expressible in form

$$A_j(\boldsymbol{x}) = act(\, ||\boldsymbol{x} - \boldsymbol{a}_j||_{\boldsymbol{b}_j}). \tag{2.6}$$

That is, it requires A_j to be a multi-dimensional radial function based on the same *act* function as A_{ji}s are; and the norm occurring in formula (2.6) to be a scaled version of some norm in $\mathcal{R}^n$ with scaling parameters b_{ji} of A_{ji}s. Similarly, it is required that central point of A_j consists of central points of A_{ji}s. The non-triviality means that not all combinations of *act* functions with t-norms exhibit the radial property. It can be shown [2], that triangular fuzzy sets ($act(z) = \max\{0, 1 - z\}$) cannot be combined by the product t-norm to (2.6) holds.

The non-triviality of the property induces a natural question: If there are any examples of radial fuzzy systems and moreover what combinations of shapes (*act* functions) with what t-norms are allowed in order to the radial property hold; and, what are the norms occurring in formula (2.3).

3 Representation theorem

The above questions will be answered in this section by the main result of the report which is the representation theorem for continuous Archimedean radial fuzzy systems. Before we state the theorem, let us recall some notions and theorems from theories of triangular norms (t-norms) and quasi-arithmetic means.

As it is well known *a triangular norm* T is an operation from the unit square to the unit interval, i.e., $T\colon [0,1]^2 \to [0,1]$, which satisfies commutativity, associativity, monotonicity and boundary conditions. Several other properties are recognized for t-norms such as continuity or the Archimedean property which requires that for any pair $(x,y) \in (0,1)^2$ there exists an n such that $T^n(x) < y$. As usual

$$T^n(x) = T(x, ..., x) = \underbrace{x \star \cdots \star x}_{n},$$

which has sense due to the associativity of T. $\star$ symbol is another symbol for a t-norm T, i.e., $T(x,y) = x \star y$.

Continuous Archimedean t-norms occupy a special place among other t-norms due to the following representation theorem:

Theorem 1 [5]: *T is a continuous Archimedean t-norm if and only if T has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $t : [0,1] \to [0,+\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant such that for all $(x,y) \in [0,1]^2$:*

$$T(x,y) = t^{(-1)}(t(x) + t(y)), \tag{3.1}$$

where $t^{(-1)}$ is the pseudo-inverse of t.

The theorem says that a t-norm is continuous Archimedean if and only if it has some additive generator. On the basis of the theorem it can be shown that for continuous Archimedean t-norms and $\boldsymbol{x} = (x_1, \ldots, x_n) \in [0,1]^n$ we have also

$$T(x_1, \ldots, x_n) = t^{(-1)}(t(x_1) + \cdots + t(x_n)). \tag{3.2}$$

The pseudo-inverse of an aditive generator is a function from $[0,+\infty]$ to $[0,1]$ defined as

$$t^{(-1)} = \begin{cases} t^{-1}(z) & \text{for} \quad z \in [0, t(0)] \\ 0 & \text{for} \quad z \in [t(0), +\infty] \end{cases} \tag{3.3}$$

where t^{-1} is the ordinary inverse of t. The above formula can be equivalently written as

$$t^{(-1)}(z) = t^{-1}(\min\{t(0), z\}), \tag{3.4}$$

which enables us to rewrite formulas (3.1), (3.2) as follows:

$$T(x,y) \quad = \quad t^{-1}(\min\{t(0), t(x) + t(y)\}), \tag{3.5}$$
$$T(x_1, \ldots, x_n) \quad = \quad t^{-1}(\min\{t(0), t(x_1) + \cdots + t(x_n)\}). \tag{3.6}$$

Let t be a generator of a t-norm. Then the t-norm is nilpotent if $t(0) < +\infty$ and strict if $t(0) = +\infty$. A continuous Archimedean t-norm is either nilpotent or strict [5].

From formula (3.3) and properties of additive generators it is easy to observe that for nilpotent t-norms the coresponding pseudo-inverses are strictly decreasing on interval $[0, t(0)]$, and constant $(t^{(-1)}(z) = 0)$ on interval $[t(0), +\infty]$. In the case of strict t-norms, the pseudo-inverses are strictly decreasing on whole domain $[0, +\infty]$. For both types we have $t^{(-1)}(0) = 1$ and $t^{(-1)}(+\infty) = 0$.

The following definition and theorem are taken from Aczél and Dhombres [1]:

Definition 2 [1]: *Let $k : \mathcal{R}_{0+} \to \mathcal{R}$ be a continuous and strictly decreasing function (injection) with inverse k^{-1}. The range of k is an interval (also domain $\mathcal{R}_{0+}$ may be replaced by other infinite or finite intervals). Then*

$$M(x,y) = k^{-1}\left(\frac{k(x) + k(y)}{2}\right) \tag{3.7}$$

is a quasi-arithmetic mean. The arithmetic mean $[k(x) = x]$, *the exponential mean* $[k(x) = \exp(px)]$, $p \neq 0]$, *the root-mean-square* $[k(x) = x^2]$, *the root-mean-power* $[k(x) = x^p,\ p > 0]$*; if* $\mathcal{R}_{0+}$ *(domain of* k*) is replaced by* $\mathcal{R}_+$ *then also the root-mean-powers with* $p < 0$*, including the harmonic mean* $[k(x) = 1/x]$ *and the geometric mean* $[k(x) = \ln(x)]$ *are examples of quasi-arithmetic means.*

Theorem 2 [1]: *Geometric mean and root-mean-powers, i.e.,*

$$\sqrt{xy} \quad \text{and} \quad \left(\frac{x^p + y^p}{2}\right)^{1/p}$$

are only homogeneous quasi-arithmetic means. That is only these satisfy

$$M(cx, cy) = cM(x, y)$$

for $x, y, c > 0$.

Now we can approach to the main result of the report.

Theorem 3 *Let the t-norm* $\star$ *which is used to represent* IF-THEN *rules of a fuzzy system, be continuous Archimedean. Then the fuzzy system is radial if and only if for* $z \in [0, +\infty]$ *its act function has form*

$$act(z) = t^{(-1)}(qz^p), \tag{3.8}$$

where $t^{(-1)}$ *is the pseudo-inverse of an additive generator of* $\star$ *and* $q > 0$, $p \geq 1$ *are parameters.*

Proof: A fuzzy system is radial if it satisfies requirements (i)-(iii) of Definition 1. The first two are relatively unrestrictive as they specify properties of an *act* function, which are rather weak, and how to determine shapes of membership functions of individual fuzzy sets. A really strong is the requirement (iii), i.e., the requirement for the validity of the radial property (2.3), as a specification of an *act* function and membership functions already gives, under the selected t-norm $\star$, the representation of antecedent.

So, if we are interested in a search for at least sufficient conditions to a fuzzy system be radial, we are, in fact, interested in a solution of functional equation (feq in short)

$$act\left(\left|\frac{x_1 - a_{j1}}{b_{j1}}\right|\right) \star \cdots \star act\left(\left|\frac{x_n - a_{jn}}{b_{jn}}\right|\right) = act\left(\left\|\frac{x_1 - a_{j1}}{b_{j1}}, \ldots, \frac{x_n - a_{jn}}{b_{jn}}\right\|\right). \tag{3.9}$$

on domain $\mathcal{R}^n$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{R}^n$. A solution consists of two functions *act* and $||\cdot||$ such that:
1) $act : [0, +\infty] \to [0, 1]$, $act(0) = 1$, $act(+\infty) = 0$ ($\lim_{z \to +\infty} act(z) = 0$), continuous, such that either (ia) there exists $z_0 \in (0, +\infty)$ such that *act* is strictly descreasing on $[0, z_0]$ and $act(z) = 0$ for $z \in [z_0, +\infty]$, or (ib) *act* is strictly decreasing on $[0, +\infty]$; and
2) $||\cdot||$ is a continuous norm on $\mathcal{R}^n$.

The paramaters of the feq are: (i) $n \in \mathcal{N}$, $n \geq 2$; (ii) $a_{ji} \in \mathcal{R}$, i.e., $\boldsymbol{a}_j \in \mathcal{R}^n$, $\boldsymbol{a}_j = (a_{j1}, \ldots, a_{jn})$; (iii) $b_{ji} > 0$, i.e., $\boldsymbol{b}_j \in \mathcal{R}^n_+$, $\boldsymbol{b}_j = (b_{j1}, \ldots, b_{jn})$.

We start with the easier part of the theorem, i.e., with showing that *act* function specified according to formula (3.8) makes the system to be radial.

First of all we check that *act* of (3.8) meets requirement (i) of Definition 1. Recalling the properties of pseudo-inverses and observing that function qz^p is the strictly increasing bijection from $[0, +\infty]$ onto $[0, +\infty]$ for $q > 0$, $p \geq 1$, we get function $t^{(-1)}(qz^p)$ either of (ia) type for nilpotent t-norms or (ib) type for strict t-norms.

Let *act* and A_{ji}s be specified according to (3.8) and (2.1), respectively. Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{R}^n$, $\boldsymbol{a}_j = (a_{j1}, \ldots, a_{jn}) \in \mathcal{R}^n$, $\boldsymbol{b}_j = (b_{j1}, \ldots, b_{jn}) \in \mathcal{R}^n_+$ and $u_{ji} = (x_i - a_{ji})/b_{ji}$ for some j and $i = 1, \ldots, n$.

Then, due to formulas (3.2) and (3.8), the representation of antecedent $A_j(\boldsymbol{x})$ reads as

$$A_j(\boldsymbol{x}) = A_{j1}(|u_{j1}|) \star \cdots \star A_{jn}(|u_{jn}|), \tag{3.10}$$

$$A_j(\boldsymbol{x}) = t^{(-1)}\left[\sum_{i=1}^{n} t(act(|u_{ji}|))\right], \tag{3.11}$$

$$A_j(\boldsymbol{x}) = t^{(-1)}\left[\sum_{i=1}^{n} t(t^{(-1)}(q|u_{ji}|^p))\right]. \tag{3.12}$$

By (3.6) the latter formula is written in form

$$A_j(\boldsymbol{x}) = t^{(-1)}\left[\sum_{i=1}^{n} t(t^{-1}(\min\{t(0), q|u_{ji}|^p\}))\right], \tag{3.13}$$

which gives

$$A_j(\boldsymbol{x}) = t^{(-1)}\left[\sum_{i=1}^{n} \min\{t(0), q|u_{ji}|^p\}\right]. \tag{3.14}$$

Now, two cases are possible. 1) If $q|u_{ji}|^p < t(0)$ for all i, then (3.14) has form $A_j(\boldsymbol{x}) = t^{(-1)}(\sum_i^n q|u_{ji}|^p)$, which can be written in an equivalent form as $A_j(\boldsymbol{x}) = t^{(-1)}(q\,(\sqrt[p]{\sum_i^n |u_{ji}|^p})^p)$, i.e., $A_j(\boldsymbol{x}) = act(||\boldsymbol{u}||_p) = act(||\boldsymbol{x}-\boldsymbol{a}_j||_{p_{\boldsymbol{b}_j}})$.

2) If there exists an i such that $q|u_i|^p \geq t(0)$, then, on one hand, the sum in (3.14) is greater or equal to $t(0)$ and therefore $A_j(\boldsymbol{x}) = 0$. On the other hand, $\sum_i^n q|u_{ji}|^p \geq t(0)$, i.e., $q\,(\sqrt[p]{\sum_i |u_i|^p})^p = q(||\boldsymbol{u}||_p)^p \geq t(0)$ and therefore $act(||\boldsymbol{x}-\boldsymbol{a}||_{p_{\boldsymbol{b}_j}}) = 0$. Thus, also in this case we have $A_j(\boldsymbol{x}) = act(||\boldsymbol{x} - \boldsymbol{a}||_{p_{\boldsymbol{b}_j}})$ and we see that the specification of act function according to (3.8) is sufficient for the validity of the radial property.

Observe an important fact here that if act is specified according to (3.8), then scaled ℓ_p norms occur in the representation of antecedents.

The proof of the necessity of act specification according to (3.8) is harder. It falls into the area of functional equations of several variables. This area is well covered in book by Aczél and Dhombres [1].

Let us start by the following observation: If a solution (a pair act, $||\cdot||$) of feq (3.9) exists, then it is also the solution of

$$act(x_1) \star \cdots \star act(x_n) = act(||x_1, \ldots, x_n||) \tag{3.15}$$

for $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{R}^n_{0+}$.

Indeed, let $s = (act, ||\cdot||)$ be a solution of (3.9). If an $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{R}^n_{0+}$ would violate (3.15) for s, then $\boldsymbol{x}'$, $x'_i = b_i x_i + a_i$, $i = 1, \ldots, n$ would violate (3.9) and s would not be the solution of (3.9). A contradiction.

In the sequel we will consider the case $n = 2$. The proof for $n > 2$ follows the case of $n = 2$. Thus, we will search for necessary conditions on "shapes" of a solution of feq

$$act(x) \star act(y) = act(n(x, y)) \tag{3.16}$$

on domain $(x, y) \in \mathcal{R}^2_{0+}$ ($x = x_1$, $y = x_2$) and parameters specified as above. A solution consists of two functions act, $||\cdot|| = n(x, y)$ with the characteristics already mentioned.

Employing an additive generator t of $\star$ and formula (3.5), we obtain (3.16) in form

$$act(n(x, y)) = t^{-1}(\min\{t(0), t(act(x)) + t(act(y))\}) \tag{3.17}$$

for $(x, y) \in \mathcal{R}^2_{0+}$.

Lemma 1 *Let an $act(z)$ with some $z_0 \in (0, +\infty]$ and a $n(x, y)$ form a solution of feq (3.16). Then for $(x, y) \in \mathcal{R}^2_{0+}$*

$$n(x, y) < z_0 \quad \text{iff} \quad t(act(x)) + t(act(y)) < t(0). \tag{3.18}$$

Proof of Lemma 1: Let us denote the left side of (3.17) by $L(x,y)$ and the right side by $P(x,y)$. We have $L(x,y) > 0$ iff $n(x,y) < z_0$ and $L(x,y) = 0$ iff $n(x,y) \geq z_0$ from properties of act; $P(x,y) > 0$ iff $t(act(x)) + t(act(y)) < t(0)$ from properties of t^{-1} (see below). $P(x,y) = 0$ iff $t(act(x)) + t(act(y)) \geq t(0)$ by definition of $P(x,y)$. If $act(z)$ and $n(x,y)$ solve (3.17) then $L(x,y) = P(x,y)$ must hold for all $(x,y) \in \mathcal{R}^2_{0+}$, which gives the assertion of the lemma. □

Let an act has some $z_0 \in (0,+\infty]$ and solves (3.17) for some norm $n(x,y)$. We denote by f the restriction of act on $[0,z_0]$, i.e., $f : [0,z_0] \to [0,1]$, $f(z) = act(z)$ for $z \in [0,z_0]$. Let us review the properties of functions f, t, their inverses and compositions:

- $f : [0,z_0] \leftrightarrow [0,1]$, bijection (indicated by symbol $\leftrightarrow$), $f(0) = 1$, $f(z_0) = 0$, continuous, strictly decreasing
- $t : [0,1] \leftrightarrow [0,t(0)]$, bijection, $t(1) = 0$, continuous, strictly decreasing
- $h = t(f(z))$, $h : [0,z_0] \leftrightarrow [0,t(0)]$, bijection, $h(0) = 0$, $h(z_0) = t(0)$, continuous, strictly increasing (if $x_1 < x_2$ then $f(x_1) > f(x_2)$ and therefore $t(f(x_1)) < t(f(x_2))$)
- $f^{-1} : [0,1] \leftrightarrow [0,z_0]$, bijection, $f^{-1}(1) = 0$, $f^{-1}(0) = z_0$, continuous, strictly decreasing
- $t^{-1} : [0,t(0)] \leftrightarrow [0,1]$, bijection, $t^{-1}(0) = 1$, continuous, strictly decreasing
- $h^{-1} = f^{-1}(t^{-1}(z))$, $h^{-1} : [0,t(0)] \leftrightarrow [0,z_0]$, bijection, $h^{-1}(t(0)) = z_0$, $h^{-1}(0) = 0$, continuous. Obviously, h^{-1} is the inverse of h.

Lemma 2 *Let $n(x,y)$ be a continuous norm in $\mathcal{R}^2_{0+}$ and $z_0 \in (0,+\infty]$. Then there exists a finite z_0^*, $0 < z_0^* < z_0$ such that $n(x,y) < z_0$ for $(x,y) \in [0,z_0^*]^2$.*

Before we give a proof, let us explain the meaning of the lemma. For a general norm $n(x,y)$ specified on $\mathcal{R}^2_{0+}$ and a positive finite z_0 the assertion "if $x < y \leq z_0$ and $n(y,y) < z_0$, then $n(x,y) < z_0$" does not hold. As an example consider norm[2] $n(x,y) = 2x + 4y - 4\sqrt{xy}$ for $z_0 = 3$, $y = 1$, $x = 0$. The lemma says that for a given z_0 there exists $0 < z_0^* < z_0$ such that for elements of square $[0,z_0^*]^2$ we have simultaneously $x < z_0$, $y < z_0$ and $n(x,y) < z_0$.

Proof of Lemma 2: If $z_0 = +\infty$, then z_0^* can be arbitrary positive real, i.e., $z_0^* \in \mathcal{R}_+$. For z_0 finite, let us consider a square $[0,u]^2$ for $u > 0$. As this square is a compact set and $n(x,y)$ is a continuous mapping, the set $\{v \,|\, v = n(x,y), (x,y) \in [0,u]^2\}$ is compact (the image of a compact set under a compact mapping is a compact set), i.e., it is a closed interval in $\mathcal{R}$. The left limit point of this interval is 0 and the right limit point we denote $g(u)$, i.e., $[0,g(u)] = \{v \,|\, v = n(x,y), (x,y) \in [0,u]^2\}$. Let us set $g(0) = 0$, function $u \to g(u)$, $u > 0$ is continuous (from right) at 0. Indeed, according to the equivalence theorem for norms, we have $n_1(x,y) \leq \alpha n_2(x,y)$, $\alpha > 0$, for arbitrary two norms. Hence, considering ℓ_1 norm $\ell_1(x,y) = |x| + |y|$, we have $0 \leq n(x,y) \leq \alpha(|x| + |y|)$ and $g(u) \leq 2\alpha u$ for any $u > 0$. Obviously, chosing an arbitrary sequence $\{u_i\}$, $u_i > 0$ such that $\lim \{u_i\} = 0$, we obtain $\lim \{2\alpha u_i\} = 0$, i.e., $\lim \{g(u_i)\} = 0$ and therefore g is continuous at 0.

From continuity of g at 0: For any $z_0 > 0$ there must exists $\epsilon_0 > 0$ such that $g(u) < z_0$ for $u \in [0,\epsilon_0)$. By this we can choose z_0^* as an arbitrary point from intersection of intervals $(0,\epsilon_0) \cap (0,z_0]$. □

Let an act has some z_0 and solves (3.17) for some $n(x,y)$. Then for $(x,y) \in [0,z_0^*]^2$ we have $x,y < z_0$, $n(x,y) < z_0$, and from (3.17), by equivalence (3.18), we obtain

$$f(n(x,y)) = t^{-1}(t(f(x)) + t(f(y))), \tag{3.19}$$
$$t(f(n(x,y))) = t(f(x)) + t(f(y)), \tag{3.20}$$
$$h(n(x,y)) = h(x) + h(y). \tag{3.21}$$

That is, if an act solves (3.17) for some $n(x,y)$ then h solves feq (3.21) on domain $(x,y) \in [0,z_0^*]^2$.

[2] See Appendix for more details about this norm.

For $(x,y) \in [0, z_0^*]^2$ we have $n(x,x) < z_0$, $n(y,y) < z_0$ and therefore

$$\begin{aligned} h(n(x,y)) &= h(x)+h(y), && (3.22)\\ h(n(x,x)) &= 2h(x) \quad \text{hence} \quad h(x) = h(n(x,x))/2, && (3.23)\\ h(n(y,y)) &= 2h(y) \quad \text{hence} \quad h(y) = h(n(y,y))/2, && (3.24)\\ h(n(x,y)) &= [h(n(x,x)) + h(n(y,y))]/2. && (3.25) \end{aligned}$$

By linearity of norms we have $n(x,x) = x \cdot n(1,1)$. Let us consider substitutions $u = n(x,x)$, $v = n(y,y)$, which gives $x = u/n(1,1)$ and $y = v/n(1,1)$; and let us denote $c = n(1,1)$, i.e.,

$$\begin{aligned} h(n(u/c, v/c)) &= [h(u)+h(v)]/2, && (3.26)\\ h(1/c \cdot n(u,v)) &= [h(u)+h(v)]/2, && (3.27)\\ n^*(u,v) &= h^{-1}\left(\frac{h(u)+h(v)}{2}\right). && (3.28) \end{aligned}$$

where $n^*(u,v) = 1/c \cdot n(x,y)$ and $(u,v) \in [0, cz_0^*]^2$, $c > 0$.

$n^*(u,v)$ is a norm and it is a homogenous quasi-arithmetic mean by (3.28). As root-mean-powers for $p < 0$ and geometric mean are not defined if domain of k contains 0, see definition 2; and, in our case h is defined at 0, $h(0) = 0$, these means cannot interpret $n^*(u,v)$. Further, root-mean-power means for $p \in (0,1)$ do not satisfy the triangle inequality (consider $x = y$). Similarly, the geometric mean does not satisfy the triangle inequality for $x = 0, y > 0$, which makes another reason why it cannot correspond to $n^*(u,v)$. Thus, the only possibility is that $n^*(u,v) = (x^p + y^p)^{1/p}$ for $p \in [1, +\infty)$, i.e., that $n^*(u,v)$ corresponds to an ℓ_p norm for $(x,y) \in \mathcal{R}^2_{0+}$, $p \in [1, +\infty)$.

We have

$$\begin{aligned} 1/c \cdot n(u,v) &= [(u^p + v^p)/2]^{1/p}, && (3.29)\\ n(x,y) &= [((xc)^p + (yc)^p)/2]^{1/p}, && (3.30)\\ n(x,y) &= \frac{c}{\sqrt[p]{2}} \cdot (x^p + y^p)^{1/p}. && (3.31) \end{aligned}$$

Now, the question is what is the value of $c = n(1,1)$.

$$\begin{aligned} h(n(x,y)) &= h(x)+h(y), && (3.32)\\ h(n(x,0)) &= h(x), && (3.33)\\ n(x,0) &= x. && (3.34) \end{aligned}$$

From (3.31) we get $\sqrt[p]{2} \cdot n(x,0) = cx$. Employing (3.34), we obtain $\sqrt[p]{2} = c$. Hence the final form of the norm is

$$n(x,y) = (x^p + y^p)^{1/p} \qquad (3.35)$$

for $(x,y) \in [0, z_0^*]^2$, $p \geq 1$.

What we have shown is that if a solution of (3.17) exists, and *act* function has some $z_0 \in (0, +\infty]$ then $n(x,y)$ is an ℓ_p norm for $(x,y) \in [0, z_0^*]^2$. As we are searching for a solution on $\mathcal{R}^2_{0+}$, we can translate this result as, if a solution of (3.17) exists, then the restriction of the norm occurring in the solution must correspond to an ℓ_p norm on square $[0, z_0^*]^2$, $z_0^* > 0$. It can be easily shown that this is satisfied only if the original norm is an ℓ_p norm having the same value of p as occurs in the restriction.

Indeed, let the restriction of a norm $n(x,y)$ on a square $[0, z_0^*]^2$, $z_0^* > 0$ be an ℓ_p norm $\ell_p(x,y) = \sqrt[p]{x^p + y^p}$, $p \geq 1$. Now, let $n(x_1, y_1) \neq \ell_p(x_1, y_1)$ for some $(x_1, y_1) \in \mathcal{R}^2_{0+}$. Then there exists $c > 0$ such that $cx_1, cy_1 < z_0^*$. By linearity we have $cn(x_1, y_1) \neq c\ell_p(x_1, y_1)$, i.e., $n(cx_1, cy_1) \neq \ell_p(cx_1, cy_1)$, which clearly contradicts the requirement on the form of the restriction of $n(x,y)$ on $[0, z_0^*]^2$.

Going back to feq (3.17) and equivalence (3.18), we can state that, if a solution of (3.17) exists, then the norm of the solution is an ℓ_p norm, and there must exist a bijection $h : [0, z_0] \leftrightarrow [0, t(0)]$ satisfying feq

$$h(\sqrt[p]{x^p + y^p}) = h(x) + h(y) \qquad (3.36)$$

for $(x,y) \in \mathcal{R}^2_{0+}$ such that $\ell_p(x,y) < z_0$ and $0 \le x,y \le z_0$. As $\ell_p(x,y) < z_0$ implies $x < z_0$, $y < z_0$, the domain of the above feq consists of $(x,y) \in \mathcal{R}^2_{0+}$ such that $x^p + y^p < z_0^p$. For $z_0 = +\infty$, $p \ge 1$ we set $z_0^p = +\infty$.

Let us introduce substitutions $u = x^p$, $v = y^p$ for $x,y \ge 0$, $x^p + y^p < z_0^p$ and put them into (3.36). We obtain feq

$$h(\sqrt[p]{u+v}) = h(\sqrt[p]{u}) + h(\sqrt[p]{v})) \tag{3.37}$$

for $u,v \ge 0$ such that $u + v < z_0^p$. Introducing function $h' : [0, z_0^p] \leftrightarrow [0,1]$, $h'(z) = h(\sqrt[p]{z})$, the above can be written as

$$h'(u+v) = h'(u) + h'(v). \tag{3.38}$$

That is, if a function h solves (3.36), then function h' must solve Cauchy's functional equation on restricted domain $Z_0 = \{(u,v) \,|\, u \ge 0, v \ge 0,\ u + v < z_0^p\}$.

In [1], there is proved that only solutions of this feq are restrictions of linear functions $g(x) = qx$, $x \in \mathcal{R}$, $q \in \mathcal{R}$ on corresponding domain. By this result and due to the specification of domain Z_0, we have $h'(u) = qu$ for $u \in [0, z_0^p)$. This reads as $h(\sqrt[p]{z^p}) = qz^p$ for $z \in [0, z_0)$, i.e., $h(z) = qz^p$. Moreover, as the range of h is considered to be non-negative, $q < 0$ is not allowed. Similarly, due to the strictly increasing character of h, $q = 0$ is not allowed.

By the composition of h function, $h(z) = t(f(z))$ and the fact that $h(z_0) = 0$ we obtain

$$t(f(z)) = qz^p \quad \text{for} \quad z \in [0, z_0),\ q > 0,\ p \ge 1. \tag{3.39}$$

Let us remark that according to the representation theorem for continuous Archimedean t-norms, the additive generator is unique up to a positive multiplicative constant. As q in (3.39) is an arbitrary positive real number, the concrete version of additive generator does not affect formula (3.39) because the used multiplicative constant can be incorporated into $q \in \mathcal{R}_+$.

As h is assumed to be continuous on its domain $[0, z_0]$, from definition of h for z_0 we get $t(0) = h(z_0) = \lim_{z \to z_0^-} h(z) = \lim_{z \to z_0^-} qz^p = qz_0^p$, which gives $z_0 = \sqrt[p]{t(0)/q}$. Thus $qz^p \in [0, t(0))$ for $z \in [0, z_0)$, and (3.39) can be written in form

$$f(z) = t^{-1}(qz^p) \quad \text{for} \quad z \in [0, z_0 = \sqrt[p]{t(0)/q}\,),\ q > 0,\ p \ge 1. \tag{3.40}$$

Again, as f must be continuous on its domain $[0, z_0]$, we get from the above formula $f(z_0) = \lim_{z \to z_0^-} f(z) = t^{-1}(qz_0^p = t(0)) = 0$, which is consistent with the specification of f as the restriction of *act* function on interval $[0, z_0]$.

As we have defined $act(z) = 0$ for $z \in [z_0, +\infty]$, we can conclude the proof by the following statement: If a solution $s = (act(z), n(x,y))$ of (3.17) exists, then $n(x,y) = \sqrt[p]{x^p + y^p}$ for some $p \ge 1$, $x,y \ge 0$, i.e., $n(x,y)$ is an ℓ_p norm on $\mathcal{R}^2_{0+}$; and *act* is specified according to formula

$$act(z) = \begin{cases} t^{-1}(qz^p) & \text{for} \quad z \in [0, z_0 = \sqrt[p]{t(0)/q}\,], \\ 0 & \text{for} \quad z \in [z_0, +\infty]. \end{cases} \tag{3.41}$$

Employing the notion of pseudo-inverse of t, the above can be written in a more elegant way as

$$act(z) = t^{(-1)}(qz^p) \quad \text{for} \quad z \in [0, +\infty],\ q > 0,\ p \ge 1, \tag{3.42}$$

which is the assertion of the theorem.

To end let us add two remarks: First, as a side-result, we have shown that the norm in a solution of feq (3.17) must be an ℓ_p norm on $\mathcal{R}^2_{0+}$. With respect to a solution of original feq (3.9) for $n = 2$, the norm must be again an ℓ_p norm on whole plane $\mathcal{R}^2$. This result follows from the direct computation of the formula for $A_j(\boldsymbol{x})$ when *act* is specified according to (3.42). See, two paragraphs bellow formula (3.14).

Second, let us comment on a solution of feq (3.15) for $n > 2$. Clearly, the proof of sufficiency of *act* specification according to (3.42) holds for any $n \ge 2$. The necessary part of the proof proceeds almost in the same way as for $n = 2$ due to the following reduction: If a solution of (3.15) exists for all $\boldsymbol{x} \in \mathcal{R}_{0+}^n$, then it is also the solution for all n-tuples $(x, y, 0, \ldots, 0)$, $(x, y) \in \mathcal{R}_{0+}^2$. Thus, if we set $x_i = 0$ for $i = 3, \ldots, n$ we obtain feq (3.15) in form

$$act(x) \star act(y) = act(n_0(x, y)), \tag{3.43}$$

where $n_0(x, y) = n(x = x_1, y = x_2, 0, \ldots, 0)$ and $(x, y) \in \mathcal{R}_{0+}^2$.

If we replace $n(x, y)$ by $n_0(x, y)$ in (3.17), then Lemma 1 and consequential considerations still work and we find out that $n_0(x, y)$ must be a quasi-arithmetic mean. If $n(x_1, \ldots, x_n)$ is a norm in $\mathcal{R}_{0+}^n$, then $n_0(x, y) = n(x, y, 0 \ldots, 0)$ is a norm in $\mathcal{R}_{0+}^2$. Thus $n_0(x, y)$ must be an ℓ_p norm on $\mathcal{R}_{0+}^2$. This completes the reduction of $n > 2$ case on $n = 2$ case. The proof then proceeds in the same way as for $n = 2$. □

4 Examples of radial fuzzy systems

In this section we show several examples of radial fuzzy systems which are based on different continuous Archimedean t-norms.

4.1 Gaussian radial fuzzy system

The first radial fuzzy system we present here is the system based on the product t-norm. In fact, the properties of the Gaussian system led us to the idea of radial fuzzy systems.

The product t-norm $T_{\mathrm{P}}(x, y) = xy$, $(x, y) \in [0, 1]^2$ is continuous Archimedean with the additive generator $t(z) = -\ln(z)$, $z \in [0, 1]$, $t(0) = +\infty$. As $t(0)$ is infinite, the pseudo-inverse of t corresponds to the ordinary inverse, i.e., $t^{(-1)}(z) = t^{-1}(z) = \exp(-z)$ for $z \in [0, +\infty]$, $\exp(-\infty) = 0$. By employing Theorem 3 and setting $q = 1$, $p = 2$ we obtain the specification of act function as follows:

$$act(z) = \exp(-z^2) \quad \text{for} \quad z \in [0, +\infty]. \tag{4.1}$$

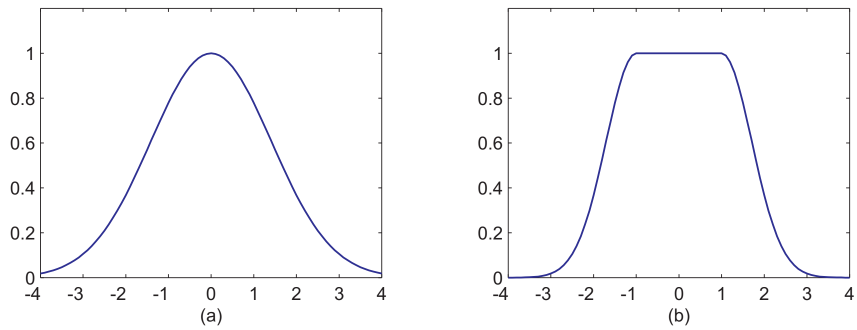

Figure 4.1: Fuzzy sets of a Gaussian radial fuzzy system: (a) antecedent $a = 0$, $b = 2$; (b) consequent $c = 0$, $d = 1$, $s = 1$.

This specification of act function determines the membership functions of one-dimensional fuzzy sets A_{ji}, B_j according to Definition 1 as

$$A_{ji}(x_i) = \exp\left[-\left(\frac{x_i - a_{ji}}{b_{ji}}\right)^2\right], \tag{4.2}$$

$$B_j(y) = \exp\left[-\left(\frac{\max\{0, |y - c_j| - s_j\}}{d_j}\right)^2\right]. \tag{4.3}$$

It can be easily observed that membership functions of A_{ji}s correspond to the well know Gaussian curves, which explains the name of this class of fuzzy systems. In the case of B_j the obtained shape is not a proper Gaussian curve, but the modified one, which we call a trapezoid-like Gaussian curve. An example of both Gaussian curves is presented in Fig. 4.1.

Due to the well known behavior of Gaussian curves with respect to the product, we can check the radial property for a Gaussian system. Indeed, the specification of an antecedent in a Gaussian system reads as follows:

$$A_j(\boldsymbol{x}) = \prod_i \exp\left[-\left(\frac{x_i - a_{ji}}{b_{ji}}\right)^2\right] = \exp\left[-\sum_i \left(\frac{x_i - a_{ji}}{b_{ji}}\right)^2\right] = \exp(-||\boldsymbol{x} - \boldsymbol{a}_j||^2_{2\boldsymbol{b}_j}), \tag{4.4}$$

where $||\cdot||_{2\boldsymbol{b}_j}$ is the scaled Euclidean norm ($p = 2$).

4.2 Łukasiewicz radial fuzzy system

This system is based on the Łukasiewicz t-norm $T_{\mathrm{L}} = \max\{0, x + y - 1\}$. The Łukasiewicz t-norm is continuous Archimedean with the aditive generator $t(z) = 1 - z$ for $z \in [0, 1]$, $t(0) = 1$. The inverse of the aditive generator is the generator itself, i.e., $t^{-1}(z) = 1 - z$ for $z \in [0, 1]$. The pseudo-inverse in this case is specified according to formula $t^{(-1)}(z) = \max\{0, 1 - z\}$, $z \in [0, +\infty]$. By setting $q = 1$, $p = 1$ we obtain the specification of *act* function according to Theorem 1 as

$$act(z) = \max\{0, 1 - z\} \quad \text{for} \quad z \in [0, +\infty]. \tag{4.5}$$

The specification determines the membership functions of fuzzy sets forming the j-th IF-THEN rule in form

$$A_{ji}(x_i) = \max\left\{0, 1 - \left|\frac{x_i - a_{ji}}{b_{ji}}\right|\right\}, \tag{4.6}$$

$$B_j(y) = \max\left\{0, 1 - \frac{\max\{0, |y - c_j| - s_j\}}{d_j^2}\right\}. \tag{4.7}$$

An example of such functions is presented in Fig. 4.2. We can see that antecedent sets correspond to the triangular and consequent to the trapezoidal fuzzy sets.

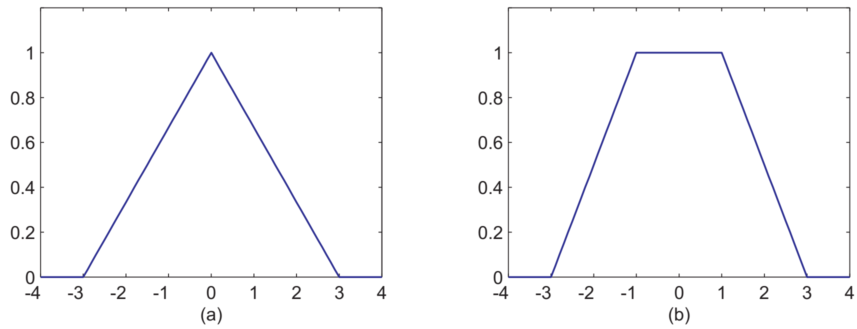

Figure 4.2: Fuzzy sets of a Łukasiewicz radial fuzzy system: (a) antecedent $a = 0$, $b = 3$; (b) consequent $c = 0$, $d = 2$, $s = 1$.

The representation of an antecedent in a Łukasiewicz system is based on the scaled octaedric ℓ_p norm $||\boldsymbol{u}||_{1\boldsymbol{b}} = \sum_i |u_i/b_i|$. That is,

$$A_j(\boldsymbol{x}) = \max\{0, 1 - ||\boldsymbol{x} - \boldsymbol{a}_j||_{1\boldsymbol{b}_j}\} = \max\left\{0, 1 - \sum_{i=1}^{n}\left|\frac{x_i - a_{ji}}{b_{ji}}\right|\right\}. \tag{4.8}$$

4.3 Schweizer-Sklar radial fuzzy system

This system is based on the Schweizer-Sklar family of t-norms and we present it here because this class of t-norms leads to a somehow uncommon shapes of membership functions.

The Schweizer-Sklar family of t-norms is specified as

$$T_\lambda(x,y) = (\max\{x^\lambda + y^\lambda - 1, 0\})^{1/\lambda} \tag{4.9}$$

for parametr $\lambda \in (-\infty, 0) \cup (0, +\infty)$. $T_{-\infty}$ corresponds to the minimum t-norm. T_0 to the product t-norm and $T_{+\infty}$ to the drastic product [5].

Schweizer-Sklar t-norms are continuous Archimedean for $\lambda = (-\infty, +\infty)$ with additive generators $t(z) = (1-z^\lambda)/\lambda$ for $\lambda \in (-\infty, 0)\cup(0, +\infty)$; and $t(z) = -\ln(z)$ for $\lambda = 0$. For the first case the inverse writes as $t^{-1}(z) = (1-\lambda z)^{1/\lambda}$ for $z = [0, t(0)]$. The form of corresponding pseudo-inverses depends on value of λ. If $\lambda > 0$, then $t(0) = 1/\lambda$ and the pseudo-inverse writes as $t^{(-1)} = (1 - \lambda \min\{1/\lambda, z\})^{1/\lambda}$. If $\lambda < 0$, then $t(0) = +\infty$ and pseudo-inverse corresponds to the inverse, i.e., $t^{(-1)} = (1 - \lambda z)^{1/\lambda}$.

In the following, we will consider the case of $\lambda > 0$. That is, by choosing $\lambda > 0$ and $q = 1$, $p = 1$ we obtain the specification of *act* function in form

$$act_\lambda(z) = (1 - \min\{1, \lambda z\})^{1/\lambda} = (\max\{0, 1 - \lambda z\})^{1/\lambda}. \tag{4.10}$$

The specification of *act* function determines the membership functions of antecedents and consequents:

$$A_{ji}(x_i) = \left(\max\left\{0, 1 - \lambda\left|\frac{x_i - a_{ji}}{b_{ji}}\right|\right\}\right)^{1/\lambda}, \tag{4.11}$$

$$B_j(y) = \left(\max\left\{0, 1 - \lambda\frac{\max\{0, |y - c_j| - s_j\}}{d_j}\right\}\right)^{1/\lambda}. \tag{4.12}$$

Examples of these function for $\lambda = 2$ are presented in Fig. 4.3. We can see that flanks of membership functions are concave other than convex, which is more usual shape in practical applications.

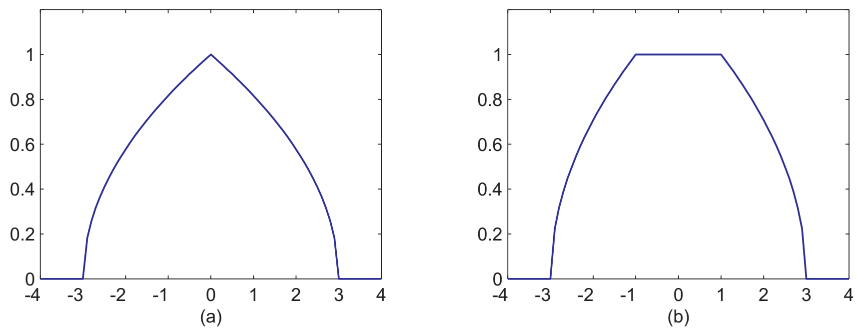

Figure 4.3: Fuzzy sets of a Schweizer-Sklar radial fuzzy system for $\lambda = 2$: (a) antecedent $a = 0$, $b = 6$; (b) consequent $c = 0$, $d = 4$, $s = 1$.

5 Conclusion

The main goal of the report was to present the full proof of sufficient and necessary conditions on a shape of the activating function in a fuzzy system to it be radial.

We have shown that the shape is determined by a composition of the pseudo-inverse of the additive generator of the t-norm used in the fuzzy system, with a polynomial qz^p, $q > 0$, $p \geq 1$. As a corollary, it was proved that the norm occurring in representation of antecedents in radial fuzzy system must be necessary an ℓ_p norm if a continuous Archimedean t-norm is employed.

As an application of this result, we can consider the problem of representation of fuzzy systems by radial basis neural networks and vice versa. If the t-norm for a (conjuctive) fuzzy system is selected and it is continuous Archimedean, the result advice us as how to design shapes of fuzzy sets so that this fuzzy system be representable in a form of an RBF network in which hidden nodes utilize activating function which corresponds to the selected shapes (membership functions) of fuzzy sets.

The stronger result can be obtained. Let the enhanced radial property be formulated by the following feq on domain $\mathcal{R}^n$:

$$A_j(\boldsymbol{x}) = A_{j1}(x_1) \star \cdots \star A_{jn}(x_n) = f(\,||\boldsymbol{x} - \boldsymbol{a}_j||_{\boldsymbol{b}_j}), \tag{5.1}$$

where $f : [0, +\infty] \to [0, 1]$ is a non-increasing continuous function such that $f(0) = 1$, $f(+\infty) = 0$. That is, in comparison with the basic radial property we do not require that f corresponds to *act* which is used in A_{ji}s specification. The used t-norm $\star$ is again considered continuous Archimedean. The question is what are the conditions on *act*, f and $||\cdot||$ in order to the enhanced radial property hold.

Solution: Let us solve the following feq for $(x, y) \in \mathcal{R}^2_{0+}$:

$$f(n(x, y)) = t^{-1}(\min\{t(0), t(act(x)) + t(act(y))\}) \tag{5.2}$$

By setting $y = 0$ we get $(t(act(0)) = 0)$:

$$f(n(x, 0)) = t^{-1}(\min\{t(0), t(act(x))\}). \tag{5.3}$$

As t is strictly decreasig on $[0, 1]$, we have $\min\{t(0), t(act(x))\}) = t(act(x))$ and the above has form

$$f(n(x, 0)) = act(x). \tag{5.4}$$

Considering substitution $x = u/n(1, 0)$ for $u \in \mathcal{R}_{0+}$ and the linearity of n, $n(x, 0) = x \cdot n(1, 0)$ for $x \in \mathcal{R}_{0+}$, we have $n(u/n(1, 0), 0) = u$ and therefore for all $u \in \mathcal{R}_{0+}$

$$f(n(u/n(1, 0), 0) = act(u/n(1, 0)), \tag{5.5}$$
$$f(u) = act(u/n(1, 0)). \tag{5.6}$$

Let us put this expression for f into (5.2). We obtain

$$act(n(x, y)/n(1, 0)) = t^{-1}(\min\{t(0), t(act(x)) + t(act(y))\}). \tag{5.7}$$

If $n(x, y)$ is a norm, then $n_1(x, y) = n(x, y)/n(1, 0)$ is also a norm. Hence we have

$$act(n_1(x, y)) = t^{-1}(\min\{t(0), t(act(x)) + t(act(y))\}) \tag{5.8}$$

where $n_1(x, y)$ is a norm on $\mathcal{R}^2_{0+}$. As we have proved, the above feq has a solution if and only if $n_1(x, y)$ is an ℓ_p norm. This gives us immediately $n(1, 0) = 1$ and (5.6) reads as

$$f(u) = act(u) \quad \text{for} \quad u \in \mathcal{R}_{0+}. \tag{5.9}$$

By this equality we see that the enhanced radial property holds if and only if the basic radial property holds and we can state that: The antecedent of an IF-THEN rule which is based on a continuous Archimedean t-norm t, can be represented by a hidden node of an RBF neural network $\Phi(||\boldsymbol{x} - \boldsymbol{a}||)$ if and only if $\Phi(z) = t^{(-1)}(qz^p)$, $q > 0$, $p \geq 1$, where $t^{(-1)}$ is the pseudo-inverse of the additive generator of the employed t-norm t.

6 Appendix

Here we discuss norm $n(x, y) = 2x + 4y - 4\sqrt{xy}$ on $\mathcal{R}^2_{0+}$. The norm is an example of the *non-monotonic norm*. Let $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$. A norm on $\mathcal{R}^2_{0+}$ is monotonic if $(x_1, y_1) \leq (x_2, y_2)$ implies $n(x_1, y_1) \leq n(x_2, y_2)$. Not all norms on $\mathcal{R}^2_{0+}$ are monotonic. The question is how to find an example of such a norm.

Our search is based on the Euler's feq on homogeneous function of degree $k = 1$. The solution of this feq states that if a function $f(x, y)$ defined on $\mathcal{R}^2_{0+}$ is homogenous of degree $k \in \mathcal{N}$, i.e., if $f(cx, cy) = c^k f(x, y)$ for $x, y, c > 0$, $k \in \mathcal{N}$, then there exists a function g such that $f(x, y) = x \cdot g(y/x)$

and vice versa. If g is a non-monotonic function we can obtain (triangle inequality must hold) a non-monotonic norm.

Our norm is obtained by chosing $g(z) = (2\sqrt{z} - 1)^2 + 1$, i.e.,

$$n(x, y) = x \cdot g(y/x) = x \cdot [(2\sqrt{y/x} - 1)^2 + 1] = 2x + 4y - 4\sqrt{xy}. \tag{6.1}$$

for $x, y > 0$. However, the right side is defined on all $x, y \geq 0$.

Let us show that (6.1) constitutes the norm on $\mathcal{R}^2_{0+}$.

(N1): $n(0, 0) = 0$. If $x > 0$ and $y = 0$ then $n(x, y) = 2x > 0$. If both $x > 0$, $y > 0$ then $g(y/x) > 0$ from the composition of quadratic function g.

(N2): Obviously, $n(cx, cy) = c \cdot n(x, y)$ for $c > 0$.

(N3) (the triangle inequality): The following chain of inequalities hold for arbitrary $x_1, x_2, y_1, y_2 \geq 0$, $a = x_1 y_2, b = x_2 y_1$

$$
\begin{aligned}
0 &\leq (\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b, && (6.2)\\
2\sqrt{ab} &\leq a + b, && (6.3)\\
2\sqrt{x_1 y_2 x_2 y_1} &\leq x_1 y_2 + x_2 y_1, && (6.4)\\
x_1 y_1 + 2\sqrt{x_1 y_2 x_2 y_1} + x_2 y_2 &\leq (x_1 + x_2)(y_1 + y_2), && (6.5)\\
(\sqrt{x_1 y_1} + \sqrt{x_2 y_2})^2 &\leq (x_1 + x_2)(y_1 + y_2), && (6.6)\\
\sqrt{x_1 y_1} + \sqrt{x_2 y_2} &\leq \sqrt{(x_1 + x_2)(y_1 + y_2)}, && (6.7)\\
-4\sqrt{x_1 y_1} - 4\sqrt{x_2 y_2} &\leq -4\sqrt{(x_1 + x_2)(y_1 + y_2)}, && (6.8)\\
n(x_1, y_1) + n(x_2, y_2) &\geq n(x_1 + x_2, y_1 + y_2). && (6.9)
\end{aligned}
$$

To demonstrate the non-monotonic character of $n(x, y)$ let us consider points $(1, 0.25)$, $(1, 0)$. We have $(1, 0) \leq (1, 0.25)$ but $n(1, 0) = 2 > n(1, 0.25) = 1$.

Obviously, since $n(x, y)$ is norm on $\mathcal{R}^2_{0+}$, $n(x, y) = 2|x| + 4|y| - 4\sqrt{|xy|}$ is norm on $\mathcal{R}^2$.

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